PRIMARY IDEALS AND VALUATION IDEALS

BY

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1. **Introduction.** Let D be an integral domain, let $\mathscr{Q}$ denote the set of primary ideals of D, and let $\mathscr{V}$ denote the set of valuation ideals of D. The object of this paper is to investigate the significance of the relationships $\mathscr{V} \subseteq \mathscr{Q}$, $\mathscr{Q} \subseteq \mathscr{V}$, and $\mathscr{Q} = \mathscr{V}$. Our point of departure was the observation in [8, p. 341, Example 2], that if D is a Dedekind domain, then $\mathscr{Q} = \mathscr{V}$. We prove here that $\mathscr{Q} = \mathscr{V}$ if and only if D is a Prüfer domain of dimension $\leqq 1$. Also, $\mathscr{V} \subseteq \mathscr{Q}$ if and only if every proper prime ideal of D is maximal (i.e., $\dim D \leqq 1$). However, these results are fairly immediate, and our main concern is with the implications of the containment $\mathscr{Q} \subseteq \mathscr{V}$. If D is a Prüfer domain, it is clear that $\mathscr{Q} \subseteq \mathscr{V}$; and under the hypothesis that D satisfy the ascending chain condition for prime ideals, we are able to prove $\mathscr{Q} \subseteq \mathscr{V}$ implies D is Prüfer. Moreover, in §5 we construct an example of a domain which satisfies the condition $\mathscr{Q} \subseteq \mathscr{V}$ but which is not Prüfer.

Our terminology adheres to the conventions of [7], [8] with two exceptions: First, the domain D is not included in the "ideals" of D, and second $\subseteq$ will denote containment and $\subset$ indicates proper containment.

2. **Preliminary results on valuation ideals.** We shall begin by reviewing some definitions (found in [8, Appendices 3, 4]).

(1) An ideal A of a domain D is called a *valuation ideal* if there exists a valuation ring $D_v \supseteq D$ and an ideal A_v of D_v such that $A_v \cap D = A$. When we want to specify the particular valuation ring D_v, we shall say A is a v-ideal. If A is a v-ideal, then $A \cdot D_v \cap D = A$.

(2) If A is an ideal of D and if S is the set of all nontrivial valuations of the quotient field K of D which are non-negative on D, then $A' = \bigcap_{v \in S} A \cdot D_v$ is called the *completion* of A. If $A = A'$, then A is called complete. $D' = \bigcap_{v \in S} D_v$ is the integral closure of D (by [8, p. 15, Theorem 6]).

We now list some fundamental properties of complete ideals and valuation ideals.

2.1. PROPOSITION. *Let D be a domain. If A is any ideal of D, denote by A' the completion of A and by A^* the intersection of those valuation ideals of D which contain A*(2). *Then*

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(2) There exists at least one valuation ideal ($\neq D$) containing A, since every prime ideal is a valuation ideal.

(a) $A' \cap D = A^*$.
(b) $(x)' = xD'$ *for any* $x \in D$.
(c) *If* $(x) = (x)'$ *for some* $x \in D$, $x \neq 0$, *then* $D = D'$. ((b) *proves the converse.*)
(d) $(x) = (x)^*$ *for all* $x \in D$ *if and only if* $D = D'$.

Proof.
(a) $A' \cap D = \bigcap_{v \in S}(A \cdot D_v) \cap D = \bigcap_{v \in S}(A \cdot D_v \cap D) \supseteq A^*$.
Conversely, if B is a v-ideal such that $B \supseteq A$, then $B \cdot D_v \cap D = B \supseteq A$ implies $A \cdot D_v \cap D \subseteq B$. Therefore, $A' \cap D \subseteq A^*$.
(b) [8, p. 348, Proposition 1, (f)].
(c) If $t \in D'$, then $tx \in D'x = (x)'$ by (b). Thus $tx = sx \in (x)$ for some $s \in D$. Since $x \neq 0$, $t = s \in D$ and $D = D'$.
(d) $D' = D$ implies $(x) = (x)'$ by (b). Therefore, $(x) = (x)^*$ by (a).
Conversely, $(x) = (x)^*$ implies $(x) = (x)' \cap D = x \cdot D' \cap D$, by (a) and (b). Then $y/x \in D'$ implies $y \in x \cdot D' \cap D = (x)$, so $y \in (x)$ and hence $y/x \in D$. q.e.d.

A *Prüfer domain* is by definition a domain which satisfies any of the equivalent assertions of the next theorem.

2.2. Theorem. *Let D be a domain. Then the following are equivalent*:
(a) *Every nonzero finitely generated ideal of D is invertible.*
(b) *D_P is a valuation ring for every prime ideal P.*
(c) *Whenever $A \neq (0)$, B, C are ideals such that A is finitely generated and $AB = AC$, then $B = C$.*
(d) *Every ideal of D is complete.*
(e) *Every ideal of D is an intersection of valuation ideals.*

Proof.
(a) $\Leftrightarrow$ (b) by [4, p. 554, Theorem 7].
(a) $\Leftrightarrow$ (c) by [5, p. 127, Krit. 3] and [2, Theorem 2*].
(b) $\Rightarrow$ (d): If A is an ideal of D and A' denotes the completion of A, then $A' \subseteq \bigcap(A \cdot D_M)$, where the intersection ranges over all maximal ideals M of D. But $\bigcap(A \cdot D_M) = A$ by [8, p. 94, Lemma]. Therefore, $A' \subseteq A$, so $A' = A$.
(d) $\Rightarrow$ (c): $AB \subseteq AC$ implies $(AB)' \subseteq (AC)'$, where $'$ denotes completion; so $B' \subseteq C'$. But $B' = B$, $C' = C$. Similarly, $AC \subseteq AB$ implies $C \subseteq B$. (The properties of $'$ used here are found in [8, p. 384, Proposition 1].)
(d) $\Rightarrow$ (e): Immediate from the definition of a complete ideal.
(e) $\Rightarrow$ (d): Apply 2.1–(d) and the remark of [8, p. 353], that every intersection of valuation ideals of an integrally closed domain is complete. q.e.d.

