REDUCED PRODUCTS AND HORN CLASSES

BY
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In [10], Horn considered sentences, in a first order predicate logic with identity, of the following form:

$$Q \bigwedge_{i<n}(\Phi_{i0} \wedge \cdots \wedge \Phi_{im_i} \rightarrow \Psi_i),$$

where Q is a string of quantifiers and the Φ_{ij}, Ψ_i are atomic formulas. We shall call a sentence of the above form a *Horn sentence*, the class of all models of a Horn sentence a *Horn class*, and the class of models of an arbitrary sentence an *elementary class* (following [5] and [26]). Horn proved that any Horn class is closed under direct products. The converse question naturally arose: is every elementary class which is closed under direct products a Horn class? Chang and Morel in [5] obtained a negative answer to that question by showing that Horn classes are also closed under another product operation, and giving an example of a sentence which is closed under direct products but not under the other product operation. The problem of finding a simple syntactical characterization of elementary classes closed under direct products is still open, although some special cases have been solved(1).

In [7], an operation called the *reduced* (direct) *product* was introduced(2), which comprehends both the direct product and the other product considered in [5]. Roughly speaking, the reduced product of the indexed system $\langle \mathfrak{A}_i \rangle_{i \in I}$ of structures modulo the filter D on I is that homomorphic image of the direct product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ which is determined in a certain way by D (see §2). Chang showed that any Horn class is closed under reduced products, and conjectured the following converse: every elementary class which is closed under reduced products is a Horn class.

The purpose of this paper is to show that, assuming the continuum hypothesis, Chang's conjecture is correct. More precisely, we shall obtain the following results concerning reduced products.

A. *Suppose there is at least one infinite cardinal α such that $2^\alpha = \alpha^+$. Then an elementary class K is a Horn class if and only if it is closed under reduced products.*

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(1) See, for example, [1], [2], [20], [22], [24], and [25].

Added in proof. This problem was recently solved by J. Weinstein [29]. His characterization is rather complicated but appears to be as simple as one could hope for.

(2) Reduced products of the field of real numbers were considered in [9], and reduced products of arbitrary algebraic systems were introduced under the name *champ logique* in [19]. For additional references see [7].

B. *Assume the GCH (generalized continuum hypothesis). Every Horn sentence holding in all members of a class K holds in the structure* $\mathfrak{B}$ *if and only if some ultrapower of* $\mathfrak{B}$ *is a reduced product of some system of members of K.*

C. *Assume the GCH. K is a Horn class if and only if K is closed under reduced products and the complement of K is closed under ultraproducts.*

D. *Assume the GCH. K can be characterized by a set of Horn sentences if and only if K is closed under reduced products and the complement of K is closed under ultrapowers.*

The results B, C, and D differ from A in that they hold for arbitrary classes of structures K rather than only for elementary classes. The notion of an ultraproduct is a special case of a reduced product (see §2) which has been studied extensively elsewhere (for references see [17]). We shall also obtain some weaker versions of the result A without assuming the GCH. For example, provided that our logic is denumerable and effectively defined by a Gödel numbering we have:

E. *In Bernays-Gödel set theory, it can be proved that* Φ *is logically equivalent to a Horn sentence if and only if it can be proved that the class of all models of* Φ *is closed under reduced products.*

E is obtained from A by a simple application of Gödel's theorem on the consistency of the continuum hypothesis.

In §1 we give the basic definitions and a few general known results which we shall need. Most of the notions from the theory of models are taken from Tarski [26]. We also state a result of Morley and Vaught [23] concerning the existence of saturated models; their theorem is used to considerably simplify the proofs of our main results.

In §2 we introduce the notion of a Γ-product, where Γ is an arbitrary set of formulas. If Γ is the set of all atomic formulas, then the Γ-products are just the homomorphic images of direct products. It is shown that if Γ is the set of all Horn formulas, then the Γ-product coincides with the reduced product, and when Γ is the set of all formulas, then the Γ-product coincides with the ultraproduct. In §3 our main results are proved for Γ-products in general, and various corollaries and improvements of our main results are obtained in §4. The theorems A, B, C, and D above follow from the results of §3 and §4 by specializing Γ-products to reduced products. In §5 we obtain some consequences involving Γ-products which do not require the continuum hypothesis, and which include the result E above as a special case.

Many of the theorems obtained below were announced in [12], [14], and in the appendix of [17]. For the special case of ultraproducts, several of our results were proved in [17]. A different mathematical characterization of Horn classes, which does not depend on the continuum hypothesis, is stated in [11]. A generalization of our main result to many-valued logic has been announced by Chang in [3].

1. We shall refer to [17] for our basic set-theoretic and model-theoretic terminology. For convenience, however, we shall mention here a few of the more specialized items of notation from [17], along with some terminology which was not introduced there.

We distinguish between sets and proper classes, and always assume the axiom of choice.

The letters β, γ, ρ, λ, η will be used for ordinal numbers, and the letter α will be reserved for cardinal numbers. If X is a set, then $|X|$ denotes the power, or cardinality of X, $S(X)$ is the set of all subsets of X, $S_\alpha(X)$ is the set of all subsets of X of power $<\alpha$, and $S^\alpha(X)$ is the set of all $Y \subseteq X$ such that $X - Y \in S_\alpha(X)$.

Throughout this paper we shall assume that μ is an arbitrary but fixed sequence of natural numbers with domain $\mathbf{D}\mu = \rho$, that $\mathfrak{A} = \langle A, R_\lambda \rangle_{\lambda < \rho}$ and $\mathfrak{B} = \langle B, S_\lambda \rangle_{\lambda < \rho}$ are structures (i.e., relational systems) of type μ, and that K, M, N are classes of structures of type μ. If β is an ordinal, we let $\mu \oplus \beta$ be the sequence μ' with domain $\rho + \beta$ such that $\mu \subseteq \mu'$ and, for all $\gamma < \beta$, $\mu'(\gamma) = 1$. If $a \in A^\beta$, we denote by $(\mathfrak{A}, a)$ the structure $\langle A, R_\lambda \rangle_{\lambda < \rho + \beta}$ or type $\mu \oplus \beta$ such that, for each $\gamma < \beta$ and $b \in A$, $R_{\rho + \gamma}(b) = 1$ if and only if $b = a_\gamma$.

We consider the applied first order predicate logic $L(\mu)$ with variables $v_n, n < \omega$, a $\mu(\lambda)$-placed predicate symbol P_λ for each $\lambda < \rho$, and identity symbol $=$. We shall depart slightly from [17] by taking, for the propositional connectives of $L(\mu)$, the true formula $\mathfrak{t}$ and the false formula $\mathfrak{f}$ along with the usual connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. The quantifiers are $\exists$ and $\forall$. By an atomic formula of $L(\mu)$ we shall mean either a (well-formed) formula which has no connectives or quantifiers, or the formula $\mathfrak{t}$, or the formula $\mathfrak{f}$.

If a formula Φ of $L(\mu)$ is denoted by $\Phi(v_0, \cdots, v_{n-1})$, it is to be understood that every free variable of Φ occurs among $v_0, \cdots, v_{n-1}$; however, we do not require that all of $v_0, \cdots, v_{n-1}$ be free variables of Φ, or even occur at all in Φ. If $\Phi(v_0, \cdots, v_{n-1})$ is satisfied in $\mathfrak{A}$ by the elements $a_0, \cdots, a_{n-1} \in A$, we shall write

$$\mathfrak{A} \models \Phi[a_0, \cdots, a_{n-1}]. \tag{1}$$

$\Phi(v_0, \cdots, v_{n-1})$ is said to be *satisfiable in* $\mathfrak{A}$ if (1) holds for some $a_0, \cdots, a_{n-1} \in A$. A set Σ of formulas is said to be *satisfiable in* $\mathfrak{A}$ if there exists $a \in A^\omega$ such that, for each $n < \omega$ and $\Phi(v_0, \cdots, v_{n-1}) \in \Sigma$, (1) holds. If Σ is a sentence, or a set of sentences, and $\Phi(v_0, \cdots, v_{n-1})$ is a formula, we write

$$\Sigma \vdash \Phi\ (v_0, \cdots, v_{n-1})$$

if every model of Σ is a model of the universal closure of Φ. We write $\vdash \Phi$ in case $\mathfrak{t} \vdash \Phi$.

By $K \restriction \alpha$ we mean the class of all structures $\mathfrak{A} \in K$ of power $\leqq \alpha$. The theory of $\mathfrak{A}$, or of K, is denoted by $\mathrm{Th}(\mathfrak{A})$, or $\mathrm{Th}(K)$. (In [17] we wrote K^* instead of $\mathrm{Th}(K)$.) Thus the Löwenheim-Skolem Theorem may be stated in the following form:

If $\omega \leqq \alpha, |\mu| \leqq \alpha$, *and* $K \in EC_\Delta$, *then* $\mathrm{Th}(K \restriction \alpha) = \mathrm{Th}(K)$.

DEFINITION 1.1. *Let* $\mathfrak{A}$ *be a structure of power* α. $\mathfrak{A}$ *is said to be saturated if it has the following property*:

If $\beta < \alpha$ *and* $a \in A^\beta$, *if* Σ *is a set of formulas of* $L(\mu \oplus \beta)$ *each having only* v_0 *free, and if every finite subset of* Σ *is satisfiable in* $(\mathfrak{A}, a)$, *then* Σ *is satisfiable in* $(\mathfrak{A}, a)$.

