Introduction. In this paper we shall study some properties of Pontryagin classes mod 3 in a way similar to that of Massey in the case of Stiefel-Whitney classes [1; 2]. In the case of Pontryagin classes mod q, where q denotes a prime number larger than 2, Hirzebruch has established a relation similar to that of Wu concerning Stiefel-Whitney classes [3;4]. In the case q = 3 the relation takes a simple form and plays a basic role in this paper. After the submission of this paper the author learned that H. Roberts had dealt with a similar problem [6].

1. Let q be a prime number larger than 2 and let $X_n$ be a compact orientable differentiable $n$-manifold. We denote by $P_q^r$ the Steenrod power [5]

$$ P_q^r : H^i(X_n, \mathbb{Z}_q) \to H^{i+2r(q-1)}(X_n, \mathbb{Z}_q). $$

For $n = i + 2r(q-1)$ an element

$$ s_q^r \in H^{2r(q-1)}(X_n, \mathbb{Z}_q) $$

is characterized by the Poincaré duality theorem and the relation

$$ P_q^r v = s_q^r v \quad [3;4] $$

for all

$$ v \in H^{n-2r(q-1)}(X_n, \mathbb{Z}_q). $$

Let $\{L_i\}$ be the multiplicative sequence of polynomials corresponding to the power series

$$ \sum_{i=0}^{\infty} a_i z^i = \frac{\sqrt{z}}{tgh \sqrt{z}}. $$

It is known that

$$ s_q^r = q^r L_{r(q-1)/2}(p_1, \cdots) \mod q $$

where $p_i \in H^{4i}(X_n, \mathbb{Z})$ denotes the Pontryagin class and we put

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It is known that the mod $q$ 4$j$-cohomology classes

$$(1.8) \quad b_{q,j} = \sum \mathcal{P}_{q}^{i} s_{q}^{r},$$

where the sum is extended over all $i, r$ with

$$(1.9) \quad 2j = i(q - 1) + r(q - 1),$$
satisfy the relation

$$(1.10) \quad \sum_{j \geq 0} b_{3,j} = \prod_{i} (1 + \gamma_{i}^{l})$$

where

$$(1.11) \quad l = \frac{1}{2}(q - 1) \quad [4].$$

When $q = 3$ (1.10) takes the form

$$(1.12) \quad \sum_{j \geq 0} b_{3,j} = \prod_{i} (1 + \gamma_{i}^{l}) = \sum_{i \geq 0} p_{i} \mod 3.$$  

Hence we have from (1.8), (1.9) and (1.12)

$$(1.13) \quad p_{j} = \sum_{j = i + r} \mathcal{P}_{3}^{i} s_{3}^{r} \mod 3.$$  

Hereafter we shall deal exclusively with the case $q = 3$ and use the following notations:

$$(1.14) \quad \mathcal{P}_{3}^{i} = \mathcal{P}^{i} \quad \text{and} \quad s_{3}^{i} = s^{r}.$$  

The following arguments will be confined to $\mathbb{Z}_{3}$ and we shall not use the symbol $\mod 3$ except in important cases. We define a class $\sum_{j \geq 0} \tilde{s}^{j} (\tilde{s}^{j} \in H^{4j}(X_{n}, \mathbb{Z}_{3}))$ by

$$(1.15) \quad \sum_{i \geq 0} s^{i} \sum_{j \geq 0} \tilde{s}^{j} = 1.$$  

The dual Pontryagin classes $\sum_{j \geq 0} \tilde{p}_{j} (\tilde{p}_{j} \in H^{4j}(X_{n}, \mathbb{Z}))$ are defined by

$$(1.16) \quad \sum_{i \geq 0} p_{i} \sum_{j \geq 0} \tilde{p}_{j} = 1.$$  

As for the Steenrod powers we shall use the following relations:

$$
\begin{align*}
(i) \quad & \mathcal{P}^{0} = \text{identity} \quad [5], \\
(ii) \quad & \mathcal{P}(uv) = \sum_{s = 0}^{r} \mathcal{P}^{s} u \mathcal{P}^{r-s} v, \\
(iii) \quad & \mathcal{P}^{r} = \sum_{i} (-1)^{r+i} \mathcal{P}^{r-s} \mathcal{P}^{i} \quad (r < 3s), \\
(iv) \quad & \mathcal{P}^{j} u_{k} = 0, \quad j > k/2, \quad u_{k} \in H^{k}(X_{n}, \mathbb{Z}_{3}).
\end{align*}
$$
From (1.13), (1.15), (1.16) and (1.17)(i),(ii),(iii) we have

\[ \bar{p}_j = \sum_{i+r=j, i, r \geq 0} p^i \bar{s}^r \mod 3 \]

because

\[ 1 = \left( \sum_{i \geq 0} p^i \right) \left( \sum_{j \geq 0} s^j \right) = \left( \sum_{i \geq 0} p^i \right) \sum_{j, k \geq 0} s^j s^k \]

\[ = \sum_{i \geq 0} \sum_{j, k \geq 0} \sum_{q=0}^{l} \sum_{j, k \geq 0} p^{i-q} s^j p^q s^k - k = \sum_{r, j \geq 0} p^r \sum_{q, k \geq 0} p^q s^k = p \sum_{q, k \geq 0} p^q s^k. \]

