VOLTERRA OPERATORS SIMILAR TO
$J: f(x) \to \int_0^x f(y)\, dy$

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Introduction. In this paper we study perturbations $J + P$ of the Volterra operator

$$J: f(x) \to \int_0^x f(y)\, dy$$

on $L^r(0,1)$ $(1 \leqq r \leqq \infty)$. Sufficient conditions—which are in a precise sense sharp—will be obtained for the similarity of J and $J + P$, where P is also a Volterra operator

$$P: f(x) \to \int_0^x p(x,y) f(y)\, dy.$$

As an important biproduct of this result it follows immediately that the lattices of invariant subspaces of certain Volterra operators are isomorphic to the (known) lattice of invariant subspaces of J[2]. But beyond the result itself and its corollaries, it is interesting to compare the methods used here with those of [4] where we were concerned with the question of similarity to the unilateral shift operator.

As in [4], we rewrite the similarity equation $J = X^{-1}(J + P)X$ as a derivational equation

$$\Delta X = -PX, \tag{1}$$

where Δ is the derivation $\Delta X = JX - XJ$. Guided now by the analogy with the classical differential equation $dX(t)/dt = -P(t)X(t)$ in a Banach algebra (see [8]), we pass to an "integral equation"

$$X = I - \Gamma(PX). \tag{2}$$

Here Γ is an integral—yet to be defined—corresponding to the derivation Δ:

$$\Delta\Gamma(Q) = Q \tag{3}$$

for all Q in a space $\mathcal{S}$ of "integrable" operators. Any solution X of (2) will then also solve (1).

The burden of the method lies in the determination of the integral Γ.

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(2) A detailed discussion of this question is found in Kalisch [5].

More precisely, spaces $\mathscr{L}$ and $\mathscr{A}$ of operators and a linear mapping Γ: $\mathscr{L} \to \mathscr{A}$ are defined satisfying (3) and the crucial condition

$$\mathscr{L}\mathscr{A} \subset \mathscr{L}. \tag{4}$$

Then, given $P \in \mathscr{L}$, a solution of (2) will have the form $X = I + \Gamma(Q)$ with $Q \in \mathscr{L}$. Substitution in (1) yields the integral equation

$$Q + P\Gamma(Q) = -P \tag{5}$$

for Q. Conversely, a simple calculation shows that if Q solves (5), then $X = I + \Gamma(Q)$ is a solution of (2).

Thus in the abstract setting described above (and to be realized in what follows for the operator J), the equations (2) and (5) are equivalent. In [4] attention was focused on an equation of the form (2) whose solution was expressible as a discrete product integral. Here, instead, we choose to work with (5). This has the advantage, among others, of avoiding the artificial adjunction of an identity to certain spaces of kernels(3). By what amounts to a successive approximations procedure, (5) will be shown to have a solution Q. Then $X = I + \Gamma(Q)$ is a nonsingular solution of (1) and hence implements the similarity $J \sim J + P$.

1. **Preliminaries.** Given Volterra operators $K: f(x) \to \int_0^x k(x,y)f(y)\,dy$ and $L: f(x) \to \int_0^x l(x,y)f(y)\,dy$, then (under mild restrictions on the kernels k and l) KL is the Volterra operator

$$KL: f(x) \to \int_0^x k * l(x,y)f(y)\,dy,$$

where

$$k * l(x,y) = \int_y^x k(x,\eta)l(\eta,y)\,d\eta. \tag{6}$$

Instead of dealing with the operators directly we will work with spaces of kernels and the composition $k * l$. To begin with, by 'kernel' we will simply mean a (measurable) complex-valued function $k(x,y)$ on $0 \leqq y < x \leqq 1$. For $\alpha > 0$ we define

$$\|k\|_{\alpha,\infty} = \operatorname*{ess\text{-}sup}_{0 \leqq y < x \leqq 1} |k(x,y)(x-y)^{1-\alpha}| \tag{7}$$

(where ess-sup denotes the essential supremum with respect to Lebesgue measure).

In order to relate $\|k\|_{\alpha,\infty}$ and $\|K\|_r$, the operator norm of

$$K: f(x) \to \int_0^x k(x,y)f(y)\,dy$$

(3) (5) might well be thought of as the Wiener-Hopf form of (2). In this connection, see [1], [2].

on $L^r(0,1)$, we need the following version of a theorem of M. Riesz (see [3, p. 518]).

Theorem 1.1. *Let (S,Σ,μ) be a positive measure space and k a measurable function on $S\times S$ with*

$$\operatorname*{ess-sup}_{x}\int_S |k(x,y)|\mu\,(dy)\leqq M<\infty,$$

and

$$\operatorname*{ess-sup}_{y}\int_S |k(x,y)|\mu\,(dx)\leqq M.$$

Then $K: f(x)\to \int_S k(x,y)f(y)\mu\,(dy)$ is a bounded operator on $L^r(S,\Sigma,\mu)$ $(1\leqq r\leqq \infty)$ with $\|K\|_r\leqq M$.

Lemma 1.2. *If $\|k\|_{\alpha,\infty}<\infty$, then $K: f(x)\to\int_0^x k(x,y)f(y)\,dy$ is a bounded operator on $L^r(0,1)$ $(1\leqq r\leqq\infty)$ and*

$$\|K\|_r\leqq \frac{1}{\alpha}\|k\|_{\alpha,\infty}.$$

Proof. We have, immediately from (7)

$$\operatorname*{ess-sup}_{0\leqq x\leqq 1}\int_0^x |k(x,y)|\,dy\leqq \|k\|_{\alpha,\infty}\sup_{0\leqq x\leqq 1}\int_0^x\frac{dy}{(x-y)^{1-\alpha}}$$

and

$$\operatorname*{ess-sup}_{0\leqq y\leqq 1}\int_y^1 |k(x,y)|\,dx\leqq \|k\|_{\alpha,\infty}\sup_{0\leqq y\leqq 1}\int_y^1\frac{dx}{(x-y)^{1-\alpha}}$$

and hence, by Theorem 1.1, we get $\|K\|_r\leqq C\|k\|_{\alpha,\infty}$ with $C=\int_0^1(x^{\alpha-1})^{-1}dx = 1/\alpha$.

Lemma 1.3. *If k and l are kernels for which $\|k\|_{\alpha,\infty}<\infty$ and $\|l\|_{\beta,\infty}<\infty$, then*

$$\|k*l\|_{\alpha+\beta,\infty}\leqq B(\alpha,\beta)\,\|k\|_{\alpha,\infty}\|l\|_{\beta,\infty}$$

(where $B(\alpha,\beta)$ is the beta function).

Proof. Since $|k(x,\eta)l(\eta,y)|\leqq(\|k\|_{\alpha,\infty}\,\|l\|_{\beta,\infty})/((x-\eta)^{1-\alpha}(\eta-y)^{1-\beta})$ essentially, it follows that (essentially),

$$|k*l(x,y)|\leqq\|k\|_{\alpha,\infty}\|l\|_{\beta,\infty}\int_y^x\frac{d\eta}{(x-\eta)^{1-\alpha}(\eta-y)^{1-\beta}}$$

$$=\|k\|_{\alpha,\infty}\|l\|_{\beta,\infty}\,(x-y)^{\alpha+\beta-1}\int_0^1\frac{dt}{t^{1-\alpha}(1-t)^{1-\beta}}.$$

Since the last integral is $B(\alpha,\beta)$, this is equivalent to the asserted inequality.

At this point we remark that all kernels encountered here will have $\|k\|_{\alpha,\infty} < \infty$ for some $\alpha > 0$ and the correspondence mentioned above between the composition $*$ and composition of operators is valid. Likewise associativity holds; $(k * l) * m = k * (l * m)$. We write

$$k^{(n)} = k * k * \cdots * k \quad (n\text{-factors}).$$

For example, we have for the iterates of J,

$$J^n : f(x) \to \int_0^x 1^{(n)}(x,y) f(y)\, dy,$$

where

$$1^{(n)}(x,y) = \frac{(x-y)^{n-1}}{(n-1)!}.$$

LEMMA 1.4. *If* $\|k\|_{\alpha,\infty} < \infty$, *then*

$$\|k^{(n)}\|_{n\alpha,\infty} \leqq \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)} \|k\|^n_{\alpha,\infty}$$

(*where* Γ *denotes the gamma function*).

Proof. This holds for $n = 1$. Assuming inductively that it holds for n, we have, by Lemma 1.3,

$$\|k^{(n+1)}\|_{(n+1)\alpha,\infty} = \|k^{(n)} * k\|_{(n+1)\alpha,\infty} \leqq B(n\alpha,\alpha)\, \|k^{(n)}\|_{n\alpha,\infty} \|k\|_{\alpha,\infty}$$
$$\leqq B(n\alpha,\alpha) \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)} \|k\|^{n+1}_{\alpha,\infty} = \frac{\Gamma(\alpha)^{n+1}}{\Gamma((n+1)\alpha)} \|k\|^{n+1}_{\alpha,\infty}.$$

The last equality follows from the identity $B(\gamma,\alpha) = \Gamma(\gamma)\Gamma(\alpha)/\Gamma(\gamma+\alpha)$.

LEMMA 1.5. *If* $\|k\|_{\alpha,\infty} < \infty$, *then the norms of the operators*

$$K^n : f(x) \to \int_0^x k^{(n)}(x,y) f(y)\, dy$$

satisfy

$$\|K^n\|_r \leqq \frac{\Gamma(\alpha)^n}{\Gamma((n+1)\alpha)} \|k\|^n_{\alpha,\infty}.$$

Thus $\lim \|K^n\|_r^{1/n} = 0$, *i.e.*, K *is a quasi-nilpotent operator on* $L^r(0,1)$, *and hence* $I + \lambda K$ *is nonsingular for every complex number* λ.

Proof. By Lemma 1.2, $\|K^n\|_r \leqq (1/\alpha)\, \|k^{(n)}\|_{n\alpha,\infty}$. By the preceding lemma, this in turn is majorized by $\Gamma(\alpha)^n \|k\|^n_{\alpha,\infty}/\alpha\Gamma(n\alpha)$. That $\lim \|K^n\|_r^{1/n} = 0$ now follows since $\lim \Gamma(n\alpha)^{1/n} = \infty$ when $\alpha > 0$.

LEMMA 1.6. *If kernels* k *and* l *are continuous on* $0 \leqq y < x \leqq 1$ *and* $\|k\|_{\alpha,\infty}$, $\|l\|_{\beta,\infty} < \infty$, *then* $k * l$ *is continuous on* $0 \leqq y < x \leqq 1$. *If* $\alpha + \beta > 1$, *then* $k * l$ *is continuous on* $0 \leqq y \leqq x \leqq 1$ *with* $k * l(x,x) \equiv 0$.

Proof. By the assumption, $k(x,y) = m(x,y)/(x-y)^{1-\alpha}$ and $l(x,y) = n(x,y)/(x-y)^{1-\beta}$, where m and n are continuous and bounded on $0 \leqq y < x \leqq 1$. When $y < x$ the variable change $\eta = y + t(x-y)$ gives

$$k * l(x,y) = (x-y)^{\alpha+\beta-1} \int_0^1 \frac{m[x, y+t(x-y)]\, n[y+t(x-y), y]}{(1-t)^{1-\alpha} t^{1-\beta}}\, dt.$$

Denote the integrand above by $f_{(x,y)}(t)$. If $0 \leqq y_0 < x_0 \leqq 1$ and (x,y) converges to (x_0, y_0), then the number $k * l(x,y)$ converges to $k * l(x_0, y_0)$, by the dominated convergence theorem. For $f_{(x,y)}(t)$ converges to $f_{(x_0,y_0)}(t)$ when $0 < t < 1$ and $|f_{(x,y)}(t)| \leqq \text{const}/(1-t)^{1-\alpha} t^{1-\beta}$. That $k * l(x,y)$ converges to 0 as (x,y) converges to (x_0, x_0) follows also from the above expression for $k * l$ providing $\alpha + \beta > 1$.

LEMMA 1.7. *If $k(x,y)$ is continuous on $0 \leqq x \leqq y \leqq 1$, $k_1(x,y) = \partial k(x,y)/\partial x$ and $l(x,y)$ are continuous on $0 \leqq y < x \leqq 1$, and $\|k_1\|_{\alpha,\infty}$, $\|l\|_{\beta,\infty} < \infty$ for some $\alpha, \beta > 0$ then,*

$$\frac{\partial}{\partial x} k * l(x,y) = k_1 * l(x,y) + k(x,x) l(x,y).$$

Proof. For $0 \leqq y < x \leqq 1$,

$$\frac{1}{h}[k * l(x+h,y) - k * l(x,y)]$$
$$= \int_y^x \frac{k(x+h,\eta) - k(x,\eta)}{h}\, l(\eta,y)\, d\eta + \frac{1}{h}\int_x^{x+h} k(x,\eta) l(\eta,y)\, d\eta$$
$$+ \int_x^{x+h} \frac{k(x+h,\eta) - k(x,\eta)}{h}\, l(\eta,y)\, d\eta.$$

As $h \to 0$, the first integral converges to $k_1 * l(x,y)$ by dominated convergence, the second to $k(x,x)l(x,y)$ by continuity of the integrand (recalling that $y < x$), and the third to 0 since the integrand is integrable, uniformly in h, in an interval about x.

We continue to use the subscript 1 to denote differentiation with respect to x. From Lemmas 1.6 and 1.7 we have the following

PROPOSITION 1.8. *If a is continuous on $0 \leqq y < x \leqq 1$ and $\|a\|_{\alpha,\infty} < \infty$ then $1 * a$ and $1^{(2)} * a$ are continuous on $0 \leqq y \leqq x \leqq 1$ and vanish identically on the diagonal. Moreover, $(1^{(2)} * a)_1 = 1 * a$ and $(1^{(2)} * a)_{11} = a$. Conversely, if q is such that*

(i) *q and q_1 are continuous on $0 \leqq y \leqq x \leqq 1$,*

(I) (ii) *$q(x,x) = q_1(x,x) \equiv 0$ on $0 \leqq x \leqq 1$, and*

(iii) *q_{11} exists and is continuous on $0 \leqq y < x \leqq 1$ and $\|q_{11}\|_{\alpha,\infty} < \infty$ for some $\alpha > 0$,* *then* $q_1 = 1 * q_{11}$ *and* $q = 1^{(2)} * q_{11}$.

2. **The "integral"** Γ. In this section we are concerned with the integration of the derivational equation $\Delta X = Q$ where Q is a Volterra operator. If we assume a solution $X = \Gamma(Q)$ of the form

$$\Gamma(Q) : f(x) \to \int_0^x \Gamma(q)(x,y)f(y)\, dy,$$

then in terms of the kernels, the equation becomes

$$1 * \Gamma(q) - \Gamma(q) * 1 = q. \tag{8}$$

Let $\mathscr{A}_\alpha$ be the class of kernels which are continuous on $0 \leqq y < x \leqq 1$ and for which $\|a\|_{\alpha,\infty} < \infty$. We set

$$\mathscr{L}_\alpha = 1^{(2)} * \mathscr{A}_\alpha, \tag{9}$$

i e., $\mathscr{L}_\alpha$ is the class of kernels of the form $q = 1^{(2)} * a$ with $a \in \mathscr{A}_\alpha$. It follows by Lemmas 1.3 and 1.6 that the spaces $\mathscr{A}_\alpha$ and $\mathscr{L}_\alpha$ decrease as α increases and satisfy the relations

$$\begin{aligned} \mathscr{A}_\alpha * \mathscr{A}_\beta &\subset \mathscr{A}_{\alpha+\beta}, \\ \mathscr{L}_\alpha * \mathscr{A}_\beta &\subset \mathscr{L}_{\alpha+\beta}. \end{aligned} \tag{10}$$

The latter is the form taken here by condition (4) of the introduction.

The space $\mathscr{L}_\alpha$ can alternately be characterized as the class of kernels q satisfying the conditions (I) of Proposition 1.8. For $q \in \mathscr{L}_\alpha$, we set

$$|q|_\alpha = \|q_{11}\|_{\alpha,\infty} + \|q_1\|_{\alpha+1,\infty}. \tag{11}$$

By Lemma 1.3 and Proposition 1.8 it follows that

$$\|q\|_{\alpha+2,\infty} = \|1^{(2)} * q_{11}\|_{\alpha+2,\infty} \leqq \frac{1}{\alpha(\alpha+1)} \|q_{11}\|_{\alpha,\infty}$$

and

$$\|q_1\|_{\alpha+1,\infty} = \|1 * q_{11}\|_{\alpha+1,\infty} \leqq \frac{1}{\alpha} \|q_{11}\|_{\alpha,\infty}$$

and hence all linear combinations of $\|q\|_{\alpha+2,\infty}, \|q_1\|_{\alpha+1,\infty}$, and $\|q_{11}\|_{\alpha,\infty}$, which include $\|q_{11}\|_{\alpha,\infty}$ give equivalent norms on $\mathscr{L}_\alpha$. Our choice (11) above is dictated by convenience.

We remark finally that $\mathscr{L}_\alpha$ and $\mathscr{A}_\alpha$ are Banach spaces under their respective norms.

THEOREM 2.1. *If* $q \in \mathscr{L}_\alpha$ *then the kernel* $\Gamma(q)$ *defined by*

$$\Gamma(q)(x,y) = \frac{\partial^2}{\partial x \partial y} \int_0^y q(\xi + x - y, \xi)\, d\xi \qquad (0 \leqq y < x \leqq 1)$$

satisfies (8), *belongs to* $\mathscr{A}_\alpha$ *and* $\|\Gamma(q)\|_{\alpha,\infty} \leqq |q|_\alpha$.

Proof. Since q_1 and q_{11} are continuous on $0 \leqq y + \epsilon \leqq x \leqq 1$ ($\epsilon > 0$) the Leibniz rule for differentiating an integral with parameter can be applied twice to $\int_0^y q(\xi + x - y, \xi)\, d\xi$. This gives (applying either $\partial^2/\partial x \partial y$ or $\partial^2/\partial y \partial x$)

$$\Gamma(q)(x,y) = -\int_0^y q_{11}(\xi + x - y, \xi)\, d\xi + q_1(x,y).$$

From this follows the continuity of $\Gamma(q)$ on $0 \leqq y < x \leqq 1$ and

$$\Gamma(q)(x,y) \leqq \int_0^y \frac{\|q_{11}\|_{\alpha,\infty}}{(x-y)^{1-\alpha}}\, d\xi + (x-y)^\alpha \|q_1\|_{\alpha+1,\infty}$$
$$\leqq \frac{\|q_{11}\|_{\alpha,\infty} + \|q_1\|_{\alpha+1,\infty}}{(x-y)^{1-\alpha}} = \frac{|q|_\alpha}{(x-y)^{1-\alpha}}.$$

Hence $\|\Gamma(q)\|_{\alpha,\infty} \leqq |q|_\alpha$. Since

$$1 * \Gamma(q)(x,y) = \int_y^x d\eta \left[\frac{\partial^2}{\partial \eta \partial y} \int_0^y q(\xi + \eta - y, \xi)\, d\xi \right]$$
$$= -\int_0^y q_1(\xi + \eta - y, \xi)\, d\xi + q(\eta, y) \Bigg|_{\eta=y}^{\eta=x}$$
$$= q(x,y) - \int_0^y q_1(\xi + x - y, \xi)\, d\xi$$
$$+ \int_0^y q_1(\xi,\xi)\, d\xi - q(y,y)$$

and

$$\Gamma(q) * 1(x,y) = \int_y^x d\eta \left[\frac{\partial^2}{\partial \eta \partial x} \int_0^\eta q(\xi + x - \eta, \xi)\, d\xi \right]$$
$$= \int_0^\eta q_1(\xi + x - \eta, \xi)\, d\xi \Bigg|_{\eta=y}^{\eta=x}$$
$$= \int_0^x q_1(\xi,\xi)\, d\xi - \int_0^y q_1(\xi + x - y, \xi)\, d\xi$$

we have

$$1 * \Gamma(q) - \Gamma(q) * 1 = q(x,y) - \int_y^x q_1(\xi,\xi)\, d\xi - q(y,y).$$

But the last two terms vanish since $q \in \mathcal{V}_\alpha$ so that (4) is satisfied by $\Gamma(q)$.

REMARK. For a kernel k of the form $k(x,y) = m(y)/m(x)$, it can be shown that the equation

$$k * \Gamma(q) - \Gamma(q) * k = q$$

is formally solved by

$$\Gamma(q)(x,y) = \frac{m(y)}{m(x)} \frac{\partial^2}{\partial x \partial y} \int_0^y q(\xi + x - y, \xi) \frac{m(\xi + x - y)}{m(\xi)} d\xi$$

provided $q(x,x) = q_1(x,x) \equiv 0$. By using this observation, results analogous to those of the present paper can be obtained for Volterra operators K with kernels k of the above type.

3. **Solution of the operator equation** $Q + P\Gamma(Q) = -P$. For Volterra operators P and Q with kernels $p, q \in \mathcal{S}_\alpha$ the operator equation (5) is equivalent to

$$q + p * \Gamma(q) = -p, \tag{12}$$

i.e., to the integro-differential equation

$$q(x,y) + \int_y^x p(x,\eta) \left[\frac{\partial^2}{\partial \eta \partial y} \int_0^y q(\xi + \eta - y, \xi)\, d\xi \right] d\eta = -p(x,y).$$

We show that, given any $p \in \mathcal{S}_\alpha$, this equation is uniquely solvable for $q \in \mathcal{S}_\alpha$.

We show in fact that, given $p \in \mathcal{S}_\alpha$, the mapping

$$\Gamma_p : q \to p * \Gamma(q) \tag{13}$$

is quasi-nilpotent on any of the spaces $\mathcal{S}_\beta$. From this follows the existence of $(I + \Gamma_p)^{-1}$ and hence the unique solvability of

$$q + p * \Gamma(q) = r \tag{14}$$

for $q \in \mathcal{S}_\beta$, given any $r \in \mathcal{S}_\beta$. The critical thing, then, is to obtain estimates of the norms of the iterates of the operator Γ_p.

Lemma 3.1. *If $p \in \mathcal{S}_\alpha$ and $q \in \mathcal{S}_\beta$ then $p * \Gamma(q) \in \mathcal{S}_{\alpha+\beta}$ and*

$$|p * \Gamma(q)|_{\alpha+\beta} \leqq B(\alpha,\beta)\, |p|_\alpha |q|_\beta. \tag{15}$$

Proof. We have

$$p * \Gamma(q) = 1^{(2)} * p_{11} * \Gamma(q) \in 1^{(2)} * \mathcal{S}_\alpha * \mathcal{S}_\beta \subset 1^{(2)} * \mathcal{S}_{\alpha+\beta} = \mathcal{S}_{\alpha+\beta}$$

and

$$(p * \Gamma(q))_1 = p_1 * \Gamma(q), \quad \text{and} \quad (p * \Gamma(q))_{11} = p_{11} * \Gamma(q).$$

Hence by Lemma 1.3

$$\| (p * \Gamma(q))_1 \|_{\alpha+\beta+1,\infty} \leqq B(\alpha + 1, \beta)\, \| p_1 \|_{\alpha+1,\infty} \| \Gamma(q) \|_{\beta,\infty}$$

and

$$\| (p * \Gamma(q))_{11} \|_{\alpha+\beta,\infty} \leqq B(\alpha,\beta)\, \| p_{11} \|_{\alpha,\infty} \| \Gamma(q) \|_{\beta,\infty}.$$

Now (15) is obtained by adding the two inequalities and then using the fact that $\| \Gamma(q) \|_{\beta,\infty} \leqq |q|_\beta$ and $B(\gamma,\beta) \leqq B(\alpha,\beta)$ when $\gamma \geqq \alpha$.

Theorem 3.2. *If $p \in \mathcal{S}_\alpha$ then $\Gamma_p : q \to p * \Gamma(q)$ is a bounded quasi-nilpotent* *operator on* $\mathscr{V}_\beta$. *Hence, given* $r \in \mathscr{V}_\beta$, *the equation* (14) *is uniquely solvable for* $q \in \mathscr{V}_\beta$.

Proof. An easy induction argument using Lemma 3.1 and the identity $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ yields

$$|\Gamma_p^n(q)|_{n\alpha+\beta} \leqq \frac{\Gamma(\alpha)^n\Gamma(\beta)}{\Gamma(n\alpha+\beta)}\,|p|_\alpha^n |q|_\beta.$$

Since the norms $|\cdot|_\gamma$ increase with γ, we surely have $|\Gamma_p^n(q)|_\beta$ dominated by the right-hand side of the above inequality. Since $\lim \Gamma(n\alpha+\beta)^{1/n} = \infty$ the theorem follows.

4. **The similarity of $J+P$ and J.** On the basis of Theorems 3.2, 2.1, and Lemma 1.5 and the general considerations of the introduction we have our main result.

THEOREM A. *If* $p \in \mathscr{V}_\alpha$ *for some* $\alpha > 0$, *then the operators* J *and* $J+P$ *where*

$$J : f(x) \to \int_0^x f(y)\,dy$$

and

$$P : f(x) \to \int_0^x p(x,y)f(y)\,dy$$

are similar on $L^r(0,1)$ $(1 \leqq r \leqq \infty)$. *This similarity* $J + P \sim J$ *is implemented by the operator* $X = I + \Gamma(Q)$, $\Gamma(Q)$ *being the Volterra operator with kernel*

$$\Gamma(q)(x,y) = \frac{\partial^2}{\partial x \partial y}\int_0^y q(\xi + x - y, \xi)\,d\xi \qquad (0 \leqq y < x \leqq 1),$$

where $q \in \mathscr{V}_\alpha$ *is the unique solution of* $q + p * \Gamma(q) = -p$.

The preceding theorem can be strengthened by a modification of a procedure used by Volterra-Pérès [9] and Kalisch [5].

Let $G : f(x) \to \int_0^x g(x,y)f(y)\,dy$ be a Volterra operator whose kernel satisfies

(i) $g(x,y)$ and $g_1(x,y)$ are continuous on $0 \leqq y \leqq x \leqq 1$.

(ii) $g(x,x) > 0$ and $\int_0^1 g(x,x)\,dx = c$.

(iii) $(d/dt)\tilde{g}(t)$ and $(d/dt)\tilde{g}_1(t)$ are continuous on $0 \leqq t \leqq 1$, where $\tilde{g}(t) = g(t,t)$ and $\tilde{g}_1(t) = g_1(t,t)$.

(iv) $g_{11}(x,y)$ is continuous on $0 \leqq y < x \leqq 1$ and $\|g_{11}\|_{\alpha,\infty} < \infty$, where $0 < \alpha \leqq 1$.

COROLLARY B. *G is similar to cJ.*

This will follow easily from the next lemmas.

Lemma 4.1. *Let G be as above with $c = 1$, and set $n(x) = \int_0^x g(t,t)\,dt$. Then $S_n: f(x) \to f(n(x))$ is a bounded nonsingular operator on $L^r(0,1)$. Moreover, $H = S_n^{-1} G S_n$ is a Volterra operator whose kernel h satisfies $h(x,x) \equiv 1$ and the conditions* (i) *to* (iv) *above.*

Proof. Since $g(t,t)$ is continuous and > 0 on $0 \leqq t \leqq 1$, and $\int_0^1 g(t,t)\,dt = 1$, both n and $m = n^{-1}$ give continuously differentiable changes of variable on $[0,1]$; $dn/dx = g(x,x)$ and $dm/dx = g(m(x), m(x))^{-1}$. Thus S_n and $S_n^{-1} = S_m$ are bounded operators on $L^r(0,1)$ (bounds $\leqq \|dm/dx\|_\infty^{1/r}$ and $\|dn/dx\|_\infty^{1/r}$, respectively). Moreover, since

$$S_n^{-1} G S_n f(x) = \int_0^{m(x)} g(m(x), y) f(m(y))\,dy = \int_0^x \frac{g(m(x), m(y))}{g(m(y), m(y))} f(y)\,dy,$$

$H = S_n^{-1} G S_n$ is a Volterra operator with kernel

$$h(x,y) = \frac{g(m(x), m(y))}{g(m(y), m(y))}$$

satisfying $h(x,x) \equiv 1$. Now

$$h_1(x,y) = \frac{g_1(m(x), m(y))}{\tilde{g}(m(y))\tilde{g}(m(x))}$$

and

$$h_{11}(x,y) = \frac{1}{\tilde{g}(m(y))} \left[\frac{g_{11}(m(x), m(y))}{\tilde{g}(m(x))^2} - \frac{g(m(x), m(y)) \dfrac{d\tilde{g}}{dt}(m(x))}{\tilde{g}(m(x))^3} \right].$$

In view of the above expression for h_1, the continuity of h_1 and $d\tilde{h}_1/dt$ follows from the continuity of g_1 and $d\tilde{g}_1/dt$. Similarly, h_{11} is continuous on $0 \leqq y < x \leqq 1$ by the assumptions (i)-(iv) on g. To see that h_{11} satisfies the proper growth condition at the diagonal, $h_{11}(x,y) = O[(x-y)^{\alpha-1}]$, notice that in the above expression for h_{11}, only the term containing $g_{11}(m(x), m(y))$ can be unbounded near $x = y$. But by the assumption (iv) on g_{11}, $g_{11}(m(x), m(y)) = O[(m(x) - m(y))^{\alpha-1}]$ which in turn is $O[(x-y)^{\alpha-1}]$ since $x - y = n(m(x)) - n(m(y)) = \int_{m(y)}^{m(x)} g(t,t)\,dt$.

Lemma 4.2. *Let H be a Volterra operator whose kernel h satisfies $h(x,x) \equiv 1$ and* (i) *to* (iv) *above and set $k(x) = \exp \int_0^x h_1(t,t)\,dt$. Then M_k: $f(x) \to k(x) f(x)$ is a bounded nonsingular operator on $L^r(0,1)$. Moreover, $Q = M_k^{-1} H M_k$ is a Volterra operator whose kernel q satisfies* (i), (iv) *and $q(x,x) \equiv 1$, $q_1(x,x) \equiv 0$.*

Proof. Since $M_k^{-1} H M_k: f(x) \to \int_0^x (k(y)/k(x)) h(x,y) f(y)\,dy$, Q is a Volterra operator with kernel $q(x,y) = h(x,y) \exp[-\int_y^x h_1(t,t)\,dt]$ so that $q(x,x) = h(x,x) \equiv 1$ and

$$q_1(x,y) = [h_1(x,y) - h_1(x,x)h(x,y)]\exp\left[-\int_y^x h_1(t,t)\,dt\right]$$

$$q_{11}(x,y) = \left[h_{11}(x,y) - h(x,y)\frac{d\tilde{h}_1}{dt}(x) + \tilde{h}_1(x)^2 h(x,y)\right]\exp\left[-\int_y^x h_1(t,t)\,dt\right].$$

Thus $q_1(x,x) = h_1(x,x) - h_1(x,x)h(x,x) \equiv 0$. That the properties (i) and (iv) hold for q follows from the above expressions for q, q_1 and q_{11} and the assumptions (i) to (iv) on h.

Proof of B. Multiplying by $1/c$, G can be normalized so that $\int_0^1 g(t,t)\,dt = 1$. Then by the lemmas, G is similar to a Volterra operator Q whose kernel satisfies $q(x,x) \equiv 1$, $q_1(x,x) \equiv 0$, and (i) and (iv). But then the operator $P = Q - J$ has kernel $p = q - 1 \in \mathscr{L}_\alpha$ and hence by Theorem A, $Q = J + P$ is similar to J.

5. **Applications.** The Volterra operator $G: f(x) \to \int_0^x g(x,y)f(y)\,dy$ is similar to J if, say,

$$g(x,y) = e^{\lambda(x-y)} \quad \text{where } \lambda \text{ is any complex number}$$

or if

$$g(x,y) = 1 + \frac{(x-y)^{\beta-1}}{\Gamma(\beta)} \quad \text{where } \beta \geqq 2.$$

This latter example shows that J is similar to $J + J^\beta$ when $\beta \geqq 2$ where J^β is the fractional integral operator,

$$J^\beta: f(x) \to \frac{1}{\Gamma(\beta)}\int_0^x (x-y)^{\beta-1} f(y)\,dy.$$

By a result of Kalisch [7], J is not similar to $J + J^\beta$ when $\beta < 2$. Thus Theorem A is sharp with respect to the allowable algebraic singularity of p_{11} at the diagonal.

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