In the above definition of saturated structures we follow Vaught [28]. Morley and Vaught in [23] study the two notions of homogeneous and of universal structures, which are based upon two more general notions of B. Jónsson. It is proved in [23], Theorem 3.4, that a structure is homogeneous and universal if and only if it is saturated. In [15] the name α-replete is used for a condition which is closely related to (but not equivalent to) being saturated and of power α. The following result is the main theorem of [23]; see Theorem 3.5 of that paper.

THEOREM 1.2. *Suppose* $|\mu| \leqq \alpha$, $\omega \leqq \alpha$, *and* $\alpha^+ = 2^\alpha$. *Then for any infinite structure* $\mathfrak{B}$ *of type* μ, *there is up to isomorphism exactly one saturated structure* $\mathfrak{A}$ *of power* α^+ *which is elementarily equivalent to* $\mathfrak{B}$.

We shall need the following set-theoretical lemma, which is proved in [17, p. 484].

LEMMA 1.3. *Let* α *be an infinite cardinal number and let* $\langle X_\beta \rangle_{\beta < \alpha}$ *be an* α*-termed sequence of sets* X_β *such that* $|X_\beta| \geqq \alpha$ *for each* $\beta < \alpha$. *Then there is a sequence* $\langle Y_\beta \rangle_{\beta < \alpha}$ *such that whenever* $\beta < \alpha$ *and* $\eta < \beta$ *we have* $Y_\eta \cap Y_\beta = 0$, $Y_\eta \subseteq X_\eta$, *and* $|Y_\eta| = \alpha$.

Another simple lemma which we shall need is the following.

LEMMA 1.4. *For any* K, *there is a cardinal* α *such that* $\mathrm{Th}(K \restriction \alpha) = \mathrm{Th}(K)$.

Proof. Let $\alpha_0 = \omega \cup |\mu|$. Then there are only α_0 formulas in $L(\mu)$. It follows that we may choose a subset $K_0 \subseteq K$ of power α_0 such that, for each sentence $\Phi \notin \mathrm{Th}(K)$, there exists $\mathfrak{B} \in K_0$ such that Φ does not hold in $\mathfrak{B}$. Then $\mathrm{Th}(K_0) = \mathrm{Th}(K)$. If we choose α such that $\alpha \geqq |B|$ for all $\mathfrak{B} \in K_0$, then we have $\mathrm{Th}(K \restriction \alpha) = \mathrm{Th}(K)$.

2. DEFINITION 2.1. *A set* Γ *of formulas of* $L(\mu)$ *is said to be a generalized atomic set (cf.* [16]) *if the following hold*:

(i) *if* $\Phi \in \Gamma$, $i, j < \omega$, v_j *does not occur bound in* Φ, *and* Ψ *is the formula obtained from* Φ *by replacing each free occurrence of* v_i *by* v_j, *then* $\Psi \in \Gamma$;

(ii) *if* $\Phi \in \Gamma$ *and* $\vdash \Phi \leftrightarrow \Psi$, *then* $\Psi \in \Gamma$;

(iii) $v_0 = v_1 \in \Gamma$;

(iv) $\mathfrak{f} \in \Gamma$.

DEFINITION 2.2. *We denote by* $\Lambda(\Gamma)$ *the least generalized atomic set* Γ' *in* $L(\mu)$ *such that*:

(i) $\Gamma \subseteq \Gamma'$;

(ii) *if* Φ, $\Psi \in \Gamma'$, *then* $\Phi \wedge \Psi \in \Gamma'$;

(iii) *if* $\Phi \in \Gamma'$ *and* $n < \omega$, *then* $\exists v_n \Phi \in \Gamma'$ *and* $\forall v_n \Phi \in \Gamma'$.

Let us assume throughout this paper that I is a nonempty set.

DEFINITION 2.3. *Let h be a function on $\prod_{i \in I} A_i$ onto B. $\mathfrak{B}$ is said to be a Γ-product of $\langle A_i \rangle_{i \in I}$ with respect to h if the following condition holds: whenever $\Phi(v_0, v_1, \cdots, v_n) \in \Gamma$ and $a_0, a_1, \cdots, a_n \in \prod_{i \in I} A_i$, if*

$$\mathfrak{A}_i \models \Phi[a_0(i), a_1(i), \cdots, a_n(i)]$$

for all $i \in I$, then

$$\mathfrak{B} \models \Phi[h(a_0), h(a_1), \cdots, h(a_n)].$$

$\mathfrak{B}$ is said to be a Γ-power of $\mathfrak{A}$ indexed by I if $\mathfrak{B}$ is a Γ-product of the sequence $\langle \mathfrak{A}_i \rangle_{i \in I}$ such that $\mathfrak{A}_i = \mathfrak{A}$ for all $i \in I$.

THEOREM 2.4.

(i) *If $\mathfrak{B}$ is a Γ-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h, and if $\Gamma_0 \subseteq \Gamma$, then $\mathfrak{B}$ is a Γ_0-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h.*

(ii) *$\mathfrak{B}$ is a Γ-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h if and only if it is a $\Lambda(\Gamma)$ product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h.*

(iii) *If $I \subseteq J$, then any Γ-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ is also a Γ-product of $\langle \mathfrak{A}_j \rangle_{j \in J}$.*

Proof. (i) is obvious. Let Γ_1 be the set of all formulas $\Phi(v_0, \cdots, v_n)$ such that, if $a_0, \cdots, a_n \in \prod_{i \in I} A_i$ and if $\mathfrak{A}_i \models \Phi[a_0(i), \cdots, a_n(i)]$ for all $i \in I$, then $\mathfrak{B} \models \Phi[h(a_0), \cdots, h(a_n)]$. It is easily seen that $\Gamma_1 = \Lambda(\Gamma_1)$, and also that $\mathfrak{B}$ is a Γ-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h if and only if $\Gamma \subseteq \Gamma_1$. From this we obtain (ii).

To prove (iii), let $\mathfrak{B}$ be a Γ-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h. Define the function h' on $\prod_{j \in J} A_j$ onto B by

$$h'(a) = h(a \cap (I \times \mathbf{\mathfrak{a}} a)) \text{ for each } a \in \prod_{j \in J} A_j.$$

Then obviously $\mathfrak{B}$ is a Γ-product of $\langle \mathfrak{A}_j \rangle_{j \in J}$ with respect to h'.

Let Γ_A be the set of all atomic formulas in $L(\mu)$.

THEOREM 2.5. *Conditions* (i) *and* (ii) *below are equivalent:*

(i) *$\mathfrak{B}$ is a Γ_A-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h.*

(ii) *h is a homomorphism on the direct product $\prod_{i \in I} \mathfrak{A}_i$ onto $\mathfrak{B}$.*

Proof. Since every formula $\Phi \in \Gamma_A$ is atomic, the result follows at once from the definitions involved.

DEFINITION 2.6. *Let Γ_H be the set of all formulas $(\Phi_0 \wedge \cdots \wedge \Phi_n) \rightarrow \Psi$ such that $\Phi_0, \cdots, \Phi_n, \Psi$ are atomic formulas in $L(\mu)$. Φ is said to be a Horn formula if $\Phi \in \Lambda(\Gamma_H)$. We shall write $K \in HC$ if K is characterized by a single Horn sentence, and $K \in HC_\Delta$ if K is characterized by some set of Horn sentences.*

DEFINITION 2.7. *Let D be a proper filter on I. $\mathfrak{B}$ is said to be a (proper) reduced product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ modulo D if there is a function h on $\prod_{i \in I} A_i$ onto B such that, for every atomic formula $\Phi(v_0, v_1, \cdots, v_n)$ and all $a_0, a_1, \cdots, a_n \in \prod_{i \in I} A_i$, we have*

$$\mathfrak{B} \models \Phi[h(a_0), \cdots, h(a_n)] \text{ if and only if } \{i \in I\colon \ \mathfrak{A}_i \models \Phi[a_0(i), \cdots, a_n(i)]\} \in D.$$

If D is an ultrafilter on I, then $\mathfrak{B}$ is said to be an ultraproduct of $\langle \mathfrak{A}_i \rangle_{i \in I}$ modulo D. If $\mathfrak{A}_i = \mathfrak{A}$ for all $i \in I$, then $\mathfrak{B}$ is said to be a reduced power, or ultrapower, respectively, of $\mathfrak{A}$ indexed by I modulo D.

Any structure $\mathfrak{B}$ is an ultrapower of itself (cf. [7]).

THEOREM 2.8. *For every $\langle \mathfrak{A}_i \rangle_{i \in I}$ and every filter D on I, there is, up to isomorphism, exactly one reduced product $\mathfrak{B}$ of $\langle \mathfrak{A}_i \rangle_{i \in I}$ modulo D.*

For the proof of the above theorem, and for a detailed discussion of reduced products, see [7]. The above Definition 2.7 is a slight departure from the usual definition of reduced products, e.g., as in [7] and [17]; usually *the* reduced product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ modulo D is defined to be the particular reduced product such that, for all $a \in \prod_{i \in I} A_i$,

$$h(a) = \{b \in \textstyle\prod_{i \in I} A_i\colon \ \{i \in I\colon \ a(i) = b(i)\} \in D\}.$$

Thus the reduced product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ modulo D in the sense of [7] is actually unique, rather than merely unique up to isomorphism.