2. Let us prove

**Lemma 1.** For any \( x \in H^{n-4k}(X, \mathbb{Z}_3) \) (\( 0 < k < n/4 \)), it holds that

\[ x \bar{p}_k = \sum_{r=1}^{k} (-p^r x) \bar{p}_{k-r} \mod 3. \]

**Proof.** We have from (1.18)

\[ x \bar{p}_k = \sum_{i+r=k} p^i \bar{s}^r = s^k + \sum_{i=1}^{k} s^i s^{k-i}. \]

On the other hand we have from (1.15)

\[ 0 = \bar{s}^k + \sum_{i=1}^{k} s^i s^{k-i}. \]

Hence we have from (2.2) and (2.3)

\[ \bar{p}_k = \sum_{i=1}^{k} (p^i \bar{s}^{k-i} - s^i s^{k-i}). \]

We have from (2.4)

\[ x \bar{p}_k = \sum_{i=1}^{k} (x p^i \bar{s}^{k-i} - x s^i s^{k-i}). \]

Meanwhile we have from (1.1) and (1.17)(ii)

\[ x s^i s^{k-i} = s^i (x s^{k-i}) = p^i (x s^{k-i}) = \sum_{r=0}^{i} p^r x p^{i-r} s^{k-i}. \]

Hence we have from (2.5), (2.6) and (1.18)

\[ x \bar{p}_k = \sum_{i=1}^{k} \sum_{r=1}^{i} (-p^r x p^{i-r} s^{k-i}) = \sum_{0 < r \leq i \leq k} (-p^r x) (p^{i-r} s^{k-i}) \]

\[ = \sum_{r=1}^{k} \left[ (-p^r x) \sum_{j=0}^{k-r} p^j s^{k-j-r} \right] = \sum_{r=1}^{k} (-p^r x) \bar{p}_{k-r}. \]

Q.E.D.

Applying the formula (2.1) many times we can express \( x \bar{p}_k \) as a product of \( x \) and a certain sum of iterated Steenrod powers:
3. Suppose that

\[ \tilde{p}_k \neq 0 \mod 3, \quad 0 < k < \frac{n}{4}. \]

Then the homomorphism

\[ x \rightarrow x\tilde{p}_k, \quad H^{n-4k}(X_n, Z_3) \rightarrow H^n(X_n, Z_3) \]

is not zero and takes the form (2.8). On the other hand any iterated Steenrod power \( \mathcal{P}^1 \cdots \mathcal{P}^r \) can be expressed as a sum of the admissible ones by virtue of (1.17)(iii):

\[ \mathcal{P}^I = \mathcal{P}^1 \cdots \mathcal{P}^r, \]

\[ n(I) = i_1 + \cdots + i_r = k, \]

\[ i_1 \geq 3i_2, \quad i_2 \geq 3i_3, \cdots, i_{r-1} \geq 3i_r > 0. \]

Hence we have

**Theorem 1.** Let \( X_n \) be a compact orientable differentiable n-manifold. If \( \tilde{p}_k \neq 0 \mod 3 \) \((0 < k < n/4)\), then it holds for some admissible iterated Steenrod power \( \mathcal{P}^I \) \((n(I)=k)\) and some nonzero \( x \in H^{n-4k}(X_n, Z_3) \) that \( \mathcal{P}^I x \neq 0 \mod 3. \)

We put as follows:

\[ i_1 = 3i_2 + \alpha_1, \quad i_2 = 3i_3 + \alpha_2, \cdots, i_{r-1} = 3i_r + \alpha_{r-1}, i_r = \alpha_r, \]

\[ e(I) = \alpha_1 + \cdots + \alpha_r. \]

Let us prove

**Lemma 2.** For any admissible iterated Steenrod power \( \mathcal{P}^I \) we have \( \mathcal{P}^I x = 0 \mod 3 \) provided that degree \( x < 2e(I) \).

