PRODUCTS OF AUTOMATA AND THE PROBLEM OF COVERING

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Introduction. The first part of this paper (Theorems 1-3) gives a short, unified treatment of (Mealy type [12]) automata (sequential machines). By associating with every input two binary relations ("next-state" and "output" relations) we obtain an easy and concise algebraic method for the description and study of complete or partial, finite or infinite automata.

In the second part (Theorems 4-7) we develop further the algebraic decomposition theory of automata, continuing previous work by J. Hartmanis [7]-[9] and M. Yoeli [16]-[18]. To make the exposition self-contained, we repeat some of the material contained in [18]. For other approaches to automata decompositions the reader is referred to [1], [3], [5], [10], [11].

In [18] the concept of semi-automaton (see Section I) was introduced and methods for its decomposition by means of overlapping partitions were derived. In the present paper these investigations are extended to (Mealy type) automata and the problems of covering specified automata by direct and cascade products are studied.

This approach leads to an interesting new algebraic concept, namely that of a weak (i.e., generalized) homomorphism defined by overlapping partitions. Recently this concept and its applicability to partial algebras has been further investigated [19] and generalizations of well-known results on homomorphisms and subdirect products of partial algebras have been obtained.

We hope that this mutual enrichment of pure algebra and automata theory will be of interest to both the applied as well as the pure algebraist. The newcomer to automata theory is referred to the introductory texts [2], [4]-[6] and the collection of papers [13], [14]. [13] also contains a very extensive bibliography on sequential machines.

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I. **Preliminaries.** We first recall some basic concepts on binary relations to be used in the sequel.

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Let R be a relation between the sets M and N, i.e., $R \subseteq M \times N$. Following [15] we denote

$$|R = \{m \mid \exists n: (m,n) \in R\} \text{ and } R| = \{n \mid \exists m: (m,n) \in R\}.$$

For $M' \subseteq M$, $M'R = \{n \mid \exists m \in M': (m,n) \in R\}$.

The following is evident:

$$R_1 \subseteq R_2 \Rightarrow R_1R \subseteq R_2R,\ RR_1 \subseteq RR_2,\ R_1^{-1} \subseteq R_2^{-1},$$

$$|R = M \Rightarrow RR^{-1} \supseteq I_M = \{(m,m) \mid m \in M\}.$$

Next we list the following basic definitions concerning automata.

We define a *semi-automaton* A as a system $\langle S_A, X_A, \Delta^A \rangle$ where S_A is a nonempty set (of *states*), X_A is a nonempty set (of *inputs*) and Δ^A (the *next-state* function) a mapping from a subset of $S_A \times X_A$ into S_A.

With every $x \in X_A$ we associate a binary relation $\bar{x}^A$ over S_A defined by:

$$(s,t) \in \bar{x}^A \Leftrightarrow \Delta^A(s,x) = t.$$

Clearly the semi-automaton A can alternatively be defined as a triple $\langle S_A, X_A, \{\bar{x}^A \mid x \in X_A\} \rangle$ where the $\bar{x}^A$ are mappings from S_A into S_A.

The following definitions are taken over from [18]. A *decomposition* π of a given set S is a family of nonempty subsets of S whose set union is S.

Let $A = \langle S_A, X_A, \Delta^A \rangle$ be a semi-automaton and π a decomposition of S_A. π is *admissible* by A, if for every $H \in \pi$ and every $x \in X_A$ there exists a $K \in \pi$ such that $H\bar{x}^A \subseteq K$. The semi-automaton $B = \langle S_B, X_B, \Delta^B \rangle$ is a *π-factor* of A if (i) $S_B = \pi$, (ii) $X_A = X_B$, (iii) for every $H \in \pi$ and every $x \in X_A$, $H\bar{x}^A = \emptyset$ implies $H\bar{x}^B = \emptyset$ and (iv) for every $H \in \pi$ and every $x \in X_A$, $H\bar{x}^A \subseteq H\bar{x}^B$.

Clearly, if $A = \langle S_A, X_A, \Delta^A \rangle$ is a semi-automaton and π a partition of S_A admissible by A, there exists exactly one π-factor B of A (notation: $B = A/\pi$).