We now establish the relationship between Γ-products and reduced products.

THEOREM 2.9. *The following two conditions are equivalent:*

(i) *$\mathfrak{B}$ is a reduced product of $\langle \mathfrak{A}_i \rangle_{i \in I}$;*

(ii) *$\mathfrak{B}$ is a $\Lambda(\Gamma_H)$-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$.*

Proof. The fact that (i) implies (ii) is due to C. C. Chang; for the proof, we refer to [7, Lemma 2.1].

Suppose $\mathfrak{B}$ is a $\Lambda(\Gamma_H)$-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h. Put $A = \prod_{i \in I} A_i$. For each atomic formula $\Phi(v_0, \cdots, v_n)$ and each $a_0, \cdots, a_n \in A$, let

$$J(\Phi, a_0, \cdots, a_n) = \{i \in I\colon \ \mathfrak{A}_i \models \Phi[a_0(i), \cdots, a_n(i)]\}.$$

Let E be the set of all sets $J(\Phi, a_0, \cdots, a_n)$ such that $n < \omega, \Phi$ is an atomic formula, $a_0, \cdots, a_n \in A$, and

$$\mathfrak{B} \models \Phi[h(a_0), \cdots, h(a_n)]. \tag{1}$$

Since

$$\Phi_0 \wedge \cdots \wedge \Phi_m \rightarrow \mathfrak{f} \in \Gamma_H$$

whenever $\Phi_0, \cdots, \Phi_m$ are atomic, and since $\mathfrak{B}$ is a Γ_H-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$, it follows that the intersection of any finite set of members of E is nonempty.

Therefore there exists a least (proper) filter D on I such that $E \subseteq D$. It suffices to prove, for each atomic formula $\Phi(v_0,\cdots,v_n)$ and $a_0,\cdots,a_n \in A$ such that $J(\Phi,a_0,\cdots,a_n) \in D$, that (1) holds. By the definition of D, there exist atomic formulas $\Psi_1(v_0,\cdots,v_{n+m}),\cdots,\Psi_p(v_0,\cdots,v_{n+m})$, and $a_{n+1},\cdots,a_{n+m} \in A$, such that

$$\bigcap_{q=1}^{p} J(\Psi_q,a_0,\cdots,a_{n+m}) \subseteq J(\Phi,a_0,\cdots,a_n), \tag{2}$$

and

$$\mathfrak{B} \models \Psi_q[h(a_0),\cdots,h(a_{n+m})] \quad \text{for } q = 1,\cdots,p. \tag{3}$$

By (2), for all $i \in I$ we have

$$\mathfrak{A}_i \models ((\Psi_1 \wedge \cdots \wedge \Psi_p) \to \Phi)[a_0(i),\cdots,a_{n+m}(i)].$$

Since

$$(\Psi_1 \wedge \cdots \wedge \Psi_p) \to \Phi \in \Gamma_H,$$

it follows that

$$\mathfrak{B} \models ((\Psi_1 \wedge \cdots \wedge \Psi_p) \to \Phi)\ [h(a_0),\cdots,h(a_{n+m})]. \tag{4}$$

From (3) and (4) we conclude that (1) holds, and our proof is complete.

THEOREM 2.10. *Let Γ_L be the set of all formulas of $L(\mu)$. Then the following are equivalent*:

(i) $\mathfrak{B}$ *is an ultraproduct of* $\langle \mathfrak{A}_i \rangle_{i \in I}$;

(ii) $\mathfrak{B}$ *is a* Γ_L*-product of* $\langle \mathfrak{A}_i \rangle_{i \in I}$.

Proof. For the proof that (i) implies (ii) we refer to [7, Lemma 2.1]. Assume (ii) and let us adopt the notation introduced in the proof of the preceding Theorem 2.9. Consider any atomic formulas

$$\Phi_1(v_0,\cdots,v_n),\cdots,\Phi_p(v_0,\cdots,v_n)$$

and elements $a_0,\cdots,a_n \in A$ such that, for $q = 1,\cdots,p$,

$$\mathfrak{B} \models \neg \Phi_q[h(a_0),\cdots,h(a_n)]. \tag{1}$$

For each $X \in D$, we must have

$$X \cap \bigcap_{q=1}^{p} \quad (I - J(\Phi_q,a_0,\cdots,a_n)) \neq 0. \tag{2}$$

Otherwise there would exist atomic formulas $\Psi_q(v_0,\cdots,v_{n+m})$, $q = 1,\cdots,r$, and elements $a_{n+1},\cdots,a_{n+m} \in A$ such that

$$\bigcap_{q=1}^{r} J(\Psi_q,a_0,\cdots,a_{n+m}) \subseteq X,$$

and hence, for all $i \in I$,

$$\mathfrak{A}_i \models ((\Psi_1 \wedge \cdots \wedge \Psi_r) \rightarrow (\Phi_1 \vee \cdots \vee \Phi_p))[a_0(i), \cdots, a_{n+m}(i)];$$

it would then follow that

$$\mathfrak{B} \models (\Psi_1 \wedge \cdots \wedge \Psi_r)[h(a_0), \cdots, h(a_{n+m})]$$

and

$$\mathfrak{B} \models ((\Psi_1 \wedge \cdots \wedge \Psi_r) \rightarrow (\Phi_1 \vee \cdots \vee \Phi_p))[h(a_0), \cdots, h(a_{n+m})],$$

contradicting (1). In view of (2), there exists an ultrafilter E on I such that $D \subseteq E$ and, whenever $\Phi(v_0, \cdots, v_n)$ is atomic, $a_0, \cdots, a_n \in A$, and

$$\mathfrak{B} \models \neg\Phi[h(a_0), \cdots, h(a_n)],$$

we have

$$I - J(\Phi, a_0, \cdots, a_n) \in E.$$

Then $\mathfrak{B}$ is an ultraproduct of $\langle \mathfrak{A}_i \rangle_{i \in I}$ modulo E, and (i) holds.

REMARK. Let K_0 be the class of all structures of type μ and of power $\geqq 2$; let Γ_J be the set of all formulas of the form

$$(\Phi_0 \wedge \cdots \wedge \Phi_n) \rightarrow (\Psi_0 \vee \Psi_1).$$

By a suitable modification of the proof of Theorem 2.10, it can be shown that, if each $\mathfrak{A}_i \in K_0$, then any $\Lambda(\Gamma_J)$-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ is an ultraproduct, and hence a Γ_L-product, of $\langle \mathfrak{A}_i \rangle_{i \in I}$. Then by Corollary 3.8 below it will follow that, for any formula Ψ, there exists a formula $\Phi \in \Lambda(\Gamma_J)$ such that

$$\mathrm{Th}(K_0) \vdash \Phi \leftrightarrow \Psi,$$

i.e., that

$$\exists v_0 v_1 (v_0 \neq v_1) \vdash \Phi \leftrightarrow \Psi.$$

This result can, however, be proved much more easily by a direct syntactical argument. For a syntactical proof of this and more general results see [4].

3. In view of Theorems 2.5, 2.9, and 2.10, results concerning Γ-products for arbitrary Γ always have corollaries concerning homomorphic images of direct products, reduced products, and ultraproducts. In this section we shall obtain some theorems concerning general Γ-products.

Throughout this section, we shall always make the following assumption:

3.0. *α is an infinite cardinal such that $|\mu| \leqq \alpha$ and $\alpha^+ = 2^\alpha$.*

THEOREM 3.1. *Let I be a set of power α, and suppose that:*

(i) $\Gamma = \Lambda(\Gamma)$;

(ii) *for each $i \in I$, $\mathfrak{A}_i$ is a structure of power $\leqq \alpha^+$;*

(iii) $\mathfrak{B}$ *is either a finite structure or a saturated structure of power* α^+;
(iv) *for every sentence* $\Phi \in \Gamma$, *we have*

$$\{i \in I : \mathfrak{A}_i \models \Phi\} \in S^\alpha(I) \textit{ implies } \mathfrak{B} \models \Phi.$$

Then $\mathfrak{B}$ *is a* Γ*-product of* $\langle \mathfrak{A}_i \rangle_{i \in I}$.