Corollary. *If every ideal of D is an intersection of primary ideals and if every primary ideal is a valuation ideal, then D is Prüfer.*

Proof. Apply (e).

If A is a valuation ideal of a domain D, the elements of A must satisfy certain relations. For example, we have the following:

2.3. LEMMA. *Let A be a valuation ideal of the domain D, and let R, S be arbitrary subsets of D. Then*
(a) *$RS \subseteq A$ implies $\{r^2 \mid r \in R\} \subseteq A$ or $\{s^2 \mid s \in S\} \subseteq A$.*
(b) *$R^2 + S^2 \subseteq A$ implies $RS \subseteq A$.*

Proof.
(a) Suppose there exists $s \in S$ such that $s^2 \notin A$, and let D_v be a valuation ring such that $A \cdot D_v \cap D = A$. Then for any $r \in R$, $v(r^2) \geqq v(rs)$ or $v(s^2) \geqq v(rs)$; and accordingly, $r^2 \in A \cdot D_v$ or $s^2 \in A \cdot D_v$. Since $s^2 \notin A$, $s^2 \notin A \cdot D_v$ and hence $r^2 \in A \cdot D_v$. Thus, $\{r^2 \mid r \in R\} \subseteq A \cdot D_v \cap D = A$.
(b) Suppose $r \in R$, $s \in S$ and D_v is a valuation ring such that $A \cdot D_v \cap D = A$. We may assume $v(r) \leqq v(s)$. Then $v(rs) \geqq v(r^2) \geqq v(r^2 + s^2)$, so $rs \in (r^2 + s^2)D_v \subseteq AD_v$. Thus, $rs \in AD_v \cap D = A$. q.e.d.

These are the only such relations which we use in this paper, so we shall not dwell on the subject. We would like to mention, however, the following general result, based essentially on the fact that the ideals of a valuation ring are linearly ordered:

THEOREM. *Let $F(x) = \sum_\sigma X_{\sigma(1)}^{m_1} \cdots X_{\sigma(s)}^{m_s}$ and $G(x) = \sum_\sigma X_{\sigma(1)}^{n_1} \cdots X_{\sigma(t)}^{n_t}$ be symmetric polynomials, σ ranging over all permutations of $1, \cdots, n$ and $1 \leqq t \leqq s \leqq n$; and suppose F, G have the same degree and $m_1 + \cdots + m_{t-j} \leqq n_1 + \cdots + n_{t-j}$ for $j = 1, \cdots, t-1$. If $A_1, \cdots, A_n$ are ideals of a domain D such that $G(A_1, \cdots, A_n)$ is an intersection of valuation ideals, then*

$$F(A_1, \cdots, A_n) \subseteq G(A_1, \cdots, A_n).$$

One might hope to characterize a valuation ideal by relations of the form $F(A_1, \cdots, A_n) \subseteq G(A_1, \cdots, A_n)$, but the above theorem tells us that such relations only characterize ideals which are intersections of valuation ideals. As a case in point, every ideal of a Prüfer domain is an intersection of valuation ideals but we shall presently see that such an ideal need not be a valuation ideal.

2.4. COROLLARY. *If every principal ideal of a domain D is a valuation ideal, then D is a valuation ring.*

Proof. By 2.3–(a), $x^2 \in (xy)$ or $y^2 \in (xy)$, for any nonzero elements $x, y \in D$. But then x/y or $y/x \in D$, so D is a valuation ring. q.e.d.

2.5. PROPOSITION. *Let Q be a primary ideal of a domain D and let M be a multiplicative system in D such that $Q \cap M = \emptyset$. Let D_0 be a domain containing D such that $Q \cdot D_0 \cap D = Q$, and let $D_0^* = (D_0)_M$, $D^* = D_M$, $Q^* = D_M \cdot Q$. Then $D_0^* \cdot Q^* \cap D^* = Q^*$.*

Proof. $Q^* \subseteq D_0^* \cdot Q^* \cap D^*$ is clear. Suppose then $x \in D_0^* \cdot Q^* \cap D^* = D_0^* \cdot Q \cap D^*$,

$$x = t/m = r/s, \qquad t \in D_0 Q,\ m,\ s \in M,\ r \in D.$$

Therefore, $st = rm$.

But $st \in D_0 \cdot Q$ and $rm \in D$, so $rm \in D_0 \cdot Q \cap D = Q$. Since $Q \cap M = \varnothing$, $m \in M$ implies $m \notin Q$. Thus $r \in Q$ and $x = r/s \in Q^*$. q.e.d.

2.6. Corollary. *Let Q, D, and M be as above. If Q is a valuation ideal, then $Q^* = Q \cdot D_M$ is also a valuation ideal.*

Proof. If D_v is a valuation ring such that $D_v \cdot Q \cap D = Q$, and if $D_v^* = (D_v)_M$, then $D_v^* \cdot Q^* \cap D^* = Q^*$ by 2.5. Thus, Q^* is a valuation ideal. q.e.d.

2.7. Corollary. *Let D be a domain and let M be a multiplicative system in D. If every primary ideal Q of D such that $Q \cap M = \varnothing$ is a valuation ideal, then every primary ideal of D_M is a valuation ideal.*

Proof. Let Q^* be any primary ideal of D_M, and let $Q = Q^* \cap D$. Then $Q \cdot D_M = Q^*$ and Q is a primary ideal of D such that $Q \cap M = \varnothing$. Therefore, by 2.6, Q^* is a valuation ideal. q.e.d.

2.8. Lemma. *Let D be a domain, and let $A_1, \cdots, A_n$ be v-ideals of D (for a fixed v). If d_i is an element of D such that $d_i \notin A_i$, $i = 1, \cdots, n$, then $d = d_1 \cdots d_n \notin A_1 \cdots A_n$.*

Proof. Since $A_i D_v \cap D = A_i$, $d_i \notin A_i$ implies $d_i \notin A_i D_v$. Therefore, $v(d_i) < v(a_i)$ for all $a_i \in A_i D_v$. But then $v(d) = v(d_1 \cdots d_n) < v(a_1 \cdots a_n)$ for any $a_1, \cdots, a_n, a_i \in A_i D_v$. This means $d \notin A_1 D_v \cdots A_n D_v$; and since $A_1 \cdots A_n \subseteq A_1 D_v \cdots A_n D_v$, $d \notin A_1 \cdots A_n$.

2.9. Corollary. *Let A, B be ideals of the domain D, such that A is a valuation ideal and $B^n \subseteq A^n$. Then $B \subseteq A$.*

Proof. If $B \nsubseteq A$, then there exists $b \in B$, $b \notin A$. Therefore, by 2.8, $b^n \notin A^n$; so $B^n \nsubseteq A^n$. q.e.d.

2.10. Lemma. *Let D be a domain, and let A be an ideal of D such that A^n is a valuation ideal for all n. Then $B = \bigcap_{n=1}^{\infty} A^n$ is prime.*

Proof. $xy \in B$ *implies* $xy \in A^{2n} = (A^n)^2$, for all n. But then by 2.8, $x \in A^n$ or $y \in A^n$. Thus, $x \in B$ or $y \in B$. q.e.d.

2.11. Corollary. *Let Q be a primary ideal of a domain D, and suppose $Q^{(i)}$ is a valuation ideal for all i (where $Q^{(i)}$ denotes the ith symbolic power of Q). Then $A = \bigcap_{i=1}^{\infty} Q^{(i)}$ is prime.*

Proof. Let $P = \sqrt{Q}$. By applying 2.6 and well-known properties of quotient rings [7, p. 223], we may assume $D = D_P$ and hence that P is maximal and $Q^{(i)} = Q$. Now apply 2.10.

2.12. LEMMA. *Let P be a prime ideal of a valuation ring D, and let A be the intersection of the primary ideals belonging to P. Then A is prime, and there exists no prime ideal P_1 such that $A \subset P_1 \subset P$.*

Proof. There is no loss of generality in assuming $D = D_P$ so that D is quasi-local and P is maximal in D. If $A = P$, the lemma holds. If $A \subset P$ so that there exists a P-primary ideal $Q \subset P$, then given $x \in P$, $x \notin Q$, $Q \subset (x) \subseteq P$. Thus if Q_λ is any P-primary ideal of D, then $x^i \in Q_\lambda$ for some i so that $(x^i) \subseteq Q_\lambda$. Further, $\sqrt{(x^i)} = \sqrt{(x)} = P$, and thus (x^i) is P-primary. It follows that $A = \bigcap_{i=1}^{\infty}(x^i)$ is prime by 2.10. Further, if B is an ideal of D such that $A \subset B \subset P$, then $B \nsubseteq (x^n)$ for some n, so that $(x^n) \subset B$. Therefore $B \subset P = \sqrt{(x^n)} \subseteq \sqrt{B}$ and B is not prime. q.e.d.

2.13. LEMMA. *Let $\{A_\lambda\} = \mathscr{S}$ be a set of valuation ideals of a domain D, and suppose for any $A_1, A_2 \in \mathscr{S}$ there exists an $A_3 \in \mathscr{S}$ such that $A_3 \subseteq A_1 \cap A_2$. If $A = \bigcap A_\lambda$, then $\sqrt{A}$ is prime.*

Proof. $xy \in \sqrt{A}$ implies $(xy)^n \in A$ for some n. Then $x^n \cdot y^n \in A_\lambda$ for all λ; so by 2.3–(a), $x^{2n} \in A_\lambda$ or $y^{2n} \in A_\lambda$. If $x^{2n} \notin A_1$ and $y^{2n} \notin A_2$ for some $A_1, A_2 \in \mathscr{S}$, then there exists $A_3 \in \mathscr{S}$ such that $A_3 \subseteq A_1 \cap A_2$; and then $x^{2n} \notin A_3$, $y^{2n} \notin A_3$, a contradiction. Thus, we may assume $x^{2n} \in A_\lambda$ for all λ. But then $x^{2n} \in A$ and hence $x \in \sqrt{A}$. q.e.d.

2.14. PROPOSITION. *Let P be a prime ideal of a domain D, and let $\{Q_\lambda\}$ be the set of primary ideals belonging to P. If $A = \bigcap Q_\lambda$ and every Q_λ is a valuation ideal, then A is prime.*

Proof. Let Q be a primary ideal of D, and suppose D_v is a valuation ring such that $Q_v \cap D = Q$, where $Q_v = Q \cdot D_v$. If $P_v = \sqrt{Q_v}$, P_v is prime and $P_v \cap D = P$. (The radical of an ideal is the intersection of all prime ideals which contain it [5, p. 9]. The prime ideals of a valuation ring are linearly ordered so that every ideal of a valuation ring has prime radical.) Let P_v^* be the intersection of the P_v-primary ideals of D_v, and let $P^* = P_v^* \cap D$. P_v^* is prime by 2.12, so P^* is also prime. Then $A \subseteq P^* \subseteq Q$, and thus $\sqrt{A} \subseteq P^* \subseteq Q$. Since this is true for any P-primary ideal Q, $\sqrt{A} \subseteq A$ and hence $\sqrt{A} = A$.

If Q_1 and Q_2 are P-primary ideals, then $Q_3 = Q_1 \cap Q_2$ is also P-primary. Therefore, we may apply 2.13 to conclude $A = \sqrt{A}$ is prime. q.e.d.

Thus, if Q is a primary ideal of a domain D having $\sqrt{Q} = P$ and if $\{Q_\lambda\}$ is the set of primary ideals belonging to P, then both $A_1 = \bigcap Q^{(i)}$ and $A_2 = \bigcap Q_\lambda$ are prime provided every Q_λ is a valuation ideal. If $Q \subset P$, $A_2 \subseteq A_1 \subset P$. If D is a valuation ring, it is easily seen that $A_2 = A_1$; but we do not know if this is true in general. More important, we know of no case where there exists a prime ideal P_1 such that $A_2 \subset P_1 \subset P$, although it seems likely that this may happen.

3. **Relationships between $\mathscr{Q}$ and $\mathscr{V}$.** Let $\mathscr{Q}(D)$ be the set of primary ideals of the domain D and let $\mathscr{V}(D)$ be the set of valuation ideals of D. $\mathscr{V}(D)$ contains, in particular, all prime ideals of D [8, p. 12, Theorem 5]. When no confusion can result, we shall simply write $\mathscr{Q}$ and $\mathscr{V}$.

The next theorem characterizes domains D with the property that $\mathscr{V} \subseteq \mathscr{Q}$.

3.1. THEOREM. *$\mathscr{V} \subseteq \mathscr{Q}$ if and only if every proper prime ideal of D is maximal.*

Proof. Suppose every proper prime ideal of D is maximal, and let A be a valuation ideal. Then there exists a valuation ring $D_v \supseteq D$ and an ideal A_v of D_v such that $A_v \cap D = A$. If P is the center of D_v on D, then $D \subseteq D_P \subseteq D_v$, and D_P is a one-dimensional quasi-local ring. Therefore, $A_v \cap D_P = A'$ is primary; and since $A' \cap D = A$, A is also primary.

Conversely, assume $\mathscr{V} \subseteq \mathscr{Q}$, and suppose there exist prime ideals P, P' of D such that $0 \subset P \subset P' \subset D$. By [6, p. 37], there exists a valuation ring D_v having prime ideals P_v, P'_v which lie over P, P', respectively. Choose $x \in P'$, $x \notin P$ and $y \neq 0$ in P, and let $A = (xy) \cdot D_v \cap D$. Then A is a valuation ideal and $A \subseteq P$. Claim: A is not primary. For, if A is primary, $xy \in A$ and $x \notin P$ implies $y \in A$. But then $y = rxy$ for some $r \in D_v$, and hence $1 = rx \in P'_v$, a contradiction. q.e.d.

3.2. LEMMA. *Let M be a prime ideal of a domain D, and suppose there exists a prime ideal $P \subset M$ such that there is no prime ideal P_1 with $P \subset P_1 \subset M$. Then P is the intersection of the M-primary ideals of D which contain P.*

Proof. By passage to D_M/PD_M, it suffices to prove the theorem under the assumption that D is a one-dimensional quasi-local domain with maximal ideal M and $P = (0)$. The proof follows easily in this case since every nonzero ideal is M-primary, and the intersection of all nonzero ideals of D, an integral domain, is (0). q.e.d.

3.3. THEOREM. *Let M be a prime ideal of a domain D, and suppose every M-primary ideal is a valuation ideal. If there exists a prime ideal $P \subset M$ such that there is no prime ideal P_1 with $P \subset P_1 \subset M$, then P is unique (and is, in fact, the intersection of all M-primary ideals).*

Proof. Let P_0 be the intersection of the M-primary ideals. By 2.14 and 3.2, P_0 is prime and $\subset M$. We shall show $P \subseteq P_0$; it then follows that $P = P_0$ and hence P is unique. By 2.6 and the 1-1 correspondence between prime (primary) ideals of D contained in M and prime (primary) ideals of D_M, we may replace D by D_M and hence assume that D is quasi-local with maximal ideal M. Let then Q be any M-primary ideal of D, and we shall show $P \subseteq Q$.

Choose $x \in Q$, $x \notin P$ and set $A = QP + (x^4)$. Then $A \subseteq Q$ and $\sqrt{A} \supseteq (P, x) \supset P$; so $\sqrt{A} = M$, and hence A is M-primary. By hypothesis, A is then a valuation ideal, so there exists a valuation ring D_v and an ideal A_v of D_v such that $A_v \cap D = A$; and we may assume $A_v = AD_v$. Let also $P_v = P \cdot D_v$, $Q_v = Q \cdot D_v$. Claim: $x^2 \notin P_v$.

For, $x^2 \in P_v$ implies $x \cdot x^2 \in Q_v \cdot P_v \cap D \subseteq A$. Then $x^3 = s + dx^4$, $s \in Q \cdot P$, $d \in D$. Therefore, $(1 - dx)x^3 = s \in P$.

Since $1 - dx$ is a unit of D, this implies $x^3 \in P$ and hence $x \in P$, a contradiction. Therefore, $x^2 \notin P_v$.

Because D_v is a valuation ring, the ideals of D_v are linearly ordered; so $x^2 \notin P_v$ implies $P_v \subseteq x^2 \cdot D_v$. Therefore, $P_v^2 \subseteq (x^2 \cdot D_v) \cdot P_v$. But $P^2 + (x^2)$ is a valuation ideal, so

$$x \cdot P \subseteq P^2 + (x^2) \quad \text{by 2.3–(b).}$$

Therefore, $x \cdot P \subseteq P^2 + (x^2) \cdot P$ since $x \notin P$.

$$(x \cdot D_v) \cdot P_v \subseteq (P_v)^2 + (x^2 D_v) \cdot P_v = (x^2 D_v) \cdot P_v.$$

Thus, $(x \cdot D_v) \cdot P_v = (x^2 D_v) \cdot P_v$; and this implies

$$P_v = (xD_v) \cdot P_v = (x^2 \cdot D_v) \cdot P_v = (x^3 D_v) \cdot P_v = \cdots.$$

Therefore, $P_v \subseteq \bigcap_{i=1}^{\infty} (x^i D_v) = P_1$. P_1 is prime by 2.10, and $x \notin P_1$ implies $P_1 \cap D \subset M$. Therefore, $P \subseteq P_v \cap D \subseteq P_1 \cap D \subset M$, so by hypothesis, $P = P_1 \cap D$. This means both A and P are v-ideals *for the same* v. Since $A \nsubseteq P$, we must have $P = P_v \cap D \subseteq A_v \cap D = A$. Thus, $P \subseteq A \subseteq Q$. q.e.d.

A domain D is said to satisfy the *ascending chain condition* for prime ideals provided any strictly ascending chain of prime ideals $P_1 \subset P_2 \subset P_3 \subset \cdots$ is finite. This is equivalent to saying that every nonempty family of prime ideals contains a maximal element. The remainder of this section is devoted to proving that if D satisfies the a.c.c. for prime ideals and $\mathscr{Q} \subseteq \mathscr{V}$, then D is Prüfer.

3.4. Lemma. *Let D be a quasi-local domain, and suppose for any nonzero prime ideal P of D there exists a prime ideal $N(P) \subset P$ such that if P_1 is a prime ideal $\subset P$, then $P_1 \subseteq N(P)$. Then D satisfies the a.c.c. for prime ideals and the prime ideals of D are linearly ordered* (*and conversely*).

Proof. If $P_1 \subset P_2 \subseteq \cdots$ is an ascending chain of prime ideals of D, then $U = \bigcup P_i$ is also prime; so if $U \neq P_i$ for all i, then $P_i \subseteq N(U)$ for all i; and we would have $U = \bigcup P_i \subseteq N(U) \subset U$, a contradiction. Therefore, D satisfies the a.c.c. for prime ideals.

Now suppose there exist prime ideals P_1, P_2 of D such that $P_1 \nsubseteq P_2$ and $P_2 \nsubseteq P_1$. Since D has the a.c.c. for prime ideals, there exists a prime ideal M, maximal with respect to the properties $P_1 \subseteq M$, $P_2 \nsubseteq M$. Since $P_2 \nsubseteq M$, M is not the maximal ideal of D and there exists a prime ideal $M_\alpha \supset M$. If $\{M_\alpha\}$ is the set of all such prime ideals, then $M \neq \bigcap M_\alpha$ since $P_2 \subseteq \bigcap M_\alpha$ and $P_2 \nsubseteq M$. Therefore, by Zorn's lemma, there is a prime ideal M_0 minimal with respect to the property that $M_0 \supset M$. Therefore, $M \subseteq N(M_0) \subset M_0$ implies $M = N(M_0)$. But then $P_2 \subset M_0$ means $P_2 \subseteq N(M_0) = M$, a contradiction to the choice of M. q.e.d.

3.5. Corollary. *Let D be a quasi-local domain such that D satisfies the a.c.c. for prime ideals. If $\mathscr{Q} \subseteq \mathscr{V}$, then the prime ideals of D are linearly ordered.*

Proof. If P is any nonzero prime ideal of D, the set of all prime ideals $P_1 \subset P$ contains a maximal element $N(P)$, since D satisfies the a.c.c. for prime ideals. By 3.3 $N(P)$ is unique and hence contains every prime ideal $P_1 \subset P$. Therefore, by 3.4 the prime ideals of D are linearly ordered. q.e.d.

3.6. Lemma. *Let D be a quasi-local domain which satisfies the a.c.c. for prime ideals, and suppose $\mathscr{Q} \subseteq \mathscr{V}$. Then D is integrally closed.*

Proof. Let $S = \{x \in D \mid (x)' \cap D \supset (x)\}$ (where $'$ denotes completion). By 2.1–(d), $S = \varnothing$ if and only if D is integrally closed; so assume $S \neq \varnothing$. By 3.5, the prime ideals of D are linearly ordered; so there exists a least prime P containing S (i.e., P is the intersection of all prime ideals which contain S). Moreover, applying the a.c.c., there exists a prime ideal $P_0 \subset P$ such that there is no prime ideal P_1 with $P_0 \subset P_1 \subset P$. Since $S \nsubseteq P_0$, there exists $x \in S$, $x \notin P_0$; and then $\sqrt{(x)} = P$. $x \cdot D_P$ is primary and hence a valuation ideal by 2.6. Therefore, $(x \cdot D_P)' \cap D_P = x \cdot D_P$, by 2.1–(a). Thus, $y \in (x)' \cap D$ implies $y \in (x \cdot D_P)' \cap D_P = x \cdot D_P$. Since $x \in S$, there exists $y \in (x)' \cap D$ and $y \notin (x)$; so

$$y = (a/b) \cdot x, \qquad b \notin P,\ a \notin (b).$$

But $y \in (x)' = x \cdot D'$ implies $y/x \in D'$, where D' is the integral closure of D (see 2.1). Therefore, $a = y/x \cdot b \in b \cdot D' = (b)'$, so $a \notin (b)$ implies $b \in S$. But $S \subseteq P$ and $b \notin P$, a contradiction. q.e.d.

3.7. Lemma. *Let D be an integrally closed, quasi-local domain; and suppose x,y are nonzero elements of D such that $xy \in (x^2, y^2)$. Then x/y or y/x is in D*(3).

Proof. If x or y is a unit of D, we are done; so assume x, y are nonunits. Then $xy = d_1x^2 + d_2y^2$, $d_i \in D$. If d_i is a unit in D, the integral closure of D gives the proposition. Therefore, we may assume d_1, d_2 are nonunits also. Let K be the quotient field of D. By [8, p. 12, Theorem 5], there exists a valuation ring $D_v \subset K$ which dominates D. Then x/y or $y/x \in D_v$, say $x/y \in D_v$.

$$d_2y/x = d_2d_1 + (d_2y/x)^2$$

(3) This result can also be proved by putting together the proofs of [3, Theorem 2.5–(f)] and [4, p. 554, Satz 6].

so $d_2 y/x$ is integral over D and hence in D.

Therefore, $d_2 = (x/y) \cdot d$ for some $d \in D$.

$$v(d_2) = v(x/y) + v(d). \tag{1}$$

But $x/y(1 - d_1 x/y) = d_2$ implies

$$v(x/y) + v(1 - d_1 x/y) = v(d_2).$$

$1 - d_1 x/y$ is a unit of D_v since d_1 is a nonunit of D_v. Therefore, $v(1 - d_1 x/y) = 0$, so

$$v(x/y) = v(d_2). \tag{2}$$

Combining (1) and (2), we get

$$v(d) = 0.$$

Therefore, d is a unit in D. Thus

$$x/y = d_2 \cdot 1/d \in D. \quad \text{q.e.d.}$$

3.8. THEOREM. *Let D be a domain which satisfies the a.c.c. for prime ideals. If $\mathcal{Q} \subseteq \mathcal{V}$, then D is a Prüfer domain (and conversely*(4)*).*

Proof. By 2.2–(b) it is sufficient to see D_P is a valuation ring for any prime ideal P of D. Therefore, by 2.7 we may assume D is quasi-local, and by 3.6 and 2.1–(d) D is integrally closed. Suppose then there exist nonzero $x, y \in D$ such that x/y and $y/x \notin D$. x, y are then nonunits of D, so the fact that the prime ideals of D are linearly ordered (by 3.5) implies $\sqrt{(x, y)}$ is prime. Consider then the set $\mathcal{S}$ of all prime ideals of D which are of the form $\sqrt{(x, y)}$ for such x, y. By the a.c.c., $\mathcal{S}$ contains a maximal element P and suppose x, y are the elements of the above type such that $P = \sqrt{(x, y)}$. $(x^2, y^2) \cdot D_P$ is then primary and hence by 2.7 a valuation ideal. Therefore, by 2.3–(b), $xy \in (x^2, y^2) \cdot D_P$. Applying 3.7, we may assume $x/y \in D_P$. Then $x/y = r/s$, $r, s \in D$, $s \notin P$. But this means r/s, $s/r \notin D$, and $s \notin P$ implies $\sqrt{(r, s)} \supset P$, a contradiction to the choice of P. q.e.d.

3.9. COROLLARY. *A noetherian domain D has the property $\mathcal{Q} \subseteq \mathcal{V}$ if and only if D is a Dedekind domain.*

Proof. D is a Dedekind domain if and only if D is a noetherian Prüfer domain (use (a) of 2.2). Now apply 3.8.

3.10. COROLLARY. *Let D be a noetherian domain and let P be a prime ideal of D such that every P-primary ideal is a valuation ideal. Then P is a*

(4) The converse follows from the fact that D_P is a valuation ring and $Q \cdot D_P \cap D = Q$ for any prime ideal P of D and any P-primary ideal Q.

minimal prime of D and D_P is a rank 1, *discrete valuation ring* (*i.e.*, D_P *is a noetherian valuation ring*).

Proof. Let N be an ideal of D maximal with respect to the property that N is a prime ideal $\subset P$. Then by 3.3, N is the intersection of all P-primary ideals. Since D is noetherian, this intersection is (0) (for example, the intersection of the symbolic powers of P is (0) [7, p. 216, Corollary 1]). Therefore, $N=(0)$ and P is minimal. Also, D_P is a noetherian domain; and by 2.6, every primary ideal of D_P is a valuation ideal. Therefore, by 3.8, D_P is Prüfer and hence a valuation ring. q.e.d.

4. **Restricted $\mathscr{V}$.** We shall now deal with some special cases which occur when the set $\mathscr{V}$ is restricted.

4.1. Proposition. *Let D be a domain with a.c.c. for prime ideals. Then the following assertions are equivalent*:

(a) *There exists a finite set $D_{v_1},\cdots,D_{v_n}$ of valuation rings such that every primary ideal of D is a v_i-ideal for some i.*

(b) *D is a Prüfer domain with $\leqq n$ maximal ideals.*

Proof. (a) $\Rightarrow$ (*b*): D is a Prüfer domain by 3.8. If M is any maximal ideal of D, M is a v_i-ideal for some v_i and hence M is the center of D_{v_i} on D. There exist at most n such distinct centers.

(b) $\Rightarrow$ (a): Let $M_1,\cdots,M_t$, $t\leqq n$, be the maximal ideals of D. Since D is Prüfer, D_{M_i} is a valuation ring, and then $D_{M_1},\cdots,D_{M_t}$ are the required valuation rings. q.e.d.

4.2. Corollary. *Let D be a domain with a.c.c. for prime ideals, and suppose every primary ideal of D is a v-ideal for a fixed v. Then D is a valuation ring* (*and conversely*).

Proof. By 4.1, D is a Prüfer domain with one maximal ideal M, and hence $D=D_M$ is a valuation ring. q.e.d.

In §5 we shall construct an example which shows this corollary does not remain true when the a.c.c. hypothesis is dropped.

4.3. Theorem. *Let D be a domain and $M\supset N$ prime ideals of D such that M is a minimal prime of $N+A$ for some finitely generated ideal A and such that every M-primary ideal is a valuation ideal. Let P be the intersection of the M-primary ideals. Then P is a prime ideal such that $N\subseteq P\subset M$ and there exists no prime ideal P_1 with $P\subset P_1\subset M$.*

Proof. Since M is a minimal prime of $N+A$, M is not a union of prime ideals properly between N and M. Therefore, we can apply Zorn's lemma to conclude there exists a prime ideal P such that $N\subseteq P\subset M$ and there exists no prime ideal P_1 with $P \subset P_1 \subset M$. By 3.3, P is the intersection of all M-primary ideals. q.e.d.

4.4. Corollary. *Let D be a domain such that $\mathscr{Q} \subseteq \mathscr{V}$, and suppose for every prime ideal P of D there exists a valuation ring D_v of rank 1 such that P is a v-ideal. Then $\dim D \leqq 1$ and D is a Prüfer domain.*

Proof. Suppose there exist prime ideals $N \subset M$ in D. We shall show $N = (0)$. Choose $x \in M$, $x \notin N$; and let M_0 be a minimal prime of $N + (x)$. Then $N \subset M_0$; and by 4.3, $N \subseteq P$, where P is the intersection of the M_0-primary ideals. There exists a rank 1 valuation ring D_v such that $M_0 \cdot D_v \cap D = M_0$; so if M_v is the maximal ideal of D_v, then $M_v \cap D = M_0$. Therefore, every M_v-primary ideal of D_v contracts to an M_0-primary ideal of D. Since D_v has rank 1, the intersection of the M_v-primary ideals of D_v is (0). Thus the intersection P of the M_0-primary ideals of D is also (0). Therefore, $N \subseteq P = (0)$, so $\dim D = 1$. D is then Prüfer by 3.8. q.e.d.

It is now natural to make the following *conjecture*:

Let D be a domain such that $\mathscr{Q} \subseteq \mathscr{V}$, and suppose for every prime ideal P of D there exists a rank n valuation ring D_v such that P is a v-ideal. Then $\dim D \leqq n$.

We have been unable to determine whether this is true or not.

4.5. Corollary. *A domain D with quotient field K is almost Dedekind*(5) *if (and only if) $\mathscr{Q} \subseteq \mathscr{V}$ and for any prime ideal P of D there exists a rank 1, discrete valuation ring $D_v \subset K$ such that P is a v-ideal.*

Proof. Suppose there exists a proper prime ideal P of D. Then by 4.4 $\dim D = 1$ and D is a Prüfer domain. Therefore, D_P is a rank 1 valuation ring, and hence D_P is a maximal subring of K. But if P is a v-ideal, then $D_P \subseteq D_v \subset K$; so $D_P = D_v$. Therefore, D_P is rank 1, discrete and thus D is almost Dedekind. q.e.d.

5. **Counterexamples.** We saw in 3.8 that when D has the a.c.c. for prime ideals, $\mathscr{Q} \subseteq \mathscr{V}$ is equivalent to the assertion that D is a Prüfer domain. We construct in this section an example to show $\mathscr{Q} \subseteq \mathscr{V}$ does not necessarily imply D is Prüfer without the additional a.c.c. hypothesis.

5.1. Proposition. *Let D be a domain with quotient field K; let $A \neq (0)$ be an ideal of D; let D_0 and D_1 be subrings of D such that $D_0 \subseteq D_1 \subseteq D$. Let $S = D_0 + A$, $T = D_1 + A$. Then*

(a) *If Q is a primary ideal of S such that $\sqrt{Q} \subset A$, then Q is an ideal in D.*

(b) *If D is quasi-local and D_1 is a field, then T is quasi-local with maximal ideal A.*

(c) *If $D_0 \subset D_1$ and D_0, D_1 are fields, then S is not a valuation ring.*

(5) A domain D is said to be almost Dedekind if for every proper prime ideal P of D, D_P is a rank 1, discrete valuation ring [1].

Proof. (a) We shall show that for any $x \in Q$ and $d \in D$, $dx \in Q$. $dx \in A$ since $x \in A$ so $dx \in S$. Now choose $a \in A$, $a \notin \sqrt{Q}$. Then

$$a(dx) = (ad) \cdot x \in Q$$

since $ad \in A \subseteq S$ and $x \in Q$. But then $a \in S$, $dx \in S$, and $a \notin \sqrt{Q}$, $a(dx) \in Q$ implies $dx \in Q$.

(b) A is clearly a maximal ideal in T. Suppose $x + a \in T$, $x \neq 0$ in D_1, $a \in A$.

$$(x + a)^{-1} = x^{-1}(1 - a(x + a)^{-1})$$

$a(x + a)^{-1} \in A$ since $x + a$ is a unit of D. Therefore, $(x + a)^{-1} \in T$.

(c) Since D_0 is a field $\subset D_1$, there exists $y \in D_1$ such that $y, 1/y \notin D_0$. If $y \in S$, then $y = z + a$ for some $z \in D_0$, $a \in A$. But $y - z = a \in D_1$ implies $a = 0$ and hence $y = z \in D_0$, a contradiction. Therefore $y \notin S$, and similarly $1/y \notin S$; so S is not a valuation ring. q.e.d.

Let k_0 and k be fields with $k_0 \subset k$, and let $x_1, x_2, \cdots, x_n, \cdots$ be elements from an extension field of k such that $x_1, x_2, \cdots, x_n, \cdots$ are algebraically independent over k. There exists a valuation v of $K = k(x_1, x_2, \cdots, x_n, \cdots)$ over k such that $v(x_i) > v(x_{i+1}^m)$ for all i, m and such that $m_v = (x_1, x_2, \cdots, x_n, \cdots)$ is the maximal ideal of the valuation ring D_v(6). Let $T = k + m_v$ and $S = k_0 + m_v$. S is quasi-local with maximal ideal m_v by (b). S is not a valuation ring by (c), so $S = S_{m_v}$ is not a valuation ring and hence S is not Prüfer.

Claim: Every primary ideal Q of S is an ideal in D_v (and hence is a v-ideal).

Proof. If $\sqrt{Q} \subset m_v$, Q is an ideal of D_v by (a). On the other hand, if $\sqrt{Q} = m_v$, $x_{i+1}^m \in Q$ for every i and some $m(i)$. But $v(x_i/x_{i+1}^m) > 0$ implies $x_i/x_{i+1}^m \in m_v \subset S$. Therefore, $x_i \in x_{i+1}^m \cdot S \subseteq Q$. Since this is true for all i, $m_v \subseteq Q$; so $m_v = Q$ and Q is an ideal of D_v. q.e.d.

Thus, S is a domain such that every primary ideal is a valuation ideal (in fact, a v-ideal for a *fixed* v), but S is not a Prüfer domain. By choosing k algebraic over k_0, we see that S is not even integrally closed, since $k \cap S = k_0 \subset k$.

The following addition to Proposition 5.1 shows that whenever P is a prime ideal of S and $P \subset m_v$, then $S_P \supseteq D_v$ and hence every such S_P is a valuation ring:

(d) (5.1 *continued*). *If P is a prime ideal of S such that $P \subset A$, then $D \subseteq S_P$.*

Proof. Choose $x \in A$, $x \notin P$. Then $1/x \in S_P$, so for any $y \in D$, $yx \in A \subseteq S$ and $y = (yx) \cdot 1/x \in S_P$. Therefore, $D \subseteq S_P$. q.e.d.

Thus, the above example has the property that for any prime ideal P of S,

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(6) Such a v, having value group the weak direct sum of the integers ordered lexicographically, may be constructed as follows: Define $v(x_1^{r_1} \cdot x_2^{r_2} \cdot \cdots \cdot x_n^{r_n}) = (r_1, \cdots, r_n, 0, \cdots)$; $v(f(x))$ = minimum value of the power products occurring in $f(x)$, for any $f(x) \in k[x_1, \cdots, x_n, \cdots]$; and $v(\xi) = v(f) - v(g)$ for any $\xi = f/g \in k(x_1, \cdots, x_n, \cdots)$.

either S_P is a valuation ring or P is the only ideal having radical P (which indeed happens for $P = m_v$). It seems reasonable then to make the following *conjecture*:

If D is a domain such that $\mathscr{Q} \subseteq \mathscr{V}$, then for any prime ideal P of D either D_P is a valuation ring or P is the only ideal having radical P(7).

To show this conjecture is false, we must make some modifications in the example. Let then D_w be a valuation ring having quotient field k_0; let $Y, x_1, \cdots, x_n, \cdots$ be quantities algebraically independent over k_0; and let $k = k_0(Y)$ and $K = k(x_1, \cdots, x_n, \cdots)$. As before, there exists a valuation v of K over k such that the valuation ring D_v has maximal ideal $m_v \neq 0$ with the property that m_v is the only m_v-primary ideal of D_v, and moreover $D_v = k + m_v$(6) (see the preceding remarks). Let $S = D_w + m_v$.

The set M of polynomials $f(Y) \in D_w[Y]$ such that the coefficients of f generate all of D_w (i.e., at least one coefficient is a unit in D_w) is multiplicatively closed in $D_w[Y]$, and the quotient ring $(D_w[Y])_M$ will be denoted by $D_w(Y)$ (see [6, p. 17]). Let $T = D_w(Y) + m_v$.

5.2. LEMMA. *Let D_w be a valuation ring and let Y be a transcendental element over D_w. Then $D_w(Y)$ is also a valuation ring.*

Proof. Any element of the quotient field of $D_w(Y)$ has the form $f(Y)/g(Y), f, g \in D_w[Y]$. Let c be the element of least value in the set of (nonzero) coefficients of f and g. Then f/c or g/c is in the multiplicative system M so $f/g = (f/c)/(g/c)$ or g/f is in $D_w(Y)$. q.e.d.

Before proceeding with the example, we shall make some further additions to the proposition of 5.1. (5.1. continued):

(e) *If D_1 is the quotient field of D_0 and if T, D_0 are valuation rings, then S is a valuation ring also.*

(f) *If D is quasi-local with maximal ideal A and B is an ideal of S such that $B \nsubseteq A$, then $B = (B \cap D_0) + A$.*

(g) *If A_0 is an ideal of D_0 such that $A_0 \cdot D_1 \cap D_0 = A_0$ and if $D_1 \cap A = 0$, then $(A_0 + A) \cdot T \cap S = A_0 + A$.*

Proof.

(e) Since T is a valuation ring, we need only see that for any $y \neq 0$ in T, either $y \in S$ or $1/y \in S$. $A \subseteq S$, so we may assume $y \notin A$. Therefore, since A is the maximal ideal of T, y is a unit in T. If $y = x + a$, $x \neq 0$ in D_1, $a \in A$, then $1/y - 1/x = 1/x(-a/(x+a)) \in A$. But D_0 is a valuation ring with quotient field D_1, so $x \in D_0$ or $1/x \in D_0$. Thus, $y \in S$ or $1/y \in S$.

(f) If $x \in S$, $x \notin A$, then $1/x \in D$. Therefore, for any $a \in A$, $a/x \in A \subseteq S$; and then $a \in x \cdot S$. In particular, $B \nsubseteq A$ implies $A \subseteq B$. For any $b \in B$, $d_0 + a = b$ with $d_0 \in D_0$, $a \in A$. Therefore, $d_0 = b - a \in B$.

(7) Note that the converse is obviously true.

(g) $(A_0+A)\cdot(D_1+A)\cap(D_0+A)\subseteq(A_0\cdot D_1+A)\cap(D_0+A)\subseteq(A_0\cdot D_1\cap D_0)+A$ since $D_0\subseteq D_1$ and $D_1\cap A=0$.

But by hypothesis, $A_0\cdot D_1\cap D_0=A_0$; so we have $(A_0+A)\cdot(D_1+A)\cap(D_0+A)\subseteq A_0+A$. The opposite inclusion is obvious. q.e.d.

Continuing with the example, if Q is any primary ideal of S, we next show that Q is a valuation ideal.

Case 1. $Q\subseteq m_v$. If $\sqrt{Q}\subset m_v$, then Q is an ideal in D_v, by 5.1–(a), and hence Q is a v-ideal. If $\sqrt{Q}=m_v$, then $Q=m_v$. For, $\sqrt{Q}=m_v$ implies $\sqrt{(Q\cdot D_v)}=m_v$ and hence $Q\cdot D_v=m_v$ since m_v is the only m_v-primary ideal of D_v. But then for any $x,y\in m_v$, $xy=x(dq)$ for some $d\in D_v$, $q\in Q$. Therefore, $xy=(xd)q\in m_v\cdot Q\subseteq Q$, so $m_v^2\subseteq Q$. But $\sqrt{m_v^2}=m_v$ implies $m_v=m_v^2$, so $m_v\subseteq Q$ and thus $m_v=Q$.

Case 2. $Q\nsubseteq m_v$. By 5.1–(b), $Q=Q_w+m_v$, where Q_w is a primary ideal of D_w. $Q_w\cdot D_w(Y)\cap D_w=Q_w$, by [6, p. 18, (6.17)].

Therefore, $Q\cdot T\cap S=Q$, by 5.1–(f). Using the fact that $D_v=k+m_v$ is a valuation ring and $D_w(Y)$ is a valuation ring (by 5.2), we can apply 5.1–(e) to conclude that T is also a valuation ring. Thus, Q is a valuation ideal.

We have therefore proved that $\mathcal{Q}(S)\subseteq\mathscr{V}(S)$. Consider then the maximal ideal $P=m_w+m_v$ of S. If D_w is chosen to be, for example, rank 1, discrete, then $m_w^2\subset m_w$. Therefore, $Q=m_w^2+m_v$ is P-primary and $Q\subset P$. However, $S_P\subseteq S_{m_v}=k_0+m_v$; and as before, k_0+m_v is not a valuation ring by 5.1–(c). Therefore, S_P is not a valuation ring either. Thus S is a domain such that $Q(S)\subseteq\mathscr{V}(S)$, yet there exists a prime ideal P of S such that S has a P-primary ideal other than P and S_P is not a valuation ring.

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