ON REGULARITY IN HUREWICZ FIBER SPACES(1)

BY
PATRICIA TULLEY(2)

1. **Introduction.** If E and B are topological spaces, p is a continuous function from E to B, $\Omega_p = \{(e,\omega) \in E \times B^I | p(e) = \omega(0)\}$, and there is a continuous function λ from Ω_p to E^I such that $\lambda(e,\omega)(0) = e$ and $p \circ \lambda(e,\omega) = \omega$ for all $(e,\omega) \in \Omega_p$, then (E,p,B) is called a fiber space (in the sense of Hurewicz). λ is said to be a lifting function for the space. If λ has the property that $\lambda(e,\omega)$ is a constant path whenever ω is a constant path, then λ is said to be regular. A fiber space is said to be regular provided it admits a regular lifting function.

It is the purpose of this paper to present results related to the question of what fiber spaces are regular. A condition on the base space which insures regularity is considered. An example of a nonregular fiber space is presented.

In [1], Hurewicz notes that not every fiber space is regular, that the fiber space of paths over a space B does not always have this property, and that, specifically, such a space fails to be regular when B is the "joint of two enumerable (infinite), connected Hausdorff spaces"(3). Here "joint" is the more commonly used "join" of two spaces (see §3). However, it is shown in §4 that such a space actually is regular. The failure of Hurewicz' example raises the question of the existence of a nonregular fiber space.

2. **Preliminaries.** Throughout, the word space means Hausdorff space, fiber space is used in the sense of Hurewicz as explained above, map means continuous function, I denotes the unit interval, and, for spaces X and Y, X^Y is the space of all maps of Y into X with the compact-open (c-o) topology. That is, the topology for X^Y has as a basis all finite intersections of sets of the form $W(C,O)$ where for any compact set C in Y and any open set O in X, $W(C,O) = \{f \in X^Y | f(C) \subset O\}$.

The following results and remarks are known and are listed here for reference.

2.1. (E,p,B) is a regular fiber space if and only if the following "Covering Homotopy Condition" is satisfied.

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(3) See p. 957 of [1].

Covering Homotopy Condition: For any space X and maps H: $X \times I \to B$ and g: $X \times \{0\} \to E$ such that $p \circ g = H$ on $X \times \{0\}$, there is a map G: $X \times I \to E$ such that G is an extension of g, $p \circ G = H$, and if, for $x_0 \in X$, $H(x_0, t)$ is independent of t, then $G(x_0, t)$ is also independent of t.

2.2. A well-known example of a fiber space is the triple (B^I, p, B) where B is an arbitrary space and $p(\omega) = \omega(1)$ for all $\omega \in B^I$. This is the space of paths over B which was mentioned in §1. For this space a lifting function λ: $\Omega_p \to (B^I)^I$ is given by $\lambda(\alpha, \omega)(t)(s) = \alpha(2s/(2-t))$ for $0 \leqq s \leqq 1 - t/2$ and $\lambda(\alpha, \omega)(t)(s) = \omega(2s - 2 + t)$ for $1 - t/2 \leqq s \leqq 1$. Note that this particular λ is not regular.

2.3. If (E, p, B) is a regular fiber space, E^* is a space, p^* is a map of E^* into B, and f is a homeomorphism of E onto E^* such that $p^* \circ f = p$, then (E^*, p^*, B) is a regular fiber space.

2.4. If f: $Z \to X^Y$, Y is locally compact, and F: $Z \times Y \to X$ is defined by $F(z, y) = f(z)(y)$, then F is a map if and only if f is a map.

2.5. If each of Y and Z is locally compact and f: $(X^Y)^Z \to (X^Z)^Y$ is defined by $f(\phi)(y)(z) = \phi(z)(y)$, then f is a homeomorphism.