Proof. Let $A = \prod_{i \in I} A_i$. By 3.0, $|A| \leqq \alpha^+$. It suffices to show that there is a function h on A onto B such that, for every formula $\Phi(v_0, \cdots, v_n) \in \Gamma$ and every $a_0, \cdots, a_n \in A$, we have:

$$\{i \in I : \mathfrak{A}_i \models \Phi[a_0(i), \cdots, a_n(i)]\} \in S^\alpha(I)$$

implies

$$\mathfrak{B} \models \Phi[h(a_0), \cdots, h(a_n)].$$

By the well-ordering principle there are α^+-termed sequences c, d such that $\mathbf{G}c = A$ and $\mathbf{G}d = B$. Let F be the set of all sets $a \times b$ such that, for some $\beta < \alpha^+$, $a \in A^\beta$, $b \in B^\beta$, and the following three conditions are satisfied:

(1) if λ is a limit ordinal, $n < \omega$, and $\lambda + 2n < \beta$, then $a_{\lambda+2n} = c_{\lambda+n}$;
(2) if λ is a limit ordinal, $n < \omega$, and $\lambda + 2n + 1 < \beta$, then $b_{\lambda+2n+1} = d_{\lambda+n}$;
(3) for every formula $\Phi(v_0, \cdots, v_{m-1}) \in \Gamma$ and all $\gamma_0, \cdots, \gamma_{m-1} < \beta$, we have:

$$\{i \in I : \mathfrak{A}_i \models \Phi[a_{\gamma_0}(i), \cdots, a_{\gamma_{m-1}}(i)]\} \in S^\alpha(I)$$

implies

$$\mathfrak{B} \models \Phi[b_{\gamma_0}, \cdots, b_{\gamma_{m-1}}].$$

(We allow the possibility $m = 0$, in which case Φ must be a sentence.)

By the maximal principle, F includes a maximal chain G with respect to set inclusion. It is easily seen that $\bigcup G$ is of the form $a \times b$, where for some $\beta \leqq \alpha^+$, $a \in A^\beta$, $b \in B^\beta$, and the conditions (1) and (2) hold. In case $\beta = 0$, (3) coincides with one of the hypotheses of the theorem, and is therefore satisfied. If $\beta > 0$, then since $\beta = \bigcup\{\mathbf{D}e : e \times f \in G\}$, we have $\gamma_0, \cdots, \gamma_{m-1} \in \mathbf{D}e$ for some $e \times f \in G$ whenever $\gamma_0, \cdots, \gamma_{m-1} < \beta$; consequently (3) also holds when $\beta > 0$.

We shall show that $\beta = \alpha^+$. Suppose $\beta < \alpha^+$, and let λ be the largest limit ordinal $\leqq \beta$. We distinguish two cases: $\beta = \lambda + 2n$, and $\beta = \lambda + 2n + 1$, where $n < \omega$.

Suppose we have the first case, $\beta = \lambda + 2n$. Let $e = a \cup \{\langle \beta, c_{\lambda+n} \rangle\}$. Then $e \in A^{\beta+1}$ and condition (1) is satisfied with e, $\beta+1$ in place of α, β. Let H be the set of all finite sequences

$$\langle \Phi(v_0, v_1, \cdots, v_m), \gamma_1, \cdots, \gamma_m \rangle$$

such that $\Phi \in \Gamma$, $\gamma_1, \cdots, \gamma_m < \beta$, and

$$\{i \in I : \mathfrak{A}_i \models \Phi[e_\beta(i), e_{\gamma_1}(i), \cdots, e_{\gamma_m}(i)]\} \in S^\alpha(I).$$

Let Σ be the set of all formulas $\Psi(v_0)$ in $L(\mu \oplus \beta)$ such that, for some $\langle \Phi(v_0, v_1, \cdots, v_m), \gamma_1, \cdots, \gamma_m \rangle \in H$, we have

$$\Psi(v_0) = \exists v_1 \cdots v_m(P_{\gamma_1}(v_1) \wedge \cdots \wedge P_{\gamma_m}(v_m) \wedge \Phi(v_0, v_1, \cdots, v_m)).$$

Any finite conjunction of members of Σ is a consequence of some member of Σ, because $\Gamma = \Lambda(\Gamma)$ and $S^\alpha(I)$ is a filter on I. Furthermore, if $\Psi \in \Sigma$, then $\Psi(v_0)$ is satisfiable in $(\mathfrak{B}, b)$; to see this we observe that if

$$\langle \Phi(v_0, v_1, \cdots, v_m),\ \gamma_1, \cdots, \gamma_m \rangle \in H,$$

then $\exists v_0(\Phi(v_0, v_1, \cdots, v_m)) \in \Gamma$ because $\Gamma = \Lambda(\Gamma)$, and also

$$\{ i \in I:\ \mathfrak{A}_i \models \exists v_0 \Phi[a_{\gamma_1}(i), \cdots, a_{\gamma_m}(i)]\} \in S^\alpha(I),$$

and therefore by (3)

$$\mathfrak{B} \models \exists v_0 \Phi[b_{\gamma_1}, \cdots, b_{\gamma_m}].$$

We now apply our hypothesis that $\mathfrak{B}$ is either finite or saturated and of power $\alpha^+ > |\beta|$, and we conclude that, because $|\Sigma| < \alpha^+$, Σ is satisfiable in $(\mathfrak{B}, b)$. Choose an element $f_\beta \in B$ which satisfies Σ in $(\mathfrak{B}, b)$, and let $f = b \cup \{\langle \beta, f_\beta \rangle\}$. Using the fact that $\Gamma = \Lambda(\Gamma)$ is a GA set and in particular has the property 2.1(i), it may now easily be shown that (3) holds with $\beta + 1$, e, f in place of β, a, b. Thus we have $e \times f \in F$. Since $\bigcup G = a \times b$ is properly included in $e \times f$, $G \cup \{e \times f\}$ is a chain which properly includes G and is included in F. But this contradicts our choice of G as a maximal chain. Therefore β cannot be of the form $\lambda + 2n$.

We are left with the second case, $\beta = \lambda + 2n + 1$. In this case let $f = b \cup \{\langle \beta, d_{\lambda+n} \rangle\}$. Thus $f \in A^{\beta+1}$ and (2) is satisfied with $f, \beta + 1$ in place of d, β. Let H' be the set of all finite sequences

$$\langle \Phi(v_0, v_1, \cdots, v_m),\ \gamma_1, \cdots, \gamma_m \rangle$$

such that $\Phi \in \Gamma$, $\gamma_1, \cdots, \gamma_m < \beta$, and Φ is not satisfied in $\mathfrak{B}$ by $f_\beta, f_{\gamma_1}, \cdots, f_{\gamma_m}$. For each

$$s = \langle \Phi, \gamma_1, \cdots, \gamma_m \rangle \in H',$$

let

$$X(s) = I - \{ i \in I : \mathfrak{A}_i \models \forall v_0 \Phi[a_{\gamma_1}(i), \cdots, a_{\gamma_m}(i)]\}.$$

Then by (3) and by $\Lambda(\Gamma) = \Gamma$, we have $|X(s)| \geqq \alpha$ for each $s \in H'$. Furthermore, since $|\mu| \leqq \alpha$ and $|\beta| \leqq \alpha$, we have $|H'| \leqq \alpha$. Hence by Lemma 1.3, there exists a function $Y \in S(I)^{H'}$ such that, whenever $s, t \in H'$ and $s \neq t$, we have $Y(s) \cap Y(t) = 0$, $Y(s) \subseteq X(s)$, and $|Y(s)| = \alpha$. Choose $e_\beta \in A$ such that, for each $s \in H'$ and $i \in Y(s)$, Φ is not satisfied in $\mathfrak{A}_i$ by $\langle e_\beta(i), a_{\gamma_1}(i), \cdots, a_{\gamma_m}(i) \rangle$. Then, if we put $e = a \cup \{\langle \beta, e_\beta \rangle\}$, it is easily seen that $e \times f \in F$. As before, we have contradicted our assumption that G is a maximal chain in F.

We thus conclude that $\beta \geqq \alpha^+$, and hence $\beta = \alpha^+$. By (1) we have $\mathbf{\alpha} a = A$, and by (2) we have $\mathbf{\alpha} b = B$. Since $v_0 = v_1 \in \Gamma$, a function h on A onto B is defined by

$$h(a_\beta) = b_\beta \text{ for all } \beta < \alpha^+.$$

In view of (3), h has the desired property, and our proof is complete.

The proof of Theorem 3.1 follows closely the pattern of the proof of Theorem 2.2 in [17]; both of these proofs, and also the proof of 1.2, resemble the classical proof of Cantor that any two countable dense simply ordered systems are isomorphic.

It may be of interest to point out exactly where the continuum hypothesis $2^\alpha = \alpha^+$ was used in the preceding proof. First, we used $2^\alpha = \alpha^+$ to conclude that $|A| \leqq \alpha^+$. Consequently, when applying the maximal principle, it was sufficient to derive a contradiction from the assumption that $\beta < \alpha^+$. In Case 1, $\beta = \lambda + 2n$, we needed the assumption $\beta < \alpha^+$ to show that $|\Sigma| \leqq \alpha$, so that the saturatedness of the structure $\mathfrak{B}$ could be used. For a finite structure $\mathfrak{B}$ the assumption $\beta < \alpha^+$ was not needed at all in Case 1. In Case 2, $\beta = \lambda + 2n + 1$, the assumption $\beta < \alpha^+$ was again needed to show that $|H'| \leqq \alpha$, so that Lemma 1.3 could be applied.

Corollary 3.2. *Let K be a class of structures such that* $\mathrm{Th}(K) = \mathrm{Th}(K \restriction \alpha^+)$, *and let $\mathfrak{B}$ be either a finite structure or a saturated structure of power α^+. Then the following two conditions are equivalent*:

(i) $(\Lambda(\Gamma) \cap \mathrm{Th}(K)) \subseteq \mathrm{Th}(\mathfrak{B})$;

(ii) $\mathfrak{B}$ *is a Γ-product of some sequence $\langle \mathfrak{A}_\beta \rangle_{\beta < \alpha}$ of members of K.*

Proof. It is obvious that (ii) implies (i).