The *direct product* $A \times B$ of the semi-automata $A = \langle S_A, X, \Delta^A \rangle$ and $B = \langle S_B, X, \Delta^B \rangle$ is the semi-automaton $C = \langle S_C, X, \Delta^C \rangle$ where $S_C = S_A \times S_B$ and $(s_A, s_B)\bar{x}^C = s_A\bar{x}^A \times s_B\bar{x}^B$.

An *automaton* $\hat{A} = \langle S_A, X_A, Z_A, \Delta^A, \Lambda^A \rangle$ is a semi-automaton $A = \langle S_A, X_A, \Delta^A \rangle$ together with a nonempty set (of *outputs*) Z_A and a mapping Λ^A (the *output* function) from a subset of $S_A \times X_A$ into Z_A.

As previously, we associate with each input $x \in X_A$ a binary relation $x_*^A \subseteq S_A \times Z_A$ defined by:

$$(s,z) \in x_*^A \Leftrightarrow \Lambda^A(s,x) = z.$$

The semi-automaton A of the automaton $\hat{A}$ is *complete* if Δ^A is completely defined, i.e., for every $s \in S_A$ and every $x \in X_A$. The automaton $\hat{A}$ is *complete* if A is complete and Λ^A is also completely defined.

As usual, we extend our considerations to *input tapes* $\xi = x_1 x_2 \cdots x_K$, $x_i \in X_A$. To each such tape ξ there correspond the following binary relations:

$$\bar{\xi}^A = \bar{x}_1^A \bar{x}_2^A \cdots \bar{x}_K^A,$$

$$\xi_*^A = \bar{x}_1^A \bar{x}_2^A \cdots \bar{x}_{K-1}^A x_{K*}^A.$$

The right-hand sides of the above formulas are the relational products of the corresponding relations. $\bar{\xi}^A$ gives the state transformation due to the tape ξ, and ξ_*^A indicates the last output due to ξ, both for any initial state.

For a partial automaton $\hat{A}$ and a certain tape ξ, $\bar{\xi}^A$ or ξ_*^A can be the *empty* relation $\emptyset$.

II. Homomorphisms of automata.

DEFINITION 1. Given two automata

$$\hat{A} = \langle S_A, X_A, Z_A, \Delta^A, \Lambda^A \rangle$$

and

$$\hat{B} = \langle S_B, X_B, Z_B, \Delta^B, \Lambda^B \rangle$$

with $X_A = X_B$ and $Z_A = Z_B$, the mapping ϕ of S_A onto S_B is a *homomorphism* of $\hat{A}$ onto $\hat{B}$ if for every $x \in X_A = X_B$

$$\text{(1)} \qquad \begin{aligned} &\text{(i)} \ \bar{x}^A \phi \subseteq \phi \bar{x}^B, \\ &\text{(ii)} \ x_*^A \subseteq \phi x_*^B. \end{aligned}$$

NOTE. $\bar{x}^A\phi$ denotes the relational product of $\bar{x}^A$ and ϕ ($\subseteq S_A \times S_B$), etc.

DEFINITION 2. Given two automata $\hat{A}$ and $\hat{B}$ as in Definition 1, and a binary relation ψ with $|\psi = S_A$ and $\psi| = S_B$, the relation ψ is a *weak homomorphism* of $\hat{A}$ onto $\hat{B}$, if for every $x \in X_A = X_B$

$$\text{(2)} \qquad \begin{aligned} &\text{(i)} \ \psi^{-1} \bar{x}^A \subseteq \bar{x}^B \psi^{-1}, \\ &\text{(ii)} \ \psi^{-1} x_*^A \subseteq x_*^B. \end{aligned}$$

If ψ in Definition 2 is a mapping of S_A onto S_B then the conditions (1) and (2) are equivalent. Indeed, the assumptions $|\psi = S_A$ and $\psi| = S_B$ imply $\psi\psi^{-1} \supseteq I_{S_A}$ and $\psi^{-1}\psi \supseteq I_{S_B}$.