2.6. A map f: $X \to Y$ induces a map $\bar{f}$: $X^Z \to Y^Z$ defined by $\bar{f}(\phi)(z) = f(\phi(z))$.

2.7. If f is a map of X into I, Y is compact, and F: $X^Y \to I$ is defined by $F(\phi) = \sup\{f(\phi(y)) \mid y \in Y\}$, then F is a map. This statement remains true if sup is replaced by inf.

3. A condition on the base space which insures regularity.

A regular lifting function can be defined for a fiber space whenever it is possible to "measure" paths in the base space in a continuous manner which gives zero "measure" to the constant paths and only to the constant paths. The following definition makes this notion precise.

DEFINITION. A space B is said to admit a ϕ-function provided there is a map ϕ: $B^I \to I$ such that $\phi(\omega) = 0$ if and only if ω is a constant path.

If B is a metric space, then $\phi(\omega) = \text{diam}(\omega(I))$ defines a ϕ-function for B. It is this property of a metric space which is used in [1] to prove that, "Every fiber space (E, p, B), where B is a metric space, is regular"(4). Therefore, essentially the same proof yields the following generalization.

THEOREM 3.1. *If the space B admits a ϕ-function, then any fiber space (E, p, B) is regular.*

Proof. By hypothesis there is a map ϕ: $B^I \to I$ such that $\phi(\omega) = 0$ if and only if ω is a constant path. Define g: $B^I \to B^I$ by $g(\omega)(t) = \omega(t/\phi(\omega))$ for $t < \phi(\omega)$ and $g(\omega)(t) = \omega(1)$ for $\phi(\omega) \leqq t \leqq 1$. It is easy to show that g is a map.

(4) See p. 957 of [1].

Now if λ is any lifting function for (E,p,B) and λ': $\Omega_p \to E^I$ is defined by $\lambda'(e,\omega)(t) = \lambda(e,g(\omega))(\phi(\omega)\cdot t)$ for all $t \in I$, then λ' is continuous and, furthermore, is a regular lifting function for (E,p,B).

The following definitions and notation concerning the join of two spaces will be used in the statement and proof of the next theorem.

If X and Y are spaces, then $J(X,Y)$, the join of X and Y, is the identification space obtained from $X \times I \times Y$ by means of the equivalence relation $\sim$ where $(x,t,y) \sim (x',t',y')$ if and only if one of the following conditions holds:

(1) $x = x'$ and $t = t' = 0$,

(2) $y = y'$ and $t = t' = 1$,

(3) $x = x'$, $y = y'$, and $t = t'$.

Elements of $J(X,Y)$ are thus equivalence classes and in general will be denoted by $[(x,t,y)]$. Additional properties and notation follow. The proofs of statements depend almost exclusively on the identification topology on $J(X,Y)$ and are omitted.

(1) $x \to [(x,0,y)]$ imbeds X in $J(X,Y)$ as a closed subspace. X will be used to denote this subspace and x to denote the element $[(x,0,y)]$. Analogous remarks are to be assumed for Y.

(2) If $M = \{[(x,t,y)] \in J(X,Y) \mid 0 < t < 1\}$, then $(x,t,y) \to [(x,t,y)]$ defines a homeomorphism of $X \times (0,1) \times Y$ onto M, and open subset of $J(X,Y)$.

(3) If p: $J(X,Y) \to I$, p_X: $J(X,Y) - Y \to X$, and p_Y: $J(X,Y) - X \to Y$ are defined by $p([(x,t,y)]) = t$, $p_X([(x,t,y)]) = x$, and $p_Y([(x,t,y)]) = y$, then p, p_X, and p_Y are all maps and induce maps $\bar{p}$: $J(X,Y)^I \to I^I$, $\bar{p}_X$: $(J(X,Y) - Y)^I \to X^I$, and $\bar{p}_Y$: $(J(X,Y) - X)^I \to Y^I$ defined as in 2.6.

THEOREM 3.2. *If each of the spaces X and Y admits a ϕ-function, then the space $J(X,Y)$ admits a ϕ-function.*

Proof. Let ϕ_X be a ϕ-function for X and ϕ_Y be a ϕ-function for Y where X and Y are considered as subspaces of $J(X,Y)$.

Define the following sets in $J(X,Y)^I$:

$$X^* = \{\omega \in J(X,Y)^I \mid \omega(I) \cap X \neq \emptyset \text{ and } \omega(I) \cap Y = \emptyset\},$$

$$Y^* = \{\omega \in J(X,Y)^I \mid \omega(I) \cap Y \neq \emptyset \text{ and } \omega(I) \cap X = \emptyset\},$$

$$Q = \{\omega \in J(X,Y)^I \mid \omega(I) \cap X \neq \emptyset \text{ and } \omega(I) \cap Y \neq \emptyset\}.$$

Note that $M^I = \{\omega \in J(X,Y)^I \mid \omega(I) \cap X = \emptyset \text{ and } \omega(I) \cap Y = \emptyset\}$ is an open set in $J(X,Y)^I$ and that $J(X,Y)^I = M^I \cup X^* \cup Y^* \cup Q$ is the union of four mutually exclusive sets.

Define the following maps using previously given notation and properties:

$$d\colon J(X, Y)^I \to I \text{ by } d(\omega) = \operatorname{diam}(\bar{p}(\omega)(I)),$$

$$d_X\colon (J(X, Y) - Y)^I \to I \text{ by } d_X(\omega) = \phi_X(\bar{p}_X(\omega)),$$

$$d_Y\colon (J(X, Y) - X)^I \to I \text{ by } d_Y(\omega) = \phi_Y(\bar{p}_Y(\omega)),$$

$$f_X\colon J(X, Y)^I \to I \text{ by } f_X(\omega) = \inf\{p(\omega(t)) \mid t \in I\},$$

$$f_Y\colon J(X, Y)^I \to I \text{ by } f_Y(\omega) = 1 - \sup\{p(\omega(t)) \mid t \in I\}.$$

Continuity of f_X and f_Y follows from 2.7.

Using these maps define a function ϕ on $J(X, Y)^I$ as follows:

For $\omega \in M^I$ let $\phi(\omega) = d(\omega) + f_Y(\omega) \cdot d_X(\omega) + f_X(\omega) \cdot d_Y(\omega)$.

For $\omega \in X^*$ let $\phi(\omega) = d(\omega) + f_Y(\omega) \cdot d_X(\omega)$.

For $\omega \in Y^*$ let $\phi(\omega) = d(\omega) + f_X(\omega) \cdot d_Y(\omega)$.

For $\omega \in Q$ let $\phi(\omega) = d(\omega) = 1$.

Note that $d(\omega) + f_X(\omega) + f_Y(\omega) = 1$ for all $\omega \in J(X, Y)^I$ and that $d_X(\omega)$ and $d_Y(\omega)$ are in I whenever they are defined. It follows that $\phi\colon J(X, Y)^I \to I$.

ϕ is continuous at points of M^I by its definition at such points and the fact that M^I is open. The following additional proofs of continuity complete the proof that ϕ is a map.

Case I. Let $\omega_0 \in X^*$. Since $Y^* \cup Q$ is closed, it is sufficient to show that $\phi|(X^* \cup M^I)$ is continuous at ω_0. Furthermore, since $d + f_Y \cdot d_X$ is continuous on $X^* \cup M^I$, the desired continuity will follow if it can be shown that for an arbitrary $\epsilon > 0$ there is an open set V containing ω_0 such that $f_X(\omega) < \epsilon$ whenever $\omega \in V \cap M^I$. (Note that $f_X(\omega) < \epsilon$ implies that $f_X(\omega) \cdot d_Y(\omega) < \epsilon$.) Such a V can be found because $f_X(\omega_0) = 0$ and f_X is continuous on all of $J(X, Y)^I$. Therefore, ϕ is continuous at ω_0.

Case II. Let $\omega_0 \in Y^*$. ϕ is continuous at ω_0 by an argument analogous to that given in Case I.

Case III. Let $\omega_0 \in Q$. Then $\phi(\omega_0) = d(\omega_0) = 1$. Since $\phi(\omega) \in I$ and $\phi(\omega) \geqq d(\omega)$ for all $\omega \in J(X, Y)^I$, continuity of ϕ at ω_0 follows from that of d at ω_0.

It remains to show that $\phi(\omega) = 0$ if and only if ω is a constant path. For $\omega \in Q$ this is clear because ω is not a constant path and $\phi(\omega) = 1$. Proofs for the other cases follow.

Case I. If $\omega \in M^I$, then $f_X(\omega) \neq 0$ and $f_Y(\omega) \neq 0$. Consequently, referring to the definitions of ϕ, it is obvious that $\phi(\omega) = 0$ is equivalent to the condition that $d(\omega) = 0$, $d_X(\omega) = 0$, and $d_Y(\omega) = 0$ hold simultaneously. Since ϕ_X and ϕ_Y are ϕ-functions, this condition is equivalent to the assertion that $\bar{p}(\omega)$, $\bar{p}_X(\omega)$, and $\bar{p}_Y(\omega)$ are all constant paths which holds for $\omega \in M^I$ if and only if ω is a constant path.

Case II. If $\omega \in X^*$, then $f_Y(\omega) \neq 0$. Thus $\phi(\omega) = 0$ is equivalent to the assertion that $d(\omega) = 0$ and $d_X(\omega) = 0$ or, as before, that $\bar{p}(\omega)$ and $\bar{p}_X(\omega)$

are both constant paths. For $\omega \in X^*$ these paths are both constant if and only if ω is a constant path.

Case III. If $\omega \in Y^*$, then a proof analogous to that in Case II can be given.

Thus ϕ is a ϕ-function for $J(X, Y)$ and the theorem is proved.

If $C(X)$ is the cone over a space X, then $C(X)$ can be imbedded homeomorphically in $J(X, X)$. Also, the cartesian product $X \times Y$ can be imbedded homeomorphically in $J(X, Y)$. Since admitting a ϕ-function is obviously a hereditary property, Theorem 3.2 has the following corollaries.

COROLLARY 3.2:1. *If the space X admits a ϕ-function and $C(X)$ is the cone over X, then $C(X)$ admits a ϕ-function.*

COROLLARY 3.2:2. *If each of the spaces X and Y admits a ϕ-function, then $X \times Y$ admits a ϕ-function.*

4. **A remark concerning Hurewicz' example.** The fiber space to be considered is the path space over $J(B, B)$ where B is a countably infinite, connected Hausdorff space (see §1). Since any countable space contains no nondegenerate paths, B admits a trivial ϕ-function and, by Theorem 3.2, $J(B, B)$ admits a ϕ-function. By Theorem 3.1, any fiber space having $J(B, B)$ as base space is regular and, in particular, the space of paths over $J(B, B)$ is regular.

5. **Regularity for fiber spaces where the base space is a function space.** Theorem 5.1 will give conditions on a function space which insure that it admit a ϕ-function. It is preceded by a lemma which will be used in the proof of the theorem.

Hereafter, $\hat{Y}$ will denote the one-point compactification of any locally compact space Y and ∞ will denote the additional point of $\hat{Y}$.

LEMMA. *Let X and Y be spaces with Y locally compact. Suppose that there exist maps $h: X \times Y \to I$ and $\gamma: \hat{Y} \to I$ such that $\gamma^{-1}(0) = \infty$. Furthermore, let h^*: $X \to I$ be defined by $h^*(x_0) = \sup\{h(x_0, y) \cdot \gamma(y) \mid y \in Y\}$. Then h^* is a map.*

Proof. Define g: $X \times \hat{Y} \to I$ by $g(x, y) = h(x, y) \cdot \gamma(y)$ for $y \in Y$ and $g(x, \infty) = 0$. g is continuous and by 2.4 induces a map G: $X \to I^{\hat{Y}}$ with $G(x)(y) = g(x, y)$. By 2.7, there is a map F: $I^{\hat{Y}} \to I$ given by

$$F(\alpha) = \sup\{\alpha(y) \mid y \in \hat{Y}\}.$$

But, since $g(x_0, \infty) = 0$ for each $x_0 \in X$,

$$F \circ G(x_0) = \sup\{g(x_0, y) \mid y \in \hat{Y}\} = \sup\{g(x_0, y) \mid y \in Y\} = h^*(y).$$

Thus, $F \circ G = h^*$ and the lemma is proved.

THEOREM 5.1. *If the space B admits a ϕ-function, Y is a locally compact* *space, and there is a map γ: $\hat{Y} \to I$ such that $\gamma^{-1}(0) = \infty$, then B^Y admits a ϕ-function.*

Proof. Let f: $(B^Y)^I \to (B^I)^Y$ be defined by $f(\omega)(y)(t) = \omega(t)(y)$ for all $\omega \in (B^Y)^I$, $t \in I$, and $y \in Y$ and g: $(B^I)^Y \times Y \to B^I$ be defined by $g(\omega, y) = \omega(y)$ for all $\omega \in (B^I)^Y$ and $y \in Y$. Since Y and I are locally compact, f is a homeomorphism and g is continuous.

Let ϕ' be a ϕ-function for B and note that $\phi' \circ g$: $(B^I)^Y \to I$.

Define G: $(B^I)^Y \to I$ by $G(\omega) = \sup\{\phi' \circ g(\omega, y) \cdot \gamma(y) \mid y \in Y\}$. By the preceding lemma G is a map.

Let $\phi = G \circ f$. That ϕ is a ϕ-function for B^Y follows easily from a consideration of the definitions involved.

COROLLARY 5.1:1. *If the space B admits a ϕ-function and Y is compact, then B^Y admits a ϕ-function.*

COROLLARY 5.1:2. *If the space X admits a ϕ-function and A is a countable discrete space, then X^A admits a ϕ-function.*

Corollary 5.1:2 asserts that if a space X admits a ϕ-function, A is countable, and $X_a = X$ for each $a \in A$, then $\prod_{a \in A} X_a$ (with the Tychonoff topology) admits a ϕ-function. In this product all of the factors are the same. However, similar methods of proof can be used to show that this need not be true, i.e., admitting a ϕ-function is a countable cartesian product invariant. The following theorem demonstrates that this statement is false if the word countable is omitted.

THEOREM 5.2. *If X is a space containing at least one nondegenerate path and A is an uncountable space with the discrete topology, then X^A does not admit a ϕ-function.*

Proof. Let ω be a nondegenerate path in X and define the following paths in X^A:

(1) For every $a \in A$ define $\omega_a \in (X^A)^I$ by $\omega_a(t)(a) = \omega(t)$ for all $t \in I$ and $\omega_a(t)(a') = \omega(0)$ for all $t \in I$ and $a' \in A$ such that $a' \neq a$.

(2) Define $\mu \in (X^A)^I$ by $\mu(t)(a) = \omega(0)$ for all $t \in I$ and $a \in A$.

Now, suppose that ϕ is a ϕ-function for X^A and note that $\phi(\omega_a) \neq 0$ holds for each $a \in A$ and that $\phi(\mu) = 0$.

By the construction of the above paths and the nature of the topology on X^A, μ is a limit point for any infinite subset of $\{\omega_a \mid a \in A\}$. Hence, the continuity of λ implies that for each positive integer i there are only a finite number of $a \in A$ such that $\phi(\omega_a) \geqq 1/i$.

But then, $A = \bigcup_{i=1}^{\infty} \{a \in A \mid \phi(\omega_a) \geqq 1/i\}$ is a countable union of finite sets and is, therefore, countable. This contradicts the hypothesis and completes the proof of the theorem.

Thus there are spaces which do not admit ϕ-functions. Simple examples show that this condition is not a necessary one on B in order that (E,p,B) be regular. Other conditions insuring regularity come from the following theorem. The proof is omitted; it results from straightforward applications of 2.4 and 2.1.

THEOREM 5.3. *If (E,p,B) is a regular fiber space, Y is a locally compact space, and $\bar{p}$: $E^Y \to B^Y$ is defined by $\bar{p}(\alpha)(y) = p(\alpha(y))$, then $(E^Y, \bar{p}, B^Y)$ is a regular fiber space.*

COROLLARY 5.3:1. *If the space of paths over a space B is regular and Y is a locally compact space, then the space of paths over B^Y is regular.*

Proof. Let (B^I, p, B) and $((B^Y)^I, p^*, B^Y)$, with $p(\omega) = \omega(1)$ for $\omega \in B^I$ and $p^*(\alpha) = \alpha(1)$ for $\alpha \in (B^Y)^I$, be the path spaces under consideration. By hypothesis, (B^I, p, B) is regular. By Theorem 5.3, if $\bar{p}$: $(B^I)^Y \to B^Y$ is defined by $\bar{p}(\beta)(y) = p(\beta(y))$ for $\beta \in (B^I)^Y$, then $((B^I)^Y, \bar{p}, B^Y)$ is a regular fiber space.

By 2.5 there is a homeomorphism f: $(B^I)^Y \to (B^Y)^I$ defined by $f(\beta)(t)(y) = \beta(y)(t)$. Furthermore, f has the property that $p^* \circ f = \bar{p}$.

Applying 2.3, gives the conclusion that $((B^Y)^I, p^*, B^Y)$ is regular.

Suppose that X is a space such that X contains a nondegenerate path and the space of paths over X is regular. Let A be any uncountable discrete space. By Theorem 5.2, X^A does not admit a ϕ-function but, by Corollary 5.3:1, the space of paths over X^A is regular.

Thus the results given so far both limit and direct efforts to construct a nonregular fiber space from known spaces. A consideration of the space of paths over some base space might be fruitful since at least the "natural" lifting function given in 2.2 is not regular. Several restrictions to this approach have been demonstrated. However, in the next section it is shown that the space of paths over a certain subset of an uncountable cartesian product of unit intervals is a fiber space which admits no regular lifting function.

6. An example of a fiber space which is not regular.

DEFINITION OF THE SPACE B. Let A be an uncountable set. The points of B are the points of $C(A)$, the cone over A, but the topology on B is not the usual cone topology (for any topology on A). The points of B will be denoted by (t,a) for $t \in I$ and $a \in A$ with the single identification that $(0,a) = (0,a')$ for all a and a' in A. The vertex of the cone will sometimes be denoted by v. For each $a \in A$ let $B_a = \{(t,a) \in B \mid 0 < t \leqq 1\}$. Define the topology on B as follows:

For any $a \in A$, any $(t,a) \in B_a$, and any positive number ϵ, let $N((t,a),\epsilon) = \{(t',a) \in B_a \mid |t - t'| < \epsilon\}$.

For any positive number ϵ and any finite subset K of A let $N(v,(K,\epsilon)) = \{(t,a) \mid t \in I \text{ and } a \in A - K\} \cup \{(t,a) \mid a \in K \text{ and } t < \epsilon\}$.

Let the collection of all possible $N((t,a),\epsilon)$ and all possible $N(v,(K,\epsilon))$, as defined above, be a basis for the topology of B.

It is easy to show that B is an arcwise connected, compact (Hausdorff) space. Note that for every $a \in A$, B_a is open in B and, in fact, that the subspace $v \cup B_a$ is homeomorphic to I by the mapping $(t,a) \to t$.

Notice that B is homeomorphic in a natural way to the closed subset S of I^A defined by $S = \{f \in I^A \mid f(a) = 0$ for all but *at most one* $a \in A\}$.

Proof that (B^I, p, B) is not regular. Here p is to be defined by $p(\mu) = \mu(1)$. Thus $\Omega_p = \{(\mu,\omega) \in B^I \times B^I \mid \mu(1) = \omega(0)\}$. Assume that the fiber space admits a regular lifting function λ.

Let a_0 be a fixed element of A and define $\mu_0 \in B^I$ by $\mu_0(t) = (1-t, a_0)$ for all $t \in I$. Also, let $A' = A - \{a_0\}$ and for $a \in A'$ define $\omega_a \in B^I$ by $\omega_a(t) = (t,a)$ for all $t \in I$. Let $\tilde{v}$ denote the constant path at v. Clearly $(\mu_0, \tilde{v}) \in \Omega_p$ and $\{(\mu_0,\omega_a) \mid a \in A'\} \subset \Omega_p$.

Now $\lambda(\mu_0,\omega_a)(1)(1) = \omega_a(1) = (1,a) \in B_a$, so there is a real number $s_a < 1$ such that $\lambda(\mu_0,\omega_a)(1)((s_a,1]) \subset B_a$. This insures that a function $\gamma\colon A' \to [0,1)$ can be defined by setting

$$\gamma(a) = \inf\{s \in I \mid \lambda(\mu_0,\omega_a)(1)((s,1]) \subset B_a\}.$$

A' is uncountable; it follows that there is an $s_0 < 1$ such that $\gamma(a) \leqq s_0$ holds for infinitely many $a \in A'$. Let $Q = \{(\mu_0,\omega_a) \mid \gamma(a) \leqq s_0\}$: Q is an infinite subset of Ω_p and in this space $(\mu_0,\tilde{v})$ is a limit point for Q.

Let $q = (s_0+1)/2$ and observe that then $\lambda(\mu_0,\omega_a)(1)(q) \in B_a$ for all $(\mu_0,\omega_a) \in Q$. But $B_{a_0} \cap B_a = \emptyset$ for all $a \in A'$ and thus $\lambda(Q) \cap W = \emptyset$ where $W = W(\{1\}, W(\{q\}, B_{a_0}))$ is an open set in the c-o topology on $(B^I)^I$. However, since λ is regular, $\lambda(\mu_0,\tilde{v})(1)(q) = \lambda(\mu_0,\tilde{v})(0)(q) = \mu_0(q) = (1-q, a_0) \in B_{a_0}$ or $\lambda(\mu_0,\tilde{v}) \in W$. Thus $\lambda(\mu_0,\tilde{v})$ is not a limit point for $\lambda(Q)$. This contradicts the continuity of λ and completes the proof.

It has been noted that the space B is homeomorphic to a closed subset S of I^A. By Theorem 5.3, the fiber space of paths over I^A is regular. If, as usual, $((I^A)^I, p, I^A)$ denotes this space with $p(\omega) = \omega(1)$, then the subfiber space $(p^{-1}(S), p, S)$ is also regular. However, the proof given above shows that (S^I, p, S) is not regular. Of course, if S^I is considered as a subspace of $(I^A)^I$, it is true that S^I is a proper subspace of $p^{-1}(S)$. The example given here fails to be regualr because the top space is too small.

7. **Questions.** (1) Is there a condition on a space B which is weaker then the existence of a ϕ-function but which will insure that any fiber space (E,p,B) be regular?

(2) It is easily verified that if $B \times B$ is a normal space in which the

diagonal $D = \{(x, x) \mid x \in B\}$ is a G_δ-set, then B admits a ϕ-function. What topological properties (other than this and those given in §3 and §5) imply the existence of ϕ-functions?

(3) Suppose that X does not admit a ϕ-function but that the space of paths over X is regular. Is the space of paths over $J(X, X)$ necessarily regular?

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The University of Maryland,
College Park, Maryland