Assume (i). Let Σ be the set of all sentences of $L(\mu)$ which are consistent with $\mathrm{Th}(K)$. We must first verify that Σ is nonempty. Suppose Σ were empty. Then in particular $\mathfrak{t} \notin \Sigma$, that is, $\mathfrak{t}$ is not consistent with $\mathrm{Th}(K)$. But this means that $K = 0$, and thus $\mathfrak{f} \in \mathrm{Th}(K)$. Since $\mathfrak{f}$ is also in $\Lambda(\Gamma)$, we have $\mathfrak{f} \in \mathrm{Th}(\mathfrak{B})$, which is impossible. Therefore $\Sigma \neq 0$.

It follows from 3.0 that Σ has power $\leqq \alpha$. For each $\Phi \in \Sigma$, choose a model $\mathfrak{A}_\Phi$ of Φ which belongs to $K \restriction \alpha^+$. Let $I = \Sigma \times \alpha$, and for each $i = \langle \Phi, \beta \rangle \in I$, let $\mathfrak{A}_i = \mathfrak{A}_\Phi$. Then a sentence Φ has the property $\{ i \in I : \mathfrak{A}_i \models \Phi \} \in S^\alpha(I)$ if and only if $\Phi \in \mathrm{Th}(K)$. Therefore, whenever $\Phi \in \Lambda(\Gamma)$, we have:

$$\{ i \in I :\ \mathfrak{A}_i \models \Phi \} \in S^\alpha(I) \text{ implies } \mathfrak{B} \models \Phi.$$

Because $\Sigma \neq 0$, I is nonempty, and in fact $|I| = \alpha$. It follows from Theorem 3.1 that $\mathfrak{B}$ is a Γ-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$, and (ii) holds.

In the next corollary we point out a special case of 3.1 and 3.2 in which not all of the hypothesis 3.0 is needed.

Corollary 3.3. *Suppose $\mathfrak{B}$ is a finite structure. Then Theorem* 3.1 *and Corollary* 3.2 *are true even without the hypothesis* $2^\alpha = \alpha^+$.

Proof. The hypothesis $2^\alpha = \alpha^+$ is not used in our derivation of 3.2 from 3.1. We shall outline the way in which the proof of 3.1 can be modified so as to avoid using the hypothesis $2^\alpha = \alpha^+$. Let $A = \prod_{i \in I} A_i$, and let $\alpha_0 = \max(\alpha^+, |A|)$. We then carry through the proof of 3.1 but with the following changes:

(a) replace α^+ everywhere by α_0;
(b) choose d so that, for some finite n, $B = \{d_0, d_1, \cdots, d_n\}$;
(c) replace conditions (1) and (2) respectively by
(1′) if $\omega + \gamma < \beta$, then $a_{\omega+\gamma} = c_\gamma$,
(2′) if $n < \omega$ and $n < \beta$, then $b_n = d_n$.

The remainder of the proof of 3.1 carries over without difficulty, with Case 1 arising when $\beta \geqq \omega$ and Case 2 when $\beta < \omega$. Since we have avoided Case 2 when $\beta \geqq \alpha^+$, the argument is valid even if $\alpha_0 > \alpha^+$.

We shall say that K is *closed under* Γ*-products* if, whenever $\mathfrak{B}$ is a Γ-product of some system $\langle \mathfrak{A}_i \rangle_{i \in I}$ of members of K, then $\mathfrak{B} \in K$. Similarly, K is *closed under* α*-termed* Γ*-products* if, whenever $\mathfrak{B}$ is a Γ-product of some α-termed sequence $\langle \mathfrak{A}_\beta \rangle_{\beta < \alpha}$ of members of K, then $\mathfrak{B} \in K$. We use analogous terminology with respect to Γ-powers, reduced products and powers, and ultraproducts and ultrapowers. The next corollary tells which $K \in EC_\Delta$ are closed under Γ-products.

Corollary 3.4. *If $K \in EC_\Delta$, then the following four conditions are equivalent*:
(i) *$K \upharpoonright \alpha^+$ is closed under α-termed Γ-products*;
(ii) *K is closed under α-termed Γ-products*;
(iii) *K is closed under Γ-products*;
(iv) *K may be characterized by a set of sentences in $\Lambda(\Gamma)$.*

Proof. It is obvious that (iv) implies (iii), that (iii) implies (ii), and that (ii) implies (i). Let $\Delta = \mathrm{Th}(K) \cap \Lambda(\Gamma)$, and let $\mathfrak{A}$ be a model of Δ. If $\mathfrak{A}$ is infinite, then by Theorem 1.2, there exists a saturated structure $\mathfrak{B}$ of power α^+ which is elementarily equivalent to $\mathfrak{A}$; if $\mathfrak{A}$ is finite let $\mathfrak{B} = \mathfrak{A}$. In either case, $\mathfrak{B}$ is a model of Δ. Since $\omega \leqq \alpha$ and $|\mu| \leqq \alpha$, we have $\mathrm{Th}(K) = \mathrm{Th}(K \upharpoonright \alpha)$. Hence

$$(\mathrm{Th}(K \upharpoonright \alpha) \cap \Lambda(\Gamma)) \subseteq \mathrm{Th}(\mathfrak{B}).$$

By 3.2, $\mathfrak{B}$ is a Γ-product of some α-termed sequence of members of $K \upharpoonright \alpha$. It follows from (i) that $\mathfrak{B} \in K$, and therefore $\mathfrak{A} \in K$. We conclude that K is characterized by the set Δ of sentences, and hence (iv) holds.

The equivalence of 3.4(iii) and 3.4(ii) is analogous to the result of Vaught [6] that an elementary class is closed under finite direct products iff it is closed under arbitrary direct products.

Corollary 3.5. *If $K \in EC_\Delta$, then the following four conditions are equivalent*:
(i) *$K \upharpoonright \alpha^+$ is closed under α-termed Γ-powers*;
(ii) *K is closed under α-termed Γ-powers*;

(iii) *K is closed under Γ-powers;*

(iv) *K may be characterized by a set of sentences of the form $\Phi_1 \vee \Phi_2 \vee \cdots \vee \Phi_n$, where $\Phi_1, \cdots, \Phi_n \in \Lambda(\Gamma)$.*

Proof. Obviously (iv) implies (iii), (iii) implies (ii), and (ii) implies (i). Let Δ be the set of all sentences $\Phi \in \mathrm{Th}(K)$ which are finite disjunctions of members of $\Lambda(\Gamma)$, and let $\mathfrak{A}$ be a model of Δ. If $\mathfrak{A}$ is infinite, then by 1.2 there is a saturated model $\mathfrak{B}$ of $\mathrm{Th}(\mathfrak{A})$ of power α^+; if $\mathfrak{A}$ is finite let $\mathfrak{B} = \mathfrak{A}$. Let Σ be the set of all sentences of the form $\neg\Phi_1 \wedge \cdots \wedge \neg\Phi_n$, where $\Phi_1, \cdots, \Phi_n \in \Lambda(\Gamma)$, which hold in $\mathfrak{B}$. Each $\Psi \in \Sigma$ is consistent with K, for otherwise $\neg\Psi$ would be logically equivalent to some sentence in Δ, and hence $\neg\Psi$ would hold in $\mathfrak{B}$. Also, any conjunction of members of Σ belongs to Σ. Hence by the compactness and Löwenheim-Skolem theorems, Σ has a model $\mathfrak{A}_0 \in K$ of power $\leqq \alpha$. Clearly $\mathrm{Th}(\mathfrak{A}_0) \cap \Lambda(\Gamma) \subseteq \mathrm{Th}(\mathfrak{B})$. Therefore, by 3.2, $\mathfrak{B}$ is a Γ-power of $\mathfrak{A}_0$ indexed by α. By (i), $\mathfrak{B} \in K$, and hence $\mathfrak{A} \in K$. Consequently K is characterized by Δ, and (iv) holds.

COROLLARY 3.6(3). *If $K \in EC$, then*

(i) *K is closed under Γ-products if and only if it is characterized by a single sentence $\Phi \in \Lambda(\Gamma)$;*

(ii) *K is closed under Γ-powers if and only if it is characterized by a single sentence of the form $\Phi_1 \vee \cdots \vee \Phi_n$, where $\Phi_1, \cdots, \Phi_n \in \Lambda(\Gamma)$.*

Proof. In view of 3.4, it is sufficient for (i) to prove that if K is characterized by $\mathrm{Th}(K) \cap \Lambda(\Gamma)$, then K is characterized by a single sentence $\Phi \in \Lambda(\Gamma)$. Let the sentence Ψ characterize K. Then by the compactness theorem, Ψ is logically equivalent to the conjunction of some finite subset of $\mathrm{Th}(K) \cap \Lambda(\Gamma)$, and hence to a single sentence $\Phi \in \mathrm{Th}(K) \cap \Lambda(\Gamma)$.

The proof of (ii) is similar, using 3.5.