The further assumption that ψ is a mapping leads to $\psi^{-1}\psi = I_{S_B}$. Now,

$$\bar{x}^A \psi \subseteq \psi \bar{x}^B \Rightarrow \psi^{-1} \bar{x}^A \psi \psi^{-1} \subseteq \psi^{-1} \psi \bar{x}^B \psi^{-1} \Rightarrow \psi^{-1} \bar{x}^A \subseteq \bar{x}^B \psi^{-1}$$

and

$$x_*^A \subseteq \psi x_*^B \Rightarrow \psi^{-1} x_*^A \subseteq \psi^{-1} \psi x_*^B = x_*^B.$$

Conversely,

$$\psi^{-1}\bar{x}^A \subseteq \bar{x}^B\psi^{-1} \Rightarrow \psi\psi^{-1}\bar{x}^A\psi \subseteq \psi\bar{x}^B\psi^{-1}\psi \Rightarrow \bar{x}^A\psi \subseteq \psi\bar{x}^B$$

and

$$\psi^{-1}x_*^A \subseteq x_*^B \Rightarrow \psi\psi^{-1}x_*^A \subseteq \psi x_*^B \Rightarrow x_*^A \subseteq \psi x_*^B.$$

However, if ψ is not a mapping, conditions (1) and (2) are not equivalent, as shown by the following

EXAMPLE.

$$S_A = \{a_1, a_2, a_3, a_4\},\ S_B = \{b_1, b_2, b_3, b_4\},$$
$$X_A = X_B = \{x\},$$
$$\bar{x}^A = \{(a_1, a_4), (a_2, a_3)\},\ \bar{x}^B = \{(b_1, b_4), (b_2, b_3)\},$$
$$\psi = \{(a_1, b_1), (a_2, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)\}.$$

In this case condition (1) is satisfied, but condition (2) is not.

Let ψ be a weak homomorphism of $\hat{A}$ onto $\hat{B}$. Then for every input tape ξ we have:

(3) (i) $\psi^{-1}\bar{\xi}^A \subseteq \bar{\xi}^B\psi^{-1}$,

(ii) $\psi^{-1}\xi_*^A \subseteq \xi_*^B$.

Indeed, let $\xi = x_1 \cdots x_K$; then

$$\psi^{-1}\bar{\xi}^A = \psi^{-1}\bar{x}_1^A \cdots \bar{x}_K^A \subseteq \bar{x}_1^B\psi^{-1}\bar{x}_2^A \cdots \bar{x}_K^A \subseteq \bar{x}_1^B \cdots \bar{x}_K^B\psi^{-1} = \bar{\xi}^B\psi^{-1}$$

and

$$\psi^{-1}\xi_*^A = \psi^{-1}\bar{x}_1^A \cdots \bar{x}_{K-1}^A x_{K*}^A \subseteq \bar{x}_1^B \cdots \bar{x}_{K-1}^B\psi^{-1}x_{K*}^A \subseteq \bar{x}_1^B \cdots \bar{x}_{K-1}^B x_{K*}^B = \xi_*^B.$$

Again, let ψ be a weak homomorphism of $\hat{A}$ onto $\hat{B}$. The relation ψ *induces* a decomposition π of S_A, namely $\pi = \{H_s = s\psi^{-1} \mid s \in S_B\}$. This decomposition π is admissible by the semi-automaton A of $\hat{A}$, i.e., for every $H_s \in \pi$ and every $x \in X_A$ there exists an $H_t \in \pi$ such that $H_s\bar{x}^A \subseteq H_t$. Indeed,

$$H_s\bar{x}^A = s\psi^{-1}\bar{x}^A \subseteq s\bar{x}^B\psi^{-1}.$$

If $s\bar{x}^B = t$, we have $H_s\bar{x}^A \subseteq t\psi^{-1} = H_t \in \pi$.

If $s\bar{x}^B = \emptyset$, then also $H_s\bar{x}^A = \emptyset$ and the admissibility condition is trivially satisfied.

Furthermore, π is an *output-consistent* decomposition of $\hat{A}$, i.e., for every $H_s \in \pi$ and every tape ξ, $H_s\xi_*^A$ contains at most one element. Indeed,

$$H_s\xi_*^A = s\psi^{-1}\xi_*^A \subseteq s\xi_*^B$$

and $s\xi_*^B$ is either the empty set or a single element of $Z_B = Z_A$.

Conversely, let $\hat{A} = \langle S_A, X_A, Z_A, \Delta^A, \Lambda^A\rangle$ be an automaton and π an admissible decomposition of A, which is output-consistent in A.