**Proof.** We have from (3.4), (3.6) and (3.7)

\[ n(I) = e(I) + 3(i_2 + \cdots + i_r). \]

Hence we have

\[ i_1 - 2i_2 - \cdots - 2i_r = e(I) > \frac{1}{2}(\text{degree} x), \]

i.e.,

\[ 2i_1 - 4i_2 - \cdots - 4i_r > \text{degree} x. \]

Therefore we have

\[ 2i_1 > \text{degree} (\mathcal{P}^{i_2} \cdots \mathcal{P}^{i_r} x). \]

Hence we have from (1.17)(iv)
We put \( q = n - 4k \) and assume that \( \mathcal{P}^l \) is admissible and

\[
e(I) \leq \frac{q}{2}, \quad n(I) = k, \quad 1 < q < n.
\]

First we consider the case where \( e(I) = q/2 \). For any \( x \in H^q(X_\nu, \mathbb{Z}_3) \) we have

\[
\text{degree } \mathcal{P}^I x = 4n(I) + q = 6k_1 - 2e(I) + q
\]

\[
= 2(3\alpha_1 + 3^2\alpha_2 + \cdots + 3^r\alpha_r) - 2e(I) + q
\]

\[
= 2(3\alpha_1 + 3^2\alpha_2 + \cdots + 3^r\alpha_r).
\]

The last bracket of (3.14) consists of powers of 3 whose number is equal to

\[
\alpha_1 + \cdots + \alpha_r = e(I) = q/2.
\]

Hence we have from (3.14)

\[
n = \text{degree } \mathcal{P}^I x = 2(3\alpha_1 + 3^2\alpha_2 + \cdots + 3^r\alpha_r) + \alpha_0 + 1
\]

\[
= \{2(3\alpha_1 + 3^2\alpha_2 + \cdots + 3^r\alpha_r) + 3\alpha_0\} - 2\alpha_0 + 1.
\]

The number of powers of 3 in the last bracket of (3.18) is equal to \( \alpha_0 + 2e(I) = q - 1 \). Hence we have

\[
n = (3^{h_1} + 3^{h_2} + \cdots + 3^{h_{q/2}}) - 2\alpha_0 + 1
\]

where \( h_1 \geq h_2 \geq \cdots \geq h_{q/2} \geq 1 \) and \( 0 \leq \alpha_0 \leq q - 3 \).

In the case \( q = 1 \) it follows from (1.17)(iv) that

\[
\mathcal{P}^I x = 0 \mod 3.
\]

for any nonidentity \( \mathcal{P}^l \). Hence we have from Theorem 1

\[
\bar{p}_k = 0 \mod 3.
\]

The same thing holds for the case \( n = 4k \) by (1.3) and (2.4). Thus we have

**Theorem 2.** Let \( X_\nu \) be a compact orientable differentiable \( n \)-manifold. If \( \bar{p}_k \neq 0 \mod 3 \) and \( n > n - 4k > 1 \) we have either

\[
n = 2(3^{h_1} + 3^{h_2} + \cdots + 3^{h_{q/2}}) \quad (h_1 \geq h_2 \geq \cdots \geq h_{q/2} \geq 1, \quad q = n - 4k)
\]
or
\[ n = 3^{h_1} + 3^{h_2} + \cdots + 3^{h_{q-1}} - 2x_0 + 1 \quad (h_1 \geq h_2 \geq \cdots \geq h_{q-1} \geq 1, \quad q = n - 4k, \quad 0 \leq x_0 \leq q - 3). \]

If \( 0 \leq n - 4k \leq 1 \), then \( \bar{p}_k = 0 \mod 3 \).

4. We can derive various corollaries from Theorems 1 and 2. For example we consider the case where \( n = 14 \). If \( \bar{p}_3 \neq 0 \mod 3 \) we have from Theorem 2
\[ (4.1) \quad 14 = 2 \cdot 3^h, \quad h \geq 1, \]
and this is impossible. The second case of Theorem 2 is also impossible because \( q = 2 \). Hence we have
\[ (4.2) \quad \bar{p}_3 = 0 \mod 3. \]
Thus we have

**Corollary 1.** For any compact orientable differentiable 14-manifold \( X_{14} \)
\[ \bar{p}_3 = 0 \mod 3. \]

We can prove the above corollary otherwise. If \( \bar{p}_3 \neq 0 \mod 3 \) we have from Theorem 1
\[ \mathcal{P}^I_x \neq 0 \mod 3 \quad (n(I) = 3) \]
for some nonzero \( x \in H^2(X_{14}, \mathbb{Z}_3) \) and some admissible \( \mathcal{P}^I \). However, the only possible \( \mathcal{P}^I \) is \( \mathcal{P}^3 \) and we have
\[ \mathcal{P}^3_x = 0 \mod 3 \]
by (1.17)(iii). Hence we have \( \bar{p}_3 = 0 \mod 3 \). In such a way we see that in most cases \( \bar{p}_k = 0 \mod 3 \) provided that \( 4k \) is close enough to \( n \).

**References**


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