We shall now obtain results which compare Γ_1-products and Γ_2-products where Γ_1 and Γ_2 are different sets of formulas.

THEOREM 3.7. *Suppose* $\mathrm{Th}(K) = \mathrm{Th}(K \restriction \alpha^+)$ *and $M \in EC_\Delta$. Then* (i) *and* (ii) *below are equivalent:*

(i) *whenever $\{\mathfrak{A}_i : i \in I\} \subseteq K$, $\mathfrak{B} \in M$, and $\mathfrak{B}$ is a Γ_1-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h, then $\mathfrak{B}$ is also a Γ_2-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h;*

(ii) *for every $\Psi \in \Gamma_2$, there exists $\Phi \in \Lambda(\Gamma_1)$ such that* $\mathrm{Th}(K) \vdash \Psi \rightarrow \Phi$ *and* $\mathrm{Th}(M) \vdash \Phi \rightarrow \Psi$.

Proof. Assume (ii) and suppose $\mathfrak{B} \in M$, $\{\mathfrak{A}_i : i \in I\} \subseteq K$, and $\mathfrak{B}$ is a Γ_1-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h. Let $\Psi(v_0, v_1, \cdots, v_n) \in \Gamma_2$, let $a_0, a_1, \cdots, a_n \in \prod_{i \in I} A_i$, and suppose that

(3) The hypothesis $|\mu| \leqq \alpha$, which we assumed as a part of condition 3.0, is not needed for Corollary 3.6. This follows easily from the fact that only finitely many predicate symbols occur in a sentence which characterizes the elementary class K.

$$\mathfrak{A}_i \models \Psi[a_0(i), a_1(i), \cdots, a_n(i)]$$

holds for all $i \in I$. By (ii) there exists $\Phi \in \Lambda(\Gamma_1)$ such that $\mathrm{Th}(K) \vdash \Psi \to \Phi$ and $\mathrm{Th}(M) \vdash \Phi \to \Psi$. Clearly such a Φ may be found all of whose free variables are also free in Ψ. Then

$$\mathfrak{A}_i \models \Phi[a_0(i), a_1(i), \cdots, a_n(i)] \text{ for all } i \in I.$$

It follows that

$$\mathfrak{B} \models \Phi[f(a_0), f(a_1), \cdots, f(a_n)],$$

since by 2.4 $\mathfrak{B}$ is a $\Lambda(\Gamma_1)$-product. Since $\mathfrak{B} \in M$ and $\Phi \to \Psi \in \mathrm{Th}(M)$, we have

$$\mathfrak{B} \models \Psi[h(a_0), h(a_1), \cdots, h(a_n)].$$

Consequently $\mathfrak{B}$ is a Γ_2-product of $\langle \mathfrak{A}_i \rangle_{i \in I}$ with respect to h, and (i) holds.

Assuming (i), we consider a formula $\Psi(v_0, \cdots, v_n) \in \Gamma_2$. Let

$$\Delta = \{\Phi \in \Lambda(\Gamma_1)\colon \mathrm{Th}(K) \vdash \Psi \to \Phi\}.$$

Clearly $\Phi_1, \Phi_2 \in \Delta$ implies $\Phi_1 \wedge \Phi_2 \in \Delta$. We shall show that

(1) $$\mathrm{Th}(M) \cup \Delta \vdash \Psi.$$

Once (1) is established, it then will follow by the compactness theorem that there exists a $\Phi \in \Delta$ such that $\mathrm{Th}(M) \vdash \Phi \to \Psi$, and thus that (ii) holds. Suppose that $\mathfrak{A}^0 \in M$, $a^0 \in A^{0\omega}$, and a^0 satisfies Δ in $\mathfrak{A}^0$. In order to complete the proof of the theorem it suffices to establish

(2) $$\mathfrak{A}^0 \models \Psi[a_0^0, \cdots, a_n^0],$$

for in view of the fact that $M \in EC_\Delta$, (1) will then be verified.

It is convenient to consider the logic $L(\mu \oplus \omega)$ instead of L. For each set N of structures of type μ, let

$$N_\omega = \{(\mathfrak{A}, a)\colon \mathfrak{A} \in N \text{ and } a \in A^\omega\};$$

for each formula $\Phi(v_0, \cdots, v_n)$ in $L(\mu)$ such that v_n actually occurs free in Φ, let Φ_ω be the sentence

$$\exists v_0 \cdots v_n (P_\rho(v_0) \wedge \cdots \wedge P_{\rho+n}(v_n) \wedge \Phi)$$

in $L(\mu \oplus \omega)$, and for each sentence Φ in $L(\mu)$, let $\Phi_\omega = \Phi$. Let Γ_ω be the least GA set in $L(\mu \oplus \omega)$ such that $\Gamma_1 \subseteq \Gamma_\omega$ and, for each $n < \omega$, $P_{\rho+n}(v_0) \in \Gamma_\omega$. Let $\Delta_\omega = \{\Phi_\omega : \Phi \in \Delta\}$.

It is not difficult to show, by a simple induction, that for every sentence $\Theta \in \Lambda(\Gamma_\omega)$ there exists a formula $\Phi \in \Lambda(\Gamma_1)$ such that, for any structure $\mathfrak{A}$ of type μ and any $a \in A^\omega$, we have

$$(\mathfrak{A}, a) \models \Theta \leftrightarrow \Phi_\omega$$

It follows that for any $\mathfrak{A}$ and $a \in A^{\omega}, (\mathfrak{A}, a)$ is a model of Δ_{ω} if and only if it is a model of

$$\Lambda(\Gamma_{\omega}) \cap (\mathrm{Th}(K_{\omega}) \cup \{\Psi_{\omega}\}).$$

In particular, $(\mathfrak{A}^0, a^0)$ is a model of

$$\Lambda(\Gamma_{\omega}) \cap (\mathrm{Th}(K_{\omega}) \cup \{\Psi_{\omega}\}).$$

By Theorem 1.2, there is a structure $\mathfrak{B}'$ of type $\mu \oplus \omega$ which is elementarily equivalent to $(\mathfrak{A}^0, a^0)$, and is either finite or saturated and of power α^+. Since M is elementarily closed, there is a structure $\mathfrak{B} \in M$ and a sequence $b \in B^{\omega}$, such that $\mathfrak{B}' = (\mathfrak{B}, b)$. Also, if N is the class of $\mathfrak{A}' \in K_{\omega}$ such that $\mathfrak{A}' \models \Psi_{\omega}$, then $\mathfrak{B}'$ is a model of $\Lambda(\Gamma_{\omega}) \cap \mathrm{Th}(N)$. It follows from 3.2 that $\mathfrak{B}'$ is a Γ_{ω}-product of some sequence $\langle \mathfrak{A}'_{\beta} \rangle_{\beta < \alpha}$ of members of N, with respect to a function h. For each $\beta < \alpha$, we have $\mathfrak{A}'_{\beta} = (\mathfrak{A}_{\beta}, a(\beta))$ for some $\mathfrak{A}_{\beta} \in K$ and $a(\beta) \in A_{\beta}^{\omega}$ such that

$$\mathfrak{A}_{\beta} \models \Psi[a(\beta)_0, \cdots, a(\beta)_n].$$

For each $n < \omega$, let a_n be the element of $\prod_{\beta < \alpha} A_{\beta}$ such that, for each $\beta < \alpha$,

$$a_n(\beta) = a(\beta)_n.$$

Then for each $n < \omega$, we have $h(a_n) = b_n$, because $P_{\rho+n}(v_0) \in \Gamma_{\omega}$, $\mathfrak{B}' \models P_{\rho+n}(v_0)[b_n]$, and

$$\mathfrak{A}'_{\beta} \models P_{\rho+n}(v_0)[a_n(\beta)], \text{ for all } \beta < \alpha.$$

Moreover, $\mathfrak{B}$ is a Γ_1-product of $\langle \mathfrak{A}_{\beta} \rangle_{\beta < \alpha}$ with respect to h. Since $\mathfrak{B} \in M$, we may apply (i), and we conclude that $\mathfrak{B}$ is also a Γ_2-product of $\langle \mathfrak{A}_{\beta} \rangle_{\beta < \alpha}$ with respect to h. Since $\mathfrak{A}'_{\beta} \models \Psi_{\omega}$ for all $\beta < \alpha$, we have $\mathfrak{A}_{\beta} \models \Psi[a_0(\beta), \cdots, a_n(\beta)]$ for all $\beta < \alpha$, and hence, since $\Psi \in \Gamma_2$, we have $\mathfrak{B} \models \Psi[b_0, \cdots, b_n]$. Finally, since $\mathfrak{B}'$ is elementarily equivalent to $(\mathfrak{A}^0, a^0)$, it follows that condition (2) holds, and our proof is complete.

Corollary 3.8. *If* $K \in EC_{\Delta}$, *then* (i) *and* (ii) *below are equivalent*:

(i) *whenever* $\{\mathfrak{A}_i : i \in I\} \subseteq K$, $\mathfrak{B} \in K$, *and* $\mathfrak{B}$ *is a* Γ_1-*product of* $\langle \mathfrak{A}_i \rangle_{i \in I}$ *with respect to h, then* $\mathfrak{B}$ *is also a* Γ_2-*product of* $\langle \mathfrak{A}_i \rangle_{i \in I}$ *with respect to h*;

(ii) *for all* $\Psi \in \Gamma_2$ *there exists* $\Phi \in \Lambda(\Gamma_1)$ *such that* $\mathrm{Th}(K) \vdash \Phi \leftrightarrow \Psi$.