Clearly an automaton $\hat{B} = \langle \pi, X_A, Z_A, \Delta^B, \Lambda^B \rangle$ can be defined such that the relation ψ defined by $(s, H) \in \psi \Leftrightarrow s \in H \in \pi$ is a weak homomorphism of $\hat{A}$ onto $\hat{B}$.

The semi-automaton B of $\hat{B}$ is a π-factor of A.

Summarizing we have the following:

THEOREM 1. *A weak homomorphism ψ of $\hat{A}$ onto $\hat{B}$ defines naturally an admissible, output-consistent decomposition in $\hat{A}$ and, conversely, every such decomposition in $\hat{A}$ determines at least one corresponding weakly homomorphic image of $\hat{A}$.*

NOTE. In the special case, where ψ is a homomorphism (Definition 1), the decomposition π becomes a partition. On the other hand we have the following known

COROLLARY. *An admissible, output-consistent partition of a complete automaton $\hat{A}$ defines naturally a unique homomorphic image of $\hat{A}$.*

III. **Covering of automata.**

DEFINITION 3. The automaton $\hat{B} = \langle S_B, X, Z, \Delta^B, \Lambda^B \rangle$ is said to cover the automaton $\hat{A} = \langle S_A, X, Z, \Delta^A, \Lambda^A \rangle$ (notation: $\hat{B} \geqq \hat{A}$) if there exists a mapping χ of S_A into S_B such that for every input tape ξ:

$$\xi_*^A \subseteq \chi \xi_*^B.$$

Using the relational techniques introduced we now provide a simple proof of the following theorem, which is in fact a modified version of a known result on partial automata [4].

THEOREM 2. *Let ψ be a weak homomorphism of $\hat{A}$ onto $\hat{B}$. Then $\hat{B} \geqq \hat{A}$.*

Proof. Clearly there exists at least one mapping χ of S_A into S_B, such that $\chi \subseteq \psi$. Using (3) we obtain for every input tape ξ:

$$\psi^{-1}\xi_*^A \subseteq \xi_*^B \Rightarrow \chi^{-1}\xi_*^A \subseteq \xi_*^B \Rightarrow \chi\chi^{-1}\xi_*^A \subseteq \chi\xi_*^B.$$

But $|\chi = S_A$; hence $\chi\chi^{-1}$ includes the identity relation I_{S_A}. Thus we obtain

$$\xi_*^A \subseteq \chi\chi^{-1}\xi_*^A \subseteq \chi\xi_*^B.$$

In [18] the concept of covering of semi-automata was introduced. Using the notations of this paper we have accordingly:

DEFINITION 4. The semi-automaton $B = \langle S_B, X, \Delta^B \rangle$ covers the semi-automaton $A = \langle S_A, X, \Delta^A \rangle$ (notation: $B \geqq A$) if there exists a mapping η of a subset of S_B onto S_A such that for every $x \in X$

$$\eta \bar{x}^A \subseteq \bar{x}^B \eta. \tag{4}$$

THEOREM 3. *Let $\hat{A} = \langle S_A, X, Z, \Delta^A, \Lambda^A\rangle$ be an automaton, and $B = \langle S_B, X, \Delta^B\rangle$ a semi-automaton covering the semi-automaton A of $\hat{A}$. Then there exists an automaton $\hat{B}$ with B as its semi-automaton, such that $\hat{B} \geqq \hat{A}$.*

Proof. $B \geqq A$ implies the existence of a mapping η with $|\eta \subseteq S_B$ and $\eta| = S_A$, satisfying (4). Now, let $\hat{B} = \langle S_B, X, Z, \Delta^B, \Lambda^B\rangle$ where Λ^B is defined by:

$$x_*^B = \eta x_*^A \quad \text{for every } x \in X.$$

In other words $\Lambda^B(s,x) = \Lambda^A(s\eta, x)$ if $s \in |\eta$ and $\Lambda^A(s\eta, x)$ is defined. Otherwise $\Lambda^B(s,x)$ is not defined.

