IMPROVING THE SIDE APPROXIMATION THEOREM[1,2]

BY

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1. **Introduction.** The following version of the Side Approximation Theorem for 2-spheres is proved as Theorem 16′ in [5].

THEOREM. *For each* 2-*sphere* S *in* E^3 *and each* $\epsilon > 0$ *there is a homeomorphism* $h: S \times [-1,1] \to E^3$ *such that*

each $h_t(S)$ *is a polyhedral* 2-*sphere,*

$D(s, h_t(s)) < \epsilon \qquad (s \in S, -1 \leqq t \leqq 1)$,

S *lies except for a finite collection of mutually exclusive* ϵ-*disks in* $h(S \times (-1,1))$,

$h_{-1}(S)$ *lies except for a finite collection of mutually exclusive* ϵ-*disks in the interior of* S, *and*

$h_1(S)$ *lies except for a finite collection of mutually exclusive* ϵ-*disks in the exterior of* S.

In the above we use D to represent the distance function and interior of S, exterior of S respectively to denote the bounded, unbounded components of $E^3 - S$. We say that $h_1(S) = h(S \times 1)$ *approximates* S *"almost" from the exterior of* S *and* $h_{-1}(S)$ *approximates* S *"almost" from the interior of* S.

In the last section of [5] the above result was generalized to open subsets of 2-spheres but the proof is complicated and should not be studied. Instead, we recommend the relatively easy treatment of the following result.

THEOREM 1.1. SIDE APPROXIMATION THEOREM FOR 2-MANIFOLDS. *Suppose* M^2 *is a connected* 2-*manifold* (*perhaps noncompact*) *in a connected* 3-*manifold* M^3 *such that*

$$M^3 - M^2 = U_1 + U_2 \text{ (mutually separated)}$$

and f *is a positive continuous function defined on* M^2. *Then there is a homeomorphism* $h: M^2 \times [-1,1] \to M^3$ *such that*

(1) *each* $h_t(M^2)$ *is tame,*

(2) $D(m, h_t(m)) < f(m) \quad (m \in M^2, -1 \leqq t \leqq 1)$,

(3) $M^2 - h(M^2 \times (-1,1))$ *is covered by the interiors of a locally finite*

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collection of mutually exclusive disks in M^2 such that the diameter of each is less than the minimum value of f on it,

(4) *for $0 < t \leqq 1$, $\bar{U}_1 \cdot h_{-t}(M^2)$ is covered by the interiors of a locally finite collection of mutually exclusive disks in $h_t(M^2)$ each of diameter less than $f(x)$ if $h_t(x)$ lies in the disk, and*

(5) *for $0 < t \leqq 1$, $\bar{U}_2 \cdot h_{-t}(M^2)$ is covered by the interiors of a locally finite collection of mutually exclusive disks in $h_{-t}(M^2)$ each of diameter less than $f(x)$* if *$h_{-t}(x)$ lies in the disk.*

In fact, if $M^{2\prime}$ is a closed subset of M^2 which is a tame 2-manifold with boundary, h can be chosen so that h_0 is the identity on $M^{2\prime}$.

The details of the proof of Theorem 1.1 are delayed until the next section. However, the idea of the description of the homeomorphism h: $M^2 \times [-1,1] \to M^3$ is so simple that we outline it here. The homeomorphism h is obtained as follows.

(1) Get a locally finite collection of tame simple closed curves $J_1, J_2, \cdots$ in M^2 so that each component of $M^2 - \sum J_i$ is small.

(2) Use the approximation theorem for open subsets of 2-spheres (Theorem 8 of [2]) to get a homeomorphism h_0: $M^2 \to M^3$ such that $h_0 = I$ (identity) on $M^{2\prime} + \sum J_i$, $h_0(M^2)$ is locally tame mod $h_0(\sum J_i)$, and h_0 does not move points far.

(3) Use a variation of Theorem 8.5 of [7] to show that $h_0(M^2)$ is tame (and hence bicollared). Get a homeomorphism h: $M^2 \times [-1,1] \to M^3$ so that h: $M^2 \times 0$ is the h_0 we have already described. If the "normals" $h(m \times [-1,1])$ are taken small, the h satisfies the conclusion of Theorem 1.1.

An *n-manifold* is a metric space each of whose points has a neighborhood homeomorphic with Euclidean n-space E^n. Each point of an *n-manifold with boundary* has a neighborhood whose closure is a topological n-cell. Hence, an n-manifold is an n-manifold with boundary but not conversely. Since each component of an n-manifold is separable, we frequently work only with their components to have the added security of separability. If M is an n-manifold with boundary, we denote the set of its points which have neighborhoods homeomorphic with E^n by $\operatorname{Int} M$ (called interior M) and denote $M - \operatorname{Int} M$ by $\operatorname{Bd} M$ (called boundary M). For a disk D in the plane, the point set boundary is $\operatorname{Bd} D$ but if D is embedded in E^3, the point set boundary is more.

A 2-manifold with boundary $M^{2\prime}$ is *tame* in a 3-manifold M^3 if there is a triangulation of M^3 such that $M^{2\prime}$ is the closed sum of elements of the triangulation. Being such a tame manifold is a local property since a closed subset of a 3-manifold is tame if it is locally tame [1], [11]. A set X is *locally tame at a point p* of X if there is a neighborhood N of p and a homeomorphism of $\bar{N}$ into a combinatorial cube that takes $\bar{N} \cdot X$ onto a polyhedron. A set is *locally tame* if it is locally tame at each of its points.

In general, a set is *tame* in a manifold if the set is a geometric complex of some combinatorial triangulation (perhaps curvilinear) of the manifold. If the manifold already has a combinatorial structure we insist that the combinatorial triangulation that makes the tame set a geometric complex is isomorphic to a subdivision of the original combinatorial triangulation. Hence, a set X in a combinatorial manifold is tame if there is a homeomorphism of the manifold onto itself that takes X onto a polyhedron.

A *Sierpiński curve* is a set homeomorphic to the set obtained by removing from a 2-sphere the interiors of a null sequence of mutually exclusive subdisks whose sum is dense in the 2-sphere. A *null* sequence is one which for each positive number ϵ has at most a finite number of elements with diameters more than ϵ. A Sierpiński curve in a 3-manifold is called *tame* if it lies on some tame disk in the 3-manifold.

2. **Proof of Theorem 1.1.** Let p_0, q_0 be points in U_1, U_2 respectively and ϵ be a positive number such that no 2ϵ-subset of $V(M^2, \epsilon)$ separates p_0 from q_0 in M^3. We use $V(A, \epsilon)$ to denote the set of all points whose distance from A is less than ϵ. We suppose that $f < \epsilon$ and so small that for each point p of M^2, $V(p, 2f(p))$ lies in an open 3-cell in M^3. The reason for making f so small is to make it easy to show that $h_1(M^2)$ and $h_{-1}(M^2)$ lie except for small holes in different ones of U_1, U_2. See Theorem 5.2 of §5.

We use $\min_f(X)$ to denote the minimum value of f on X.

The proof is broken into four pieces—the first three following our outline in the introduction and the fourth showing that the homeomorphism h we describe satisfies the conditions of the theorem.

1. *Chopping M^2 into small pieces.* It is known that M^2 can be triangulated into small simplexes. It follows further from Theorem 5 of [4] that there is a locally finite collection $\{D_i\}$ of disks in M^2 such that

$\{\operatorname{Int} D_i\}$ covers M^2 but no subcollection does,

diameter $D_i < \min_f(D_i)/2$,

for each D_i there is an open 3-cell O_i in M^3 and a 2-sphere S_i so that $D_i \subset S_i \subset O_i$.

Let $\{D_i''\}$ be a collection of disks such that

$$D_i'' \subset \operatorname{Int} D_i \quad \text{and} \quad \{\operatorname{Int} D_i''\} \text{ covers } M^2.$$

It follows from Theorem 1 of [6] that for each positive integer i there is a tame Sierpiński curve X_i in S_i such that each component of $S_i - X_i$ is of diameter less than $D(\operatorname{Bd} D_i, \operatorname{Bd} D_i'')/2$. Let J_i be a simple closed curve in $X_i \cdot (\operatorname{Int} D_i - D_i'')$ that separates D_i'' from $\operatorname{Bd} D_i$ in D_i. To find such a J_i, let g_i be a map of S_i onto a 2-sphere $g_i(S_i)$ such that the inverse of each point of $g_i(S_i)$ is either the closure of a component of $S_i - X_i$ or a point of X_i not on such a closure. Let J_i' be a simple closed curve on $g_i(S_i)$ that separates $g_i(D_i'')$ from $g_i(S_i - \operatorname{Int} D_i)$ and which misses each of the countable set of points of $g_i(S_i)$ which have nondegenerate inverses. Then $J_i = g_i^{-1}(J_i')$. Denote the disk in D_i bounded by J_i by D_i'. Then $D_i'' \subset \operatorname{Int} D_i' \subset D_i' \subset \operatorname{Int} D_i$. Also, $J_i = \operatorname{Bd} D_i'$ is tame.

2. *Defining* h_0. We first get a function f' which tells us how close to approximate M^2. Let f' be a continuous function defined on M^2 such that for each element D_i of $\{D_i\}$,

$$p \in D_i \Rightarrow f'(p) < \min_f(D_i)/4,$$

$$p \notin \operatorname{Int} D_i \Rightarrow D(p, D_i') > f'(p), \text{ and}$$

$$D(p,q) < f'(q) \Rightarrow f(p) < 2f(q).$$

Let U be an open set containing $M^2 - (M^{2\prime} + \sum J_i)$ such that each component of U lies in a 3-cell in M^3 and no component of U contains two components of $M^2 - (M^{2\prime} + \sum J_i)$. It follows from Theorem 7 of [2] that there is a homeomorphism h_0 of M^2 into M^3 such that

$$h_0(M^2) \text{ is locally tame at } h_0(p) \text{ if } p \in M^2 - (\operatorname{Bd} M^{2\prime} + \textstyle\sum J_i),$$

$$D(m, h_0(m) < f'(m)/2 < f(m)/8,$$

$$h_0 = I \text{ (identity) on } M^{2\prime} + \textstyle\sum J_i,$$

and h_0 is so close to the identity that there is a homotopy g: $M^2 \times [0,1] \rightarrow M^3$ such that

$$g_0 = h_0,$$

$$g_1 = I,$$

$$g_t = I \text{ on } M^{2\prime} + \textstyle\sum J_i, \text{ and}$$

$$g_t(M^2 \cdot U) \subset U.$$

Although Theorem 7 of [2] says nothing about a homotopy g, it does imply that h_0 can be taken close to the identity. To see that taking h_0 near the identity implies a g, consider a locally finite collection of topological cubes such that each component u of U lies in one of them (say C_u). Let h_u be a homeomorphism of C_u onto a canonical cube. Then there is a homotopy g if we take h_0 close enough to the identity that for each u and for each point $p \in M^2 \cap u$, the point $h_0(p)$ lies in C_u and the segment from $h_u(p)$ to $h_u h_0(p)$ lies in $h_u(u)$.

Let h_i' be the homeomorphism that is h_0 on $\sum_1^i D_j'$ and the identity on $M^2 - \sum_1^i D_j'$. It follows from Corollary 7.2 of §7 that each $h_i'(M^2)$ separates p from q in M^3. Hence $h_0(M^2)$ is two sided in M^3.

3. *Defining* h. It follows from a modification of Theorem 8.5 of [7] that $h_0(M^2)$ is locally tame. This modification is spelled out in Theorem 3.1 of the next section. Hence $h_0(M^2)$ locally has a cartesian product neighborhood and it follows from [8] that there is a homeomorphism h: M^2

$\times[-1,1]\to M^3$ such that h_0 is the homeomorphism previously defined and each $h_t(M^2)$ is locally tame. Suppose that for each point p of M^2, diameter $h(p\times[-1,1])<f'(p)/2$. This implies that

$$D(p,h_t(p))<f'(p)<f(p)/4.$$

Except for a possible exchange of t and $-t$ this homeomorphism h satisfies the conclusion of Theorem 1.1.

4. *Showing that h satisfies Conditions* 2, 3, 4, 5. Condition 2 of the conclusion of Theorem 1 is satisfied because

$$D(m,h_0(m))<f(m)/8 \text{ and}$$

$$\text{diameter } h(m\times[-1,1])<f(m)/8.$$

Each component of $M^2-h(M^2\times(-1,1))$ lies on the interior of an element of $\{D_i\}$ since

$$\sum J_i\subset h_0(M^2)\subset h(M^2\times(-1,1)) \text{ and}$$

$$\text{each component of } M^2-\sum J_i \text{ lies in a disk } D_i' \text{ of } \{D_i'\}.$$

It follows from Theorem 4.1 that there is a locally finite collection of mutually exclusive disks $G_1,G_2,\cdots$ in M^2 such that the interiors of the G_i's cover $M^2-h(M^2\times(-1,1))$. The G_i's are small since if $G_i\subset D_j$,

$$\text{diameter } G_i\leqq\text{diameter } D_j<\min_f(D_j)/2\leqq\min_f(G_i)/2.$$

Hence Condition 3 is satisfied.

As a step toward showing that Conditions 4 and 5 are satisfied we show that each component A of $M^2\cdot h_1(M^2)$ lies in an $h_1(\operatorname{Int}D_i)$ for some element D_i of $\{D_i\}$. Since $\sum J_i\cdot h_1(M^2)=0$, A lies in a component of $M^2-\sum J_i$ and hence in a D_i'. Then A lies in the corresponding $h_1(\operatorname{Int}D_i)$ because $\mathrm{Cl}(h_1(M^2-D_i))$ misses D_i' as can be seen from the facts that $D(p,h_1(p))<f'(p)$ and $f'(q)>D(q,D_i')$ if $q\notin\operatorname{Int}D_i$.

The $h_1(D_i)$'s give a locally finite collection of disks whose interiors cover $M^2\cdot h_1(M^2)$. It follows from Theorem 4.1 that there is a locally finite collection of mutually exclusive disks $E_1^1,E_2^1,\cdots$ in $h_1(M^2)$ whose interiors cover $M^2\cdot h_1(M^2)$ such that each of these disks lies in an $h_1(D_i)$. If $E_i^1\subset h_1(D_j)$, then

$$\text{diameter } E_i^1<\text{diameter } h_1(D_j)\leqq\text{diameter } D_j+2\min_f(D_j)/4$$
$$\leqq\min_f(D_j)\leqq\min_f h^{-1}(E_i^1).$$

Since $h_1(M^2)-\sum E_i^1$ is connected, it lies in one of U_1,U_2—so with a possible exchange of U_1,U_2, Condition 4 is satisfied for $t=1$.

Similarly, it follows that for $-1\leqq t<0$ or $0<t\leqq 1$ there is a collection of mutually exclusive small disks $E_1^tE_2^t,\cdots$ in $h_t(M^2)$ such that $h_t(M^2)-\sum E_i^t$ lies in one of U_1,U_2. If t,t' are of the same sign, $h_t(M^2)-\sum E_i^t$

and $h_{t'}(M^2) - \sum E_i^{t'}$ lie in the same one of U_1, U_2 or else there is a $t_0 \neq 0$ and a sequence $t_1, t_2, \cdots$ converging to t_0 such that $h_{t_0}(M^2) - \sum E_i^{t_0}$ and $h_{t_j}(M^2) - \sum E_i^{t_j}$ $(j > 0)$ lie in different ones of U_1, U_2. This is impossible since for j sufficiently large, $h_{t_j}\{h_{t_0}^{-1}[h_{t_0}(M^2) - \sum E_i^{t_0}]\}$ lies in the one of U_1, U_2 containing $h_{t_0}(M^2) - \sum E_i^{t_0}$ and there would not be a finite collection of small disks in $h_{t_j}(M^2)$ covering $h_{t_j}\{h_{t_0}^{-1}[h_{t_0}(M^2) - \sum E_i^{t_0}]\}$.

Condition 5 is established when we show that it is not the same one of U_1, U_2 containing $h_1(M^2) - \sum E_i^1$ and $h_{-1}(M^2) - \sum E_i^{-1}$. This follows from Theorem 5.2 of §5. The only difficult condition to check in applying Theorem 5.2 is to see that Condition 5 of the hypothesis of that theorem is satisfied. We do this in the next paragraph.

Suppose G_k is one of the disks of Condition 3 that intersects the disk E_i of Condition 4 in a point p and $h_1(q) = p$. We show that $G_k + E_i$ lies in an open 3-cell in M^3 by showing that each lies in $V(q, 2f(q))$. First, there is a D_j such that $E_i \subset h_1(D_j)$. Then since h_1 does not move x farther than $f'(x)$,

$$E_i \subset h_1(D_j) \subset V(D_j, \min_f(D_j)/4) \subset V(q, \text{diameter}\, D_j + \min_f(D_j)/4)$$
$$\subset V(q, \min_f(D_j)/2 + \min_f(D_j)/4) \subset V(q, f(q)).$$

Also, since $D(p,q) < f'(q), f(p) < 2f(q)$ and

$$G_k \subset V(p, f(p)/2) \subset V(p, f(q)) \subset V(q, D(p,q) + f(q))$$
$$\subset V(q, f'(q) + f(q)) \subset V(q, f(q)/4 + f(q)) \subset V(q, 2f(q)).$$

Using the fact that any homeomorphism of a triangulated 3-manifold with boundary into a triangulated 3-manifold can be approximated with a piecewise linear homeomorphism as shown by Theorem 9 of [3] and Theorem 2 of [10], we obtain the following variation of Theorem 1.1.

THEOREM 1.1′. *Suppose M^2 is a connected polyhedral 2-manifold, h' is a homeomorphism of M^2 into a triangulated 3-manifold M^3 such that $h'(M^2)$ separates M^3, and f is a positive continuous function defined on M^2. Then there are three locally finite collections $\{G_i\}$, $\{E_i\}$, $\{F_i\}$ of mutually exclusive disks in M^2 and a piecewise linear homeomorphism h: $M^2 \times [-1,1] \to M^3$ such that*

$$D(h'(m), h_t(m)) < f(m) \qquad (m \in M^2, -1 \leqq t \leqq 1),$$

$$\text{diameter}\, h'(G_i) < \min_f(G_i),$$

$$\text{diameter}\, h_{-1}(E_i) < \min_f(E_i),$$

$$\text{diameter}\, h_1(F_i) < \min_f(F_i),$$

$$h'(M^2 - \sum \text{Int}\, G_i) \subset h(M^2 \times (-1, 1)),$$

and

$$h'(M^2) \text{ separates } h_{-1}(M^2 - \sum \operatorname{Int} E_i) \text{ from } h_1(M^2 - \sum \operatorname{Int} F_i) \text{ in } M^3.$$

In fact, if P is a polyhedron in M^2 on which h' is piecewise linear, h can be chosen so that $h_0 = h'$ on P.

3. **Tameness** mod **tame sets.** It was shown in [7] that a 2-sphere in E^3 is tame if it is locally tame mod the sum of a finite number of sets each of which is either a tame arc or a tame Sierpiński Curve. (This result was used in showing that the 2-manifold $h_0(M^2)$ considered in the preceding two sections was locally tame.) The following is a mild extension of Theorem 8.5 of [7].

THEOREM 3.1. *Suppose M^2 is a 2-manifold embedded in a 3-manifold M^3, $\{X_i\}$ is a countable collection each of whose elements is either a tame arc or a tame Sierpiński curve in M^2, U is an open subset of M^2 such that M^2 is locally tame at each point of $U - \sum X_i$. Then M^2 is locally tame at each point of U.*

Proof. We first prove the theorem in the case where there is only one element X_1 in $\{X_i\}$. Let p be a point of $X \cdot U$ under consideration, D a disk in U, O^3 an open 3-cell, and S a 2-sphere, such that $p \in \operatorname{Int} D \subset S \subset O^3 \subset M^3$. Then there is an arc or Sierpiński curve X_1' in $X_1 \cdot D$ such that for some open subset N of $\operatorname{Int} D$ $p \in N \cdot X_1 \subset X_1'$. It follows from Theorem 8.5 of [7] S is locally tame at each point of N. Since $p \in N \subset M^2$, M^2 is locally tame at p.

Now that we have disposed of the special case, let Y be the set of all points of U at which M^2 is not locally tame. Then Y is a relatively closed subset of M^2 which lies in $\sum X_i$. It follows from the Baire Category Theorem that there is a point q of Y and an open subset N of U such that $q \in N$ and a single X_i contains $Y \cdot N$. It follows from the special case treated in the preceding paragraph that M^2 is locally tame at each point of N. But then M^2 is locally tame at q.

4. **Building disks about sets.** In the proof of Theorem 1.1 we showed that $h_1(M^2)$ lay except for a set with small components in $M^3 - M^2$. In this section we point out why this implies that $h_1(M^2)$ lies except for a locally finite collection of mutually exclusive small disks on one side of M^2.

THEOREM 4.1. *Suppose M^2 is a connected 2-manifold, X is a closed subset of M^2, and $\{D_i\}$ is a locally finite collection of disks in M^2 such that each component of X lies on the interior of one of the disks. Then there is a locally finite collection of mutually exclusive disks $E_1, E_2, \cdots$ in M^2 such that each component of X lies on the interior of an E_i and each E_i lies in some $\operatorname{Int} D_j$.*

Proof. If C is a component of X in $\operatorname{Int} D_1$, there is a disk D_C such that $C \subset \operatorname{Int} D_C \subset D_1$ and $X \cdot \operatorname{Bd} D_C = 0$. By so covering each component of X not in any $\operatorname{Int} D_2, \operatorname{Int} D_3, \cdots$, we find that there is a finite collection of disks $D_{11}, D_{12}, \cdots, D_{1n_1}$ in D_1 such that each $X \cdot \mathrm{Bd}\, D_{1i} = 0$, and each component of X lies in the interior of one of $D_{11}, D_{12}, \cdots, D_{1n_1}, D_2, D_3, \cdots$.

We next replace D_2 with a finite collection of disks $D_{21}, D_{22}, \cdots, D_{2n_2}$ in D_2 such that each $X \cdot \mathrm{Bd}\, D_{2i} = 0$ and each component of X lies in the interior of one of $D_{11}, D_{12}, \cdots, D_{1n_1}, D_{21}, D_{22}, \cdots, D_{2n_2}, D_3, D_4, \cdots$. Continuing in this fashion we get a locally finite collection of D_{ij}'s such that each D_{ij} lies in D_i, each $X \cdot \mathrm{Bd}\, D_{ij} = 0$, and each component of X lies in some $\mathrm{Int}\, D_{ij}$.

Throw away each D_{ij} that lies in a larger one. This causes the closure of each component of $D_{ij} - \sum \mathrm{Bd}\, D_{rs}$ to be a disk rather than an annulus or something worse. The collection of closures of such components may not be a locally finite collection but the subcollection of those which intersect X is a locally finite collection. If two of these disks intersect, the intersection is on the boundary so if each is shrunk slightly but not enough to uncover any point of X or even bring it to the boundary, we obtain a collection of disks satisfying the conditions of the theorem.

The following result follows from Theorem 4.1 and the fact that a connected 2-manifold minus the sum of a locally finite collection of mutually exclusive disks is connected.

COROLLARY 4.2. *Under the hypotheses of Theorem* 4.1 *there is a component U of $M^2 - X$ and a locally finite collection of mutually exclusive disks E_1, $E_2, \cdots$ in M^2 such that each E_i lies in a D_j and the E_i's cover $M^2 - U$.*

EXAMPLE. Corollary 4.2 is not true if we weaken the hypothesis by supposing that each component of X lies in some D_i rather than in some $\mathrm{Int}\, D_i$. One could let M^2 be a 2-sphere, D_1 and D_2 be two disks in M^2 with a common boundary, and X be the sum of D_1 and a sequence of mutually exclusive simple closed curves in $\mathrm{Int}\, D_2$ converging homeomorphically to $\mathrm{Bd}\, D_1 = \mathrm{Bd}\, D_2$.

The above is essentially the only counterexample to the modified Corollary 4.2 as may be seen by the following theorems which are not used elsewhere in this paper but included since they show an extent to which Theorem 4.1 and Corollary 4.2 can be strengthened.

THEOREM 4.3. *Suppose M^2 is a 2-manifold, $\{D_i\}$ is a locally finite collection of disks in M^2, and X is a closed subset of M^2 such that each component of X lies in a D_i. Then $M - X$ has at least one component which does not lie in any D_i unless possibly M^2 is a 2-sphere which is the sum of two D_i's.*

Proof. Suppose each component of $M^2 - X$ lies in some D_i. We show that under this condition there are two D_i's whose sum is M^2.

Let U_0 be a component of $M^2 - X$ and $D_1, D_2, \cdots, D_n$ be the D_i's containing U_0. Let Y_i be the set of all points p of D_i such that each arc in

M^2 from p to $\mathrm{Bd}\, D_i$ intersects X, and C_i be the component of Y_i containing U_0. Denote the point set boundary in M^2 of C_i by F_i—that is, $F_i = C_i \cdot \overline{(M^2 - C_i)}$. Then F_i is connected and lies in X.

Consider the case where there is a D_j containing an F_i such that $C_i \not\subset D_j$. We show that under this condition that $D_i + D_j = M^2$. Suppose not. Let $q_0 \in C_i - D_j$. There would be a disk E_j in M^2 slightly larger than D_j such that $D_j \subset \mathrm{Int}\, E_j$, $q_0 \notin E_j$, and $D_i + E_j \neq M^2$. Since $F_i \subset \mathrm{Int}\, E_j$, there is a disk $E_i \subset C_i$ such that $q_0 \in \mathrm{Int}\, E_i$ and $\mathrm{Bd}\, E_i \subset E_j$. Let E_j' be the disk in E_j bounded by $\mathrm{Bd}\, E_i$. The disks E_i, E_j' have the same boundary but not the same interiors since $q_0 \in \mathrm{Int}\, E_i$, $q_0 \notin \mathrm{Int}\, E_j'$. Hence $E_i + E_j'$ is a 2-sphere. This is contrary to the condition that $D_i + E_j \neq M^2$ implied by the false assumption that $D_i + D_j \neq M^2$. We suppose henceforth that $F_i \subset D_j$ implies that $C_i \subset D_j$.

Since there are only a finite number of C_i's, we pick one of these (say C_1) such that it (C_1) is not properly contained in any other C_i. Then C_1 contains the component of X containing F_1 or else there is a larger C_i and C_1 is a component of $C_1 + X$. Let $G_1, G_2, \cdots$ be a decreasing sequence of disks in M^2 such that $G_{i+1} \subset \mathrm{Int}\, G_i$, $G_1 \cdot G_2 \cdot \cdots = C_1$, and $X \cdot \mathrm{Bd}\, G_i = 0$. Let U_i be the component of $M^2 - X$ containing $\mathrm{Bd}\, G_i$. It follows from the local finiteness of $\{D_i\}$ that there is one of the D_i's (say D_j) such that D_j contains infinitely many of the U_i's. For convenience we suppose D_j contains all the U_i's. Unless $D_1 + D_j$ covers M^2, there is an integer k such that $G_k + D_j$ does not cover M^2. Since $\mathrm{Bd}\, G_k$ bounds a disk in D_j, this implies that $G_k \subset D_j$. But since U_k lies in D_j, each arc in M^2 from $\mathrm{Bd}\, G_k$ to $\mathrm{Bd}\, D_j$ intersects X. Also, each arc in M^2 from G_k to $\mathrm{Bd}\, D_j$ intersects X and C_1 is not as large as supposed—it should have contained G_k.

Example. Theorem 4.3 is not true if we do not insist that $\{D\}$ is locally finite as can be seen by letting M^2 be the plane, D_i be the round disk with center at the origin and radius i, and $X = \sum \mathrm{Bd}\, D_i$. However, the following result shows that even without local finiteness on $\{D_i\}$, there cannot be two large components of $M^2 - X$.

Theorem 4.4. *Suppose M^2 is a connected 2-manifold, $\{D_i\}$ is a collection of disks in M^2, X is a closed subset of M^2 such that each component of X lies in some D_i, and U is a component of $M^2 - X$ that does not lie in any D_i. Then each component of $M^2 - U$ lies in some $\{D_i\}$.*

Proof. Suppose the component C of $M^2 - U$ fails to lie in any D_i. Let p be a point of C such that each neighborhood of p intersects U. Let C' be the component of X containing p. Then C' lies in some D_i (say D_1). Since D_1 does not contain C, there is a disk E in M_2 such that $C' \subset D_1 \subset \mathrm{Int}\, E$ but $C \not\subset E$ and $U \not\subset E$. Since the component of $X \cdot E$ containing C' does not intersect $\mathrm{Bd}\, E$, there is a disk E' in E such that $C' \subset \mathrm{Int}\, E'$

and $X \cdot \mathrm{Bd}\, E' = 0$. Then $\mathrm{Bd}\, E' \subset U$ and $C \subset E' \subset E$. The assumption that the component C of $M^2 - U$ fails to lie in any D_i led to the contradiction that $C \subset E$ and $C \not\subset E$.

5. **Separating 3-manifolds with disks.** If S^2 is a 2-sphere in E^3, h a homeomorphism of $S^2 \times [-1,1] \to E^3$ such that $S^2 \subset h(S^2 \times (-1,1))$, then one of $h_{-1}(S^2)$, $h_1(S^2)$ lies in the bounded component of $E^3 - S^2$ and the other lies in the unbounded component unless $h(S^2 \times (-1,1))$ contains the bounded component of $E^3 - S^2$. This result follows from the Invariance of Domain Theorem. Something can be said about the matter in the case $h(S^2 \times (-1,1))$ contains all of S^2 except for some small holes as evidenced by Theorem 15 of [5]. In this section we generalize Theorems 14, 15 of [5] to see what can be concluded about M^2 in M^3.

THEOREM 5.1. *Suppose $\{A_i\}$, $\{B_i\}$ are locally finite collections of mutually exclusive disks in a 3-manifold M^3 such that if an element A_r of $\{A_i\}$ intersects an element B_s of $\{B_i\}$, then $A_r + B_s$ lie in an open 3-cell in M^3. Then if $\sum A_i + \sum B_i$ separates two points p, q in M^3, there are elements A_u, B_v in $\{A_i\}$, $\{B_i\}$ respectively such that $A_u \cdot B_v \neq 0$ and $A_u + B_v$ separates p from q in M^3.*

Proof. This theorem is an extension of Theorem 14 of [5] and is proved the same way. We suppose that $\sum B_i + \sum_{i \geq 2} A_i$ does not separate p from q but $\sum A_i + \sum B_i$ does. Let pq be an arc from p to q in $\sum B_i + \sum_{i\geq 2} A_i$ which intersects A_1 as few times as possible. As pointed out in the proof of Theorem 14 of [5], pq intersects A_1 in only a finite number of points and pierces it at each point at which it intersects it. Also, as pointed out in that same proof, there is an element B_t of $\{B_i\}$ and a component U of $A_1 - B_t$ such that $U \cdot \mathrm{Bd}\, A_1 = 0$ and $U \cdot pq$ is precisely one point.

Let O^3 be an open 3-cell in M^3 containing $A_1 + B_t$. If $A_1 + B_t$ does not separate p from q, there is a simple closed curve J' in M^3 such that $pq \subset J'$ and $(J' - pq) \cdot (A_1 + B_t) = 0$. Adjust J' near $M^3 - O^3$ to get a simple closed curve J in O^3 such that for some neighborhood N of $A_1 + B_t$, $J' \cdot N = J \cdot N$. However, this violates Theorem 13 of [5]. Hence, the assumption that $A_1 + B_t$ does not separate p from q is false.

THEOREM 5.2. *Suppose*

M^2 is a connected 2-manifold (perhaps noncompact) in a connected 3-manifold M^3,

p, q are points in different components of $M^3 - M^2$,

ϵ is a positive number such that no 2ϵ-subset of $V(M^2, \epsilon)$ separates p from q in M^3, and

h is a homeomorphism of $M^2 \times [-1,1] \to M^3$ such that

(1) $h(M^2 \times [-1,1]) \subset V(M^2, \epsilon)$,

(2) M^2 contains a locally finite collection $\{D_i\}$ of mutually exclusive ϵ-disks

such that $M^2 - \sum D_i \subset h(M^2 \times (-1, 1))$,

(3) $h_{-1}(M^2)$ *contains a locally finite collection* $\{E_i\}$ *of mutually exclusive* ϵ*-disks such that* $M^2 \cdot h_{-1}(M^2) \subset \sum E_i$,

(4) $h_1(M^2)$ *contains a locally finite collection* $\{F_i\}$ *of mutually exclusive* ϵ*-disks such that* $M^2 \cdot h_1(M^2) \subset \sum F_i$, *and*

(5) *if an element* D *of* $\{D_i\}$ *intersects an element* G *of* $\{E_i\} + \{F_i\}$, *then* $D + G$ *lies in an open* 3*-cell in* M^3.

Then $h_{-1}(M^2) - \sum E_i$ *lies in one component of* $M^3 - M^2$ *and* $h_1(M^2) - \sum F_i$ *lies in the other.*

Proof. This theorem is an analogue of Theorem 15 of [5] and is proved in the same way. We note that $\{E_i\} + \{F_i\}$ is a locally finite collection of mutually exclusive disks and $\{D_i\}$ is another such collection. It follows from Theorem 5.1 that there is an arc from p to q in $M^3 - (\sum D_i + \sum E_i + \sum F_i)$. The first point of $M^2 + h(M^2 \times [-1, 1])$ on this arc in the order from p to q belongs to one of $h_{-1}(M^2) - \sum E_i$, $h_1(M^2) - \sum F_i$ and the last such point belongs to the other.

6. **Triangulations with tame skeletons.** One might consider our proof of Theorem 1.1 neater if instead of chopping M^2 up helter-skelter by the J_i's we had taken a triangulation of it with a tame 1-skeleton. In this section we show that any 2-manifold M^2 embedded in a 3-manifold can be given a fine triangulation with a tame 1-skeleton. The 1-skeleton is picked in a certain tame Sierpiński-like set to ensure that it is tame.

THEOREM 6.1. *Suppose* M^2 *is a connected* 2*-manifold embedded in a* 3*-manifold* M^3 *and* f *is a positive continuous function defined on* M^2. *Then there is a null sequence of mutually exclusive disks* $E_1, E_2, \cdots$ *in* M^2 *such that*

$$\text{diameter } E_i < \min_f E_i,$$

$$\textstyle\sum E_i \textit{ is dense in } M_2, \textit{ and}$$

$$M^2 - \textstyle\sum \operatorname{Int} E_i \textit{ lies in a tame 2-manifold in } M^3.$$

Proof. As in the proof of Theorem 1.1 we let $\{D_i'\}$ be a locally finite collection of disks in M^2 such that

$$\{\operatorname{Int} D_i'\} \text{ covers } M^2 \text{ but no subcollection does,}$$

$$\operatorname{Bd} D_i' \text{ is tame,}$$

$$\text{diameter } D_i' < \min_f D_i', \text{ and}$$

$$\text{each } D_i' \text{ lies on a 2-sphere in an open 3-cell in } M^3.$$

The closure of each component of $M^2 - \sum \operatorname{Bd} D_i'$ is a disk with a tame boundary. Let $F_1, F_2, \cdots$ be the collection of these closures and S_i, O_i be a 2-sphere and open 3-cell respectively such that $F_i \subset S_i \subset O_i \subset M^3$.

It follows from Theorem 9.1 of [7] that there is a tame Sierpiński curve X_i' in S_i such that each component of $S_i - X_i'$ is of diameter less than $1/i$ and $\operatorname{Bd} F_i$ belongs to the set of inaccessible points of X_i'. Then $X_i = X_i' \cdot F_i$ is a tame Sierpiński curve in F_i that contains $\operatorname{Bd} F_i$.

The E_i's are chosen so that $M^2 - \sum \operatorname{Int} E_i = \sum \operatorname{Bd} F_i + \sum X_i$. It follows from the Approximation Theorem for Surfaces (Theorem 7 of [2]) that there is a homeomorphism h of M^2 into M^3 such that h is the identity on $\sum \operatorname{Bd} F_i + \sum X_i$ and $h(M^2)$ is locally tame off $\sum \operatorname{Bd} F_i + \sum X_i$. It follows from Theorem 3.1 that $h(M^2)$ is tame.

THEOREM 6.2. *The E_i's of Theorem 6.1 may be chosen so that there is a monotone map g of M^2 onto itself such that for each point m of M^2*

$$D(m,g(m)) < f(m) \text{ and}$$

$$g^{-1}(m) \text{ is either an } E_i \text{ or a point of } M^2 - \sum E_i.$$

Proof. It follows from [12] that there is a monotone map g_i of F_i onto itself that is the identity on $\operatorname{Bd} F_i$ and such that for each point x of F_i, $g_i^{-1}(x)$ is either a point of $F_i - \sum E_j$ or an E_j in F_i. The map g required by Theorem 6.2 is g_i on each F_i and the identity elsewhere.

THEOREM 6.3. *Suppose M^2 is a 2-manifold embedded in a 3-manifold and f is a positive continuous function defined on M^2. Then there is a triangulation T of M^2 such that the 1-skeleton of T is tame and for each simplex σ of T,* diameter $\sigma < \min_f(\sigma)$.

Proof. Since we can operate on the components of M^2 one at a time, we suppose with no loss of generality that M^2 is connected.

Let f_1 be a positive continuous function defined on M^2 such that

$$D(p,q) < f_1(p) \Rightarrow f(q) < 2f(p).$$

It follows from Theorem 6.2 that there is a null sequence of mutually exclusive disks $E_1, E_2, \cdots$ and a monotone map g of M^2 onto itself such that

$$M^2 - \sum \operatorname{Int} E_i \text{ lies on a tame 2-manifold in } M^3,$$

$$\text{diameter } E_i < \min_{f_1} E_i,$$

$$D(m,g(m)) < f(m)/5,$$

$$g^{-1} \text{ is either an } E_i \text{ or a point of } M^2 - \sum E_i.$$

Let f_2 be a continuous function defined on M^2 such that

$$D(g(p),g(a)) < f_2 g(a) \Rightarrow f(p) < 2f(a).$$

Also, let f_3 be a continuous positive function defined on M^2 such that

$$f_3 g(p) < f(p)/5.$$

Consider a triangulation T' of M^2 such that the 1-skeleton of T' misses each $g(E_i)$ and for each simplex s of T'

$$\text{diameter of } s \text{ is less than either } \min_{f_2}(s) \text{ or } \min_{f_3}(s).$$

Let T be the triangulation of M^2 such that for each simplex s of T', $g^{-1}(s)$ is a simplex of T. To see that diameter $g^{-1}(s) < \min_f g^{-1}(s)$, consider three points p, q, a of $g^{-1}(s)$ such that $D(p,q) = \text{diameter}\, g^{-1}(s), f(a) = \min_f g^{-1}(s)$. Then

$$\begin{aligned}\text{diameter } g^{-1}(s) &= D(p,q) < f(p)/5 + f(q)/5 + D(g(p),g(q))\\ &< f(p)/5 + f(q)/5 + f_3 g(a) \leqq 2f(a)/5 + 2f(a)/5 + f(a)/5\\ &= f(a) = \min_f g^{-1}(s).\end{aligned}$$

THEOREM 6.4. *Suppose T_1 is a triangulation of a 2-manifold M^2 embedded in a 3-manifold M^3 so that the 1-skeleton of T_1 is tame. Then for each positive continuous function f defined on M^2 there is a triangulation T_2 of M^2 refining T_1 such that the 1-skeleton of T_2 is tame and for each simplex σ of T_2,* diameter $\sigma < \min_f(\sigma)$.

Proof. We pick f_1, $\{E_i\}$, g, f_2, f_3 as in the proof of Theorem 6.3 but with the added precaution that the E_i's miss the 1-skeleton of T_1 (as we picked the E_i's to miss $\sum \mathrm{Bd}\, F_i$ in the proof of Theorem 6.1) and in defining g we pick it to be the identity on the 1-skeleton of T_1 (as we picked g to be the identity on $\sum \mathrm{Bd}\, F_i$ in the proof of Theorem 6.2). The triangulation T' is taken to refine T_1 so that the 1-skeleton of T' misses the $g(E_i)$'s and each simplex s of T' is of diameter less than the minimum value of $\min_{f_2}(s)$ or $\min_{f_3}(s)$. The simplexes of T_2 are the $g^{-1}(s)$'s.

7. **Maintaining separation.** We finally give the theorem which shows that the tame 2-manifold $h_0(M)$ constructed in the proof of Theorem 1.1 is two-sided.

THEOREM 7.1. *Suppose X is a subset of E^n that separates the point p from the point q, U is a bounded subset of X, and h_t $(0 \leqq t \leqq 1)$ is a homotopy of $X \to E^n - (\{p\} + \{q\})$ such that h_0 is the identity and h_t is fixed on $X - U$. Then $h_1(X)$ separates p from q.*

Proof. We suppose that X is closed since if a set separates two points of E^n, some closed set in it does.

We alter E^n and X by adding the point at infinity to each. This makes X compact. We then recover E^n by deleting the point q. We now have that p belongs to a bounded component of $E^n - X$ and wish to show that p belongs to a bounded component of $E^n - h_1(X)$.

Let S^{n-1} be the unit $(n-1)$-sphere with center at p and let r be the retraction of $E^n - \{p\}$ onto S^{n-1} that takes open rays from p onto the point where they pierce S^{n-1}. It follows from Theorem VI 10 on page 97 of [9] that $r: X \to S^{n-1}$ is not homotopic to a constant map. It also follows from this same Theorem VI 10 that p belongs to a bounded component of $E^n - h_1(X)$ unless $r: h_1(X) \to S^{n-1}$ is homotopic to a constant map.

Assume that p does not belong to a bounded component of $E^n - h_1(X)$ and f_t $(0 \leqq t \leqq 1)$ is a homotopy of $h_1(X) \to S^{n-1}$ such that $f_0 = r$ and $f_1 =$ constant map. Then

$$g_t = \begin{cases} rh_{2t} & (0 \leqq t \leqq 1/2), \\ f_{2t-1}h_1 & (1/2 \leqq t \leqq 1) \end{cases}$$

shows that $r: X \to S^{n-1}$ is homotopic to a constant map since $g_0 = r$ and $g_1 =$ constant map. The assumption that p does not belong to a bounded component of $E^n - h_1(X)$ led to this contradiction.

COROLLARY 7.2. *Suppose M^2 is a 2-manifold embedded as a closed set in a 3-manifold M^3, U is an open subset of M^3 homeomorphic to E^3, X is a compact subset of $M^2 \cdot U$ and h is a homeomorphism of M^2 into M^3 such that h is fixed on $M^2 - X$ and $H(X) \subset U$. Then $h(M^2)$ separates M^3 if and only if M^2 does. If p, q are two points of $M^3 - U$ separated in M^3 by M^2, they are also separated in M^3 by $h(M^2)$.*

QUESTION. Can Theorem 7.1 be generalized by replacing E^n by an n-manifold where it is understood that U has a compact closure?

8. **Corrections to [5].** While studying [5], Stephen Slack pointed out some minor errors in it. Although the proof given in [5] contains some simplifications over the proof given in [2], three of the figures used in [5] do not reflect these simplifications and are more applicable to [2]. For example, the assumption for the *very, very special* case (pp. 151-154 of [5]) is so strong that no component of $S - R_2$ has a triod in its boundary. Hence, the situation could not be as bad as depicted in Figure 3 of [5]. Since each component of $S \cdot R_2$ intersects R_1, Figure 5 is inapplicable. Also, the assumption for the *very special case* is enough to prevent anything as bad as that shown in Figure 7.

The proof of the *very special case* (pp. 154-161 of [5]) overlooked the fact that two X_i's as defined on page 155 might intersect in two points and the corresponding $K(X_i)$'s as defined on page 156 might intersect in a nondegenerate subcontinuum of each. This is remedied if one replaces the definition of $K(X_i)$ on page 156 with the following.

"For each X_i that does not share two points with any preceding X_i, let $K(X_i)$ be the sum of X_i and all components of $S - X_i$ with diameters less than $\epsilon/2$. Other K's are defined inductively. If X_i shares two points with some preceding X's and the K's on these preceding X's have been defined, let X_i' be the closure of X_i minus the K's associated with these preceding X's and let $K(X_i)$ be the sum of X_i' and all components of $S - X_i'$ with diameters less than $\epsilon/2$."

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