Proof. By Theorem 3.7 with $K = M$.

4. As we have pointed out at the beginning of §3, most of the results of §3 have as special cases results concerning homomorphic images of direct products, reduced products, and ultraproducts. We shall state explicitly only four of these special cases; the first three concern reduced products and Horn sentences, and the fourth is a result concerning ultraproducts which is useful for further applications.

In this section we shall continue to assume the hypothesis 3.0 *at all times.*

Corollary 4.1. *If $K \in EC_\Delta$, then the following four conditions are equivalent*:
(i) $K \upharpoonright \alpha^+$ *is closed under α-termed reduced products*;
(ii) K *is closed under α-termed reduced products*;
(iii) K *is closed under reduced products*;
(iv) $K \in HC_\Delta$.

Proof. By 2.9 and 3.4.

Remark. Corollary 4.1 above has an analogue for homomorphic images of direct products, but a much better result than that analogue follows easily from the literature. Thus Lyndon [21] showed that a $K \in EC_\Delta$ is preserved under homomorphic images if and only if K can be characterized by a set of positive sentences; moreover, it can be seen from the proof of Theorem 2 in Bing [2] that if K is characterized by a set of positive sentences and K is closed under 2-termed direct products, then K can be characterized by a set of positive Horn sentences, i.e., sentences in $\Lambda(\Gamma_A)$. Hence, even without the continuum hypothesis, one can conclude that the following three conditions are equivalent if $K \in EC_\Delta$:
(i) K is closed under homomorphic images and 2-termed direct products;
(ii) K is closed under homomorphic images and direct products;
(iii) K can be characterized by a set of sentences in $\Lambda(\Gamma_A)$.

Corollary 4.2. *If $K \in EC_\Delta$, then the following are equivalent*:
(i) $K \upharpoonright \alpha^+$ *is closed under α-termed reduced powers*;
(ii) K *is closed under α-termed reduced powers*;
(iii) K *is closed under reduced powers*;
(iv) K *may be characterized by a set of finite disjunctions of Horn sentences.*

Proof. By 2.9, 3.5.

Corollary 4.3(4). *Let $K \in EC$. Then $K \in HC$ if and only if K is closed under reduced products. Furthermore, K is a finite union of members of HC if and only if K is closed under reduced powers.*

Proof. By 2.9, 3.6.

Corollary 4.4. *Any infinite structure $\mathfrak{A}$ of power $\leq \alpha^+$ has a saturated ultrapower $\mathfrak{B}$ of power α^+.*

Proof. By 1.2, 2.10, and 3.2 with $K = \{\mathfrak{A}\}$.

We shall now require a stronger hypothesis than 3.0. In fact,

For the remainder of §4 *we shall assume the GCH.*

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(4) Corollary 4.3, like 3.6, does not require the hypothesis $|\mu| \leq \alpha$. (See footnote 3.)

COROLLARY 4.5. *Let K be an arbitrary class of structures, and let* $\mathfrak{B}$ *be any structure. Then the following two conditions are equivalent*:

(i) $(\Lambda(\Gamma) \cap \mathrm{Th}(K)) \subseteq \mathrm{Th}(\mathfrak{B})$;

(ii) *some ultrapower of* $\mathfrak{B}$ *is a* Γ*-product of some sequence of members of K.*

Proof. There exists a cardinal α which is sufficiently large that

$$\omega \leqq \alpha, \ |\mu| \leqq \alpha, \ |B| \leqq \alpha^+,$$

and $\mathrm{Th}(K) = \mathrm{Th}(K \restriction \alpha^+)$. The result then follows by 3.2 and 4.4.

COROLLARY 4.6. *Let K be an arbitrary class of structures. Then the following two conditions are equivalent*:

(i) $K \in EC_\Delta$ *and K is characterized by* $\mathrm{Th}(K) \cap \Lambda(\Gamma)$;

(ii) *K is closed under* Γ*-products and* $\bar{K}$ *is closed under ultrapowers.*

Proof. Obviously (i) implies (ii). If (ii) holds, then by 4.5 we have

$$K = \{\mathfrak{B} : (\Lambda(\Gamma) \cap \mathrm{Th}(K)) \subseteq \mathrm{Th}(\mathfrak{B})\},$$

and hence (i) holds.

COROLLARY 4.7. *Let K be an arbitrary class of structures. Then the following are equivalent*:

(i) $K \in EC$ *and K is characterized by a single sentence* $\Phi \in \Lambda(\Gamma)$;

(ii) *K is closed under* Γ*-products and* $\bar{K}$ *is closed under ultraproducts.*

Proof. Obviously (i) implies (ii).

Assume (ii). By 4.6, condition 4.6(i) holds. By 2.10, $\bar{K}$ is closed under Γ_L-products. Also, since any ultrapower is a Γ-product, K is closed under ultrapowers. Hence we may apply 4.6 with $\Gamma = \Gamma_L$ to conclude that $\bar{K} \in EC_\Delta$. Since both K and $\bar{K}$ are EC_Δ classes, it follows by a well-known argument using the compactness theorem that $K \in EC$ (see [26]), and so K is characterized by some sentence Ψ. Consequently

$$(\Lambda(\Gamma) \cap \mathrm{Th}(K)) \vdash \Psi.$$

By the compactness theorem, there exists $\Phi \in \Lambda(\Gamma) \cap \mathrm{Th}(K)$ such that $\Phi \vdash \Psi$. Since we also have $\Psi \vdash \Phi$, K is characterized by Φ, and (i) holds.

COROLLARY 4.8. *For an arbitrary class K of structures, the following are equivalent*:

(i) $K \in EC_\Delta$ *and K is characterized by a set of sentences of the form* $\Phi_1 \vee \cdots \vee \Phi_n$, *where* $\Phi_1, \cdots, \Phi_n \in \Lambda(\Gamma)$.

(ii) *K is closed under* Γ*-powers and ultraproducts, and* $\bar{K}$ *is closed under ultrapowers.*

Proof. Obviously (i) implies (ii). Assume (ii). By 2.10 and by 4.6 with $\Gamma = \Gamma_L$, we have $K \in EC_\Delta$. (i) now follows by 3.5.

COROLLARY 4.9. *For an arbitrary class K of structures,* (i) *and* (ii) *below are equivalent*:

(i) *$K \in EC$ and K is characterized by a single sentence of the form $\Phi_1 \vee \cdots \vee \Phi_n$, where $\Phi_1, \cdots, \Phi_n \in \Lambda(\Gamma)$;*

(ii) *K is closed under Γ-powers and ultraproducts, and $\bar{K}$ is closed under ultraproducts.*

Proof. Since any conjunction of sentences of the form

$$\Phi_1 \vee \cdots \vee \Phi_n, \quad \Phi_1, \cdots, \Phi_n \in \Lambda(\Gamma),$$

is also of that form, we may argue in the same way as in the proof of Corollary 4.7.

COROLLARY 4.10. *For arbitrary classes K, M of structures, the following are equivalent*:

(i) $(\mathrm{Th}(K) \cap \Lambda(\Gamma_1)) \cup (\mathrm{Th}(M) \cap \Lambda(\Gamma_2))$ *is consistent;*

(ii) *there exists a structure $\mathfrak{B}$ which is both a Γ_1-product of some sequence of members of K and a Γ_2-product of some sequence of members of M.*

Proof. If (ii) holds, then by 4.5 we have

$$(\mathrm{Th}(K) \cap \Lambda(\Gamma_1)) \cup (\mathrm{Th}(M) \cap \Lambda(\Gamma_2)) \subseteq \mathrm{Th}(\mathfrak{B}),$$

and hence (i) holds.

Assume (i), and let $\mathfrak{A}$ be a model of $(\mathrm{Th}(K) \cap \Lambda(\Gamma_1)) \cup (\mathrm{Th}(M) \cap \Lambda(\Gamma_2))$. Choose α sufficiently large that $\omega \leqq \alpha$, $|\mu| \leqq \alpha$, $\mathrm{Th}(K) = \mathrm{Th}(K \restriction \alpha^+)$, and $\mathrm{Th}(M) = \mathrm{Th}(M \restriction \alpha^+)$. By 1.2, there is a structure $\mathfrak{B}$ which is elementarily equivalent to $\mathfrak{A}$ and is either finite or is saturated and of power α^+. Then by 3.2, condition (ii) is satisfied by $\mathfrak{B}$.