The automaton $\hat{B}$ covers $\hat{A}$. Indeed, evidently, there exists a mapping χ of S_A into S_B such that $\chi \subseteq \eta^{-1}$. Now, for any input tape $\xi = x_1 x_2 \cdots x_K$

$$\chi^{-1}\xi_*^A \subseteq \eta\xi_*^A = \eta\overline{x}_1^A \cdots \overline{x}_{K-1}^A x_{K*}^A \subseteq \overline{x}_1^B \cdots \overline{x}_{K-1}^B \eta x_{K*}^A = \overline{x}_1^B \cdots \overline{x}_{K-1}^B x_{K*}^B = \xi_*^B.$$

Hence

$$\xi_*^A \subseteq \chi\chi^{-1}\xi_*^A \subseteq \chi\xi_*^B.$$

Theorem 3 is immediately applicable to the direct product of automata, which is defined as follows:

DEFINITION 5. The *direct product* $\hat{A} \times \hat{B}$ of the automata $\hat{A} = \langle S_A, X, Z_A, \Delta^A, \Lambda^A\rangle$ and $\hat{B} = \langle S_B, X, Z_B, \Delta^B, \Lambda^B\rangle$ is the automaton $\hat{C} = \langle S_C, X, Z_C, \Delta^C, \Lambda^C\rangle$ where

$$S_C = S_A \times S_B, \; Z_C = Z_A \times Z_B$$

and

$$(s_A, s_B)\overline{x}^C = s_A\overline{x}^A \times s_B\overline{x}^B,$$

$$(s_A, s_B)x_*^C = s_A x_*^A \times s_B x_*^B.$$

Definition 5 implies that $C = A \times B$.

In order to apply Theorem 3 to direct products of automata we shall need the following extension of the covering concept.

DEFINITION 6. The automaton $\hat{B} = \langle S_B, X, Z_B, \Delta^B, \Lambda^B\rangle$ is said to *cover widely* the automaton $\hat{A} = \langle S_A, X, Z_A, \Delta^A, \Lambda^A\rangle$ (notation: $\hat{B} \geqq \hat{A}$), if there exists a mapping χ of S_A into S_B and a mapping τ of a subset of Z_B into Z_A, such that for every input tape ξ

$$\xi_*^A \subseteq \chi\xi_*^B\tau.$$

We now have the following

THEOREM 4. *Let C be the semi-automaton of $\hat{C} = \langle S_C, X, Z_C, \Delta^C, \Lambda^C\rangle$ and A, B semi-automata such that $A \times B \geqq C$. Then there exist automata $\hat{A}$, $\hat{B}$ with A and B as semi-automata, respectively, such that*

$$\hat{A} \times \hat{B} \geqq \hat{C}.$$

Proof. By Theorem 3 there exists an automaton $\hat{D}$ such that $D = A \times B$ and $\hat{D} \geqq \hat{C}$.

We define $Z_A = S_A \times X$ and $Z_B = S_B \times X$. Next, we define Λ^A. If for a given state $s_A \in S_A$ and a given input $x \in X$, there exists an $s_B \in S_B$, such that $(s_A, s_B)\, x_*^D \neq \emptyset$, let $s_A x_*^A = (s_A, x)$. Otherwise $s_A x_*^A = \emptyset$.

Λ^B is defined in an analogous way.

Introducing the map τ by

$$(s_A, s_B)\, x_*^D \neq \emptyset \Rightarrow ((s_A, x), (s_B, x))\, \tau = (s_A, s_B)\, x_*^D,$$

we obtain

$$\xi_*^D = \xi_*^E \tau,$$

where

$$\hat{E} = \hat{A} \times \hat{B}.$$

Let χ be the mapping corresponding to the covering $\hat{D} \geqq \hat{C}$; then

$$\xi_*^C \subseteq \chi \xi_*^D = \chi \xi_*^E \tau, \text{ i.e., } \hat{E} = \hat{A} \times \hat{B} \geqq \hat{C}.$$

Evidently the above discussion of direct products of two automata can be extended to any finite number of automata.

IV. **Cascade products of automata.** The following definition generalizes the concept of cascade connection in [16].

DEFINITION 7. Let $\hat{A} = \langle S_A, X_A, Z_A, \Delta^A, \Lambda^A \rangle$ and $\hat{B} = \langle S_B, X_B, Z_B, \Delta^B, \Lambda^B \rangle$ be automata with $Z_A \subseteq X_B$. The *cascade product* $\hat{C} = \hat{A} \circ \hat{B}$ is defined as the automaton $\hat{C} = \langle S_C, X_C, Z_C, \Delta^C, \Lambda^C \rangle$ where $S_C = S_A \times S_B$, $X_C = X_A$, $Z_C = Z_B$ and for every $x \in X_C$

$$(s_A, s_B)\, \bar{x}^C = s_A \bar{x}^A \times s_B \bar{y}^B,$$

$$(s_A, s_B)\, x_*^C = s_B y_*^B,$$

where

$$Z_A \ni y = s_A x_*^A.$$

(If $y = \emptyset$, then $\bar{y}$ and y_* is to be understood as $\emptyset$.)

Definition 7 is naturally modified to the case $\hat{A} \circ B$, where B is a semi-automaton. We thus obtain

$$\hat{C} = \hat{A} \circ \hat{B} \Rightarrow C = \hat{A} \circ B.$$

With any semi-automaton $A = \langle S_A, X_A, \Delta^A \rangle$ one may associate the automaton $A^* = \langle S_A, X_A, Z_A, \Delta^A, \Lambda^A \rangle$ where $Z_A = S_A \times X_A$ and $s_A x_*^A = (s_A, x)$ for every $s_A \in S_A$ and every $x \in X_A$. If $B = \langle S_B, X_B, \Delta^B \rangle$ is a semi-automaton, such that $X_B \supseteq S_A \times X_A$, we define (following [18]):

$$A \circ B = A^* \circ B.$$

We now prove

THEOREM 5. *Let $\hat{A}$ and $\hat{C}$ be automata and B a semi-automaton such that $\hat{A} \circ B \geqq C$. Then there exist automata $\hat{D}$ and $\hat{E}$ such that $\hat{D} \circ \hat{E} \geqq \hat{C}$, where $D = A$, $\hat{D} \geqq \hat{A}$, $S_E = S_B$.*

Proof. By Theorem 3 there exists an automaton $\hat{F}$ such that $F = \hat{A} \circ B$ and $\hat{F} \geqq \hat{C}$. We now define $\hat{D}$ as follows:
$D = A$, $Z_D = S_D \times X_D$, and for every $s_D \in S_D$ and every $x \in X_D$, $s_D x_*^D = (s_D, x)$.

Introducing the map τ of Z_D into Z_A determined by $(s_D, x)\tau = s_D x_*^A$ we clearly obtain:

$$\hat{D} \geqq \hat{A}.$$

Next, we define $\hat{E}$ as follows:

$$S_E = S_B,\ X_E = Z_D,\ Z_E = Z_F;$$

for every $(s_D, x) \in X_E$ and every $s_E \in S_E$

$$\overline{(s_D, x)}^E = \overline{(s_D, x)\,\tau}^B,$$

$$s_E (s_D, x)_*^E = (s_D, s_E)\, x_*^F.$$

Denoting $\hat{G} = \hat{D} \circ \hat{E}$, we proceed to show that $\hat{F} = \hat{G}$. Indeed:

$$S_F = S_A \times S_B = S_D \times S_E = S_G,$$

$$X_F = X_A = X_D = X_G,\ Z_F = Z_E = Z_G,$$

and

$$s_G \overline{x}^G = (s_D, s_E)\overline{x}^G = s_D \overline{x}^D \times s_E \overline{w}^E,$$

where

$$w = s_D x_*^D = (s_D, x).$$

Now

$$\overline{w}^E = \overline{(s_D, x)}^E = \overline{(s_D, x)\,\tau}^B = \overline{y}^B,$$

where

$$y = (s_D, x)\,\tau = s_D x_*^A.$$

Hence $s_G \overline{x}^G = s_D \overline{x}^D \times s_E \overline{w}^E = s_D \overline{x}^A \times s_E \overline{y}^B = (s_D, s_E)\, \overline{x}^F = s_G \overline{x}^F$.
Furthermore,

$$s_G x_*^G = (s_D, s_E)\, x_*^G = s_E w_*^E = s_E (s_D, x)_*^E = (s_D, s_E)\, x_*^F = s_G x_*^F.$$

Thus $\hat{D} \circ \hat{E} = \hat{G} = \hat{F} \geqq \hat{C}$.

Referring to the definition of cascade-product of semi-automata, we obtain the following:

COROLLARY. *Given an automaton $\hat{C}$ and two semi-automata A, B such that $A \circ B \geqq C$ then there exist automata $\hat{A}$, $\hat{B}$ such that $\hat{A} \circ \hat{B} \geqq \hat{C}$, where A, B are the semi-automata of $\hat{A}$, $\hat{B}$, respectively.*

In particular cases the automaton $\hat{D}$ required in Theorem 5 can be constructed with a reduced output set (cf. [17]).

We now restrict our considerations to complete, finite automata. For such an automaton $\hat{A}$ we denote by $\# A$ the number of its states. If π is a decomposition of the finite set S, $|\pi|$ will denote the number of elements in the largest class of π.

The following result was obtained in [18]:

THEOREM 6. *Let $A = \langle S_A, X_A, \Delta^A \rangle$ be a complete, finite semi-automaton, π an admissible decomposition of A, and B a π-factor of A. Then there exist semi-automata C and D, such that C is isomorphic to B, $\# D = |\pi|$, and $C \circ D \geqq A$.*

We now prove

THEOREM 7. *Let $\hat{A}$, $\hat{B}$, $\hat{C}$ be complete, finite automata, such that there exists a homomorphism ϕ from $\hat{A} \circ \hat{B}$ onto $\hat{C}$. Then there exist complete, finite automata $\hat{D}$, $\hat{E}$ and an admissible decomposition β of C such that D is isomorphic to a β-factor of C and*

$$\hat{D} \circ \hat{E} \geqq \hat{C}\ , \ \# D \leqq \# A,\ \# E \leqq \# B.$$

Proof. Denote $\hat{A} \circ \hat{B} = \hat{F}$, and let ρ be the admissible partition of F determined by

$$(s_A, s_B) \equiv (s'_A, s'_B)(\rho) \Leftrightarrow s_A = s'_A.$$

Then the unique ρ-factor F/ρ is isomorphic to $A (F/\rho \cong A)$.

Let π be the natural partition of F determined by ϕ. Then $F/\pi \cong C$.

We now consider the decomposition α of π defined by:

$$\alpha = \{ \alpha_H = \{ \pi(s) \mid s \in H \} \mid H \in \rho \}$$

where $\pi(s)$ denotes the class of π containing s.

α is admissible by the semi-automaton F/π. Indeed, let $\alpha_1 \in \alpha$, i.e., there exists an $H \in \rho$, such that $\alpha_1 = \{ \pi(s) \mid s \in H \}$.

For any input x of F we have

$$\alpha_1 \bar{x}^{F/\pi} = \{ \pi(s) \mid s \in H \} \bar{x}^{F/\pi} = \{ \pi(s\bar{x}^F) \mid s \in H \} \subseteq \{ \pi(t) \mid t \in H\bar{x}^{F/\rho} \in \rho \} \in \alpha.$$

Furthermore, $\# \alpha \leqq \# \rho = \# A$.

For every $H \in \rho$, $\# \alpha_H \leqq \# H \leqq |\rho|$.

Hence $|\alpha| \leqq |\rho| = \# B$.

The isomorphism $C \cong F/\pi$ implies the existence of an admissible decomposition β of C corresponding to the decomposition α of F/π.

We thus have

$$\# \beta \leqq \# A, \quad |\beta| \leqq \# B.$$

Now we apply Theorem 6 to obtain semi-automata D, E such that $D \circ E \geqq C$, where D is isomorphic to a β-factor of C, and $\# E = |\beta| \leqq \# B$.

D being isomorphic to a β-factor of C, satisfies $\# D = \# \beta \leqq \# A$.

As final step of this proof we apply the Corollary to Theorem 5 and obtain automata $\hat{D}$ and $\hat{E}$ such that $\hat{D} \circ \hat{E} \geqq \hat{C}$ with D, E as semi-automata of $\hat{D}$, $\hat{E}$ respectively.

J. Hartmanis and R. E. Stearns have pointed out in [9] that state reduction may destroy possibilities of cascade decompositions. Our Theorem 7 shows that even after state reduction of a complete automaton, i.e., its replacement by a homomorphic image, the original possibilities for cascade decompositions may be reconstructed in a certain sense.

We shall illustrate this point by using the example in [9].

Let $\hat{A}$, $\hat{B}$, $\hat{C}$ be the automata given by Tables I, II, III respectively:

Table I — $\hat{A}$

state \ input	0	1
a_1	a_1/y_1	a_2/y_2
a_2	a_1/y_3	a_1/y_4

next state/output

Table II — $\hat{B}$

state \ input	y_1	y_2	y_3	y_4	output
b_1	b_3	b_3	b_2	b_4	1
b_2	b_4	b_4	b_1	b_3	0
b_3	b_1	b_2	b_4	b_4	0
b_4	b_2	b_1	b_3	b_3	1

next state

Table III — $\hat{C}$

state	input 0	input 1	output
c_1	c_3	c_6	1
c_2	c_4	c_7	0
c_3	c_1	c_5	0
c_4	c_2	c_4	1
c_5	c_1	c_3	0
c_6	c_4	c_4	0
c_7	c_3	c_3	1

next state

The automaton $\hat{F} = \hat{A} \circ \hat{B}$ is given in Table IV.

Table IV — $\hat{F}$

state	input 0	input 1	output
$(a_1, b_1) = f_1$	f_3	f_7	1
$(a_1, b_2) = f_2$	f_4	f_8	0
$(a_1, b_3) = f_3$	f_1	f_6	0
$(a_1, b_4) = f_4$	f_2	f_5	1
$(a_2, b_1) = f_5$	f_2	f_4	1
$(a_2, b_2) = f_6$	f_1	f_3	0
$(a_2, b_3) = f_7$	f_4	f_4	0
$(a_2, b_4) = f_8$	f_3	f_3	1

next state

The automata $\hat{F}$ and $\hat{C}$ are identical with machines B^* and B of Figure 4 in [9].

The mapping

$$\phi = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ c_1 & c_2 & c_3 & c_4 & c_4 & c_5 & c_6 & c_7 \end{pmatrix}$$

is a homomorphism of $\hat{F}$ onto $\hat{C}$.

Indeed,

$$\bar{0}^F\phi = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ c_3 & c_4 & c_1 & c_2 & c_2 & c_1 & c_4 & c_3 \end{pmatrix} = \phi\bar{0}^C,$$

$$0^F_* = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} = \phi 0^C_*,$$

and similarly for the input 1.

The partitions ρ and π of F become:

$$\rho = \{ H_1 = \{f_1, f_2, f_3, f_4\}, H_2 = \{f_5, f_6, f_7, f_8\}\},$$

$$\pi = \{\pi_1 = \{f_1\}, \pi_2 = \{f_2\}, \pi_3 = \{f_3\},$$

$$\pi_4 = \{f_4, f_5\}, \pi_5 = \{f_6\}, \pi_6 = \{f_7\}, \pi_7 = \{f_8\}\}.$$

The decomposition α of π becomes:

$$\alpha = \{\alpha_1 = \{\pi_1, \pi_2, \pi_3, \pi_4\}, \ \alpha_2 = \{\pi_4, \pi_5, \pi_6, \pi_7\}\}.$$

The decomposition β of C is therefore

$$\beta = \{\beta_1 = \{c_1, c_2, c_3, c_4\}, \ \beta_2 = \{c_4, c_5, c_6, c_7\}\}.$$

Let D be the β-factor of C given in Table V.

Table V — D

state \ input	0	1
β_1	β_1	β_2
β_2	β_1	β_1

next state

The application of Theorem 6 leads to a semi-automaton E isomorphic to B.

Using the construction of Theorem 5 we obtain automata $\hat{D}$ and $\hat{E}$ isomorphic with $\hat{A}$ and $\hat{B}$ respectively and $\hat{D} \circ \hat{E} \geqq \hat{C}$.

On the other hand, as shown in [9], the usual methods of cascade-decomposition applied to $\hat{C}$ do not lead to the discovery of the above cascade-decomposition of $\hat{C}$.

Conclusion. This paper shows the convenience of the relational techniques introduced for the study of automata.

Furthermore, a generalized approach to the problem of automata decomposition has been developed. Namely, in order to synthesize $\hat{A}$, it is frequently convenient to construct a cover of $\hat{A}$ which is a direct or cascade product of simpler automata. (The previous approach was restricted to equality or inclusion instead of covering.)

It appears that the results obtained in this paper will have actual engineering applications to the synthesis of sequential machines.

However, in order to extend this applicability, the development of an efficient technique for obtaining suitable admissible decompositions of automata is required.

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