COROLLARY 4.11. (*Separation principle*). *Suppose $K \cap M = 0$, K is closed under Γ_1-products, and M is closed under Γ_2-products. Then there exist classes K', M' such that*:

(i) $K \subseteq K'$, $M \subseteq M'$;

(ii) $K' \cap M' = 0$;

(iii) *$K' \in EC$ and K' is characterized by a single sentence $\Phi \in \Lambda(\Gamma_1)$;*

(iv) *$M' \in EC$ and M' is characterized by a single sentence $\Psi \in \Lambda(\Gamma_2)$.*

Proof. By 4.10, the set $(\mathrm{Th}(K) \cap \Lambda(\Gamma_1)) \cup (\mathrm{Th}(M) \cap \Lambda(\Gamma_2))$ is inconsistent. By the compactness theorem, there exists $\Phi \in \mathrm{Th}(K) \cap \Lambda(\Gamma_1)$ and $\Psi \in \mathrm{Th}(M) \cap \Lambda(\Gamma_2)$ such that Φ and Ψ, are inconsistent. If we now let K', M' be the classes of all models of Φ, Ψ respectively, then (i)–(iv) are satisfied.

We have in particular proved the results A, B, C, D, concerning reduced products which were stated in the introduction; see 4.3, 4.5, 4.6, and 4.7, respectively.

5. In this section we shall give an informal discussion of some metamathematical characterizations of Horn sentences involving reduced products which do not depend on the continuum hypothesis.

Let us assume that either $\rho < \omega$, or $\rho = \omega$ and μ is a recursive function. Suppose that some effective Gödel numbering of all the formulas of $L(\mu)$, and also a Gödel numbering of all proofs in $L(\mu)$, have been introduced.

In addition to the formal system $L(\mu)$, let us consider $L(\langle 2\rangle)$, which has just one binary predicate symbol. As is well known, the familiar "Bernays-Gödel" system of set theory as described in [8] (as well as various other familiar systems of set theory) may be formulated as a set of axioms in $L(\langle 2\rangle)$. Let us denote by (BG) the set of axioms introduced in [8], including the axiom of choice (modified to conform to the notation of $L(\langle 2\rangle)$, with P_0 for ε). In the usual manner the logic $L(\mu)$ may be formalized within (BG) by means of our Gödel numberings. Many intuitive statements about sets and about models of $L(\mu)$ may then be translated into formal expressions in $L(\langle 2\rangle)$ with respect to (BG); the formal expression which is the translation of an intuitive statement s will be denoted by $\ll s \gg$.

In [8] Gödel established the following famous result:

If (BG) *is consistent, so is* (BG) $\cup \{\ll 2^{\omega} = \omega^{+} \gg\}$.

From its proof it follows (cf. [18]) that

LEMMA. *Let X be a recursively enumerable subset of ω and let $n < \omega$. Then the following are equivalent*:

(i) (BG)$\vdash \ll n \in X \gg$;

(ii) (BG)$\vdash \ll 2^{\omega} = \omega^{+}$ *implies* $n \in X \gg$.

THEOREM 5.1. *Let Γ be a recursively enumerable generalized atomic set of formulas in $L(\mu)$ (thus the set of Gödel numbers of members of Γ is recursively enumerable). Then, for each sentence Φ in $L(\mu)$, the following three conditions are equivalent*:

(i) (BG)$\vdash \ll \Phi \in \Lambda(\Gamma) \gg$;

(ii) (BG)$\vdash \ll$ *The class of all models of Φ is closed under Γ-products*$\gg$;

(iii) (BG)$\vdash \ll 2^{\omega} = \omega^{+}$ *implies that the class of all models of Φ is closed under Γ-products*$\gg$.

Proof. Since μ is recursive and Γ is recursively enumerable, $\Lambda(\Gamma)$ is recursively enumerable.

Assume (i). By formalizing the proof of Theorem 2.4 (ii) in (BG), it can be shown that

(1) (BG)$\vdash \ll \Phi \in \Lambda(\Gamma)$ implies that the class of all models of Φ is closed under Γ-products$\gg$.

By (1) we have (ii).

Obviously, (iii) follows from (ii).

Finally, let us assume (iii). By formalizing the proof of Corollary 3.6(i) in (BG), it can be shown that

$$\text{(2)} \qquad (\mathrm{BG}) \vdash \ll \text{If } 2^{\omega} = \omega^{+} \text{ and the class of all models of } \Phi \text{ is closed under } \Gamma\text{-products, then } \Phi \in \Lambda(\Gamma) \gg .$$

From (iii) and (2), it follows that

$$(\mathrm{BG}) \vdash \ll 2^{\omega} = \omega^{+} \text{ implies } \Phi \in \Lambda(\Gamma) \gg .$$

Then, by the lemma, (i) holds. Our proof is complete.

We shall say that a model $\mathfrak{A} = \langle A, R\rangle$ of (BG) is *ω-standard* if it has the following property: for each $a \in A$, we have

$$\mathfrak{A} \models \ll v_0 \in \omega \gg [a]$$

if and only if there exists $n < \omega$ such that

$$\mathfrak{A} \models \ll v_0 = n \gg [a].$$

It follows from the compactness theorem that, if (BG) is consistent, then there exist models of (BG) which are not ω-standard.

THEOREM 5.2. *Assume the hypotheses of Theorem* 5.1 *and in addition that* $(\mathrm{BG}) \cup \{\ll 2^{\omega} = \omega^{+} \gg\}$ *has at least one ω-standard model. Then each of the conditions* 5.1(i)–(iii) *are equivalent to the following condition*:

(iv) *there exists an ω-standard model* $\mathfrak{A}$ *of* $(\mathrm{BG}) \cup \{\ll 2^{\omega} = \omega^{+} \gg\}$ *such that* $\mathfrak{A} \models \ll$ *the class of all models of* Φ *is closed under* Γ*-products*$\gg$.

Proof. If 5.1 (ii) holds, then any ω-standard model $\mathfrak{A}$ of

$$(\mathrm{BG}) \cup \{\ll 2^{\omega} = \omega^{+} \gg\}$$

obviously satisfies the requirement in (iv).

Assume that 5.1(i) fails, so that $\Phi \notin \Lambda(\Gamma)$, and let $\mathfrak{A}$ be any ω-standard model of $(\mathrm{BG}) \cup \{\ll 2^{\omega} = \omega^{+} \gg\}$. Since $\mathfrak{A}$ is ω-standard, we have

$$\mathfrak{A} \models \ll \Phi \notin \Lambda(\Gamma) \gg .$$

By formula (2) in the proof of 5.1, we have $(\mathrm{BG}) \vdash \ll$if $2^{\omega} = \omega^{+}$, and $\Phi \notin \Lambda(\Gamma)$, then the class of all models of Φ is not closed under Γ-products$\gg$.

Consequently, we have

$$\mathfrak{A} \models \ll \text{the class of all models of } \Phi \text{ is not closed under } \Gamma\text{-products} \gg .$$

Hence (iv) fails, and the proof is complete.

COROLLARY 5.3. *Theorems* 5.1 *and* 5.2 *remain valid if* "$\Phi \in \Lambda(\Gamma)$" *is everywhere replaced by* "Φ *is a Horn sentence,*" *and* "Γ-*products*" *is everywhere replaced by* "*reduced products.*"

Proof. One need only formalize the proof of Theorem 2.9 in (BG) and apply 5.1 and 5.2.

The analogues of Theorems 5.1 and 5.2 for Γ-powers and finite disjunctions of members of $\Lambda(\Gamma)$ may be proved in the same way. Corollary 5.3 above includes the result E stated in the introduction.

BIBLIOGRAPHY

1. K. I. Appel, *Horn sentences in identity theory*, J. Symbolic Logic **24** (1959), 306–310.

2. K. Bing, *On arithmetical classes not closed under direct union*, Proc. Amer. Math. Soc. **6** (1955), 836–846.

3. C. C. Chang, *Theory of models for infinite-valued logic*. IV, Notices Amer. Math. Soc. **8** (1961), 141.

4. C. C. Chang and H. J. Keisler, *An improved prenex normal form*, J. Symbolic Logic **27** (1962), 228–317.

5. C. C. Chang and A. C. Morel, *On closure under direct product*, J. Symbolic Logic **23** (1958), 149–154.

6. S. Feferman and R. Vaught, *The first order properties of products of algebraic systems*, Fund. Math. **47** (1959), 57–103.

7. T. Frayne, A. C. Morel and D. Scott, *Reduced direct products*, Fund. Math. **51** (1962/63), 195–228; errata, ibid. **53** (1963), 117.

8. K. Gödel, *The consistency of the continuum hypothesis*, Princeton, 1940.

9. E. Hewitt, *Rings of real-valued continuous functions*. I, Trans. Amer. Math. Soc. **64** (1948), 45–99.

10. A. Horn, *On sentences which are true of direct unions of algebras*, J. Symbolic Logic **16** (1951), 14–21.

11. H. J. Keisler, *A characterization of Horn classes*, Notices Amer. Math. Soc. **6** (1959), 521.

12. ———, *A type of product more general than reduced products, and elementary classes closed under them*, Notices Amer. Math. Soc. **7** (1960), 364.

13. ———, *Conditions for inclusion between certain classes of algebraic operations sending members of one elementary class into members of another*, Notices Amer. Math. Soc. **7** (1960), 365.

14. ———, *Properties preserved under reduced products*. I, II, Notices Amer. Math. Soc. **7** (1960), 71; ibid. **8** (1961), 64.

15. ———, *Replete relational systems*, Notices Amer. Math. Soc. **8** (1961), 63.

16. ———, *Theory of models with generalized atomic formulas*, J. Symbolic Logic **25** (1960), 1-26.

17. ———, *Ultraproducts and elementary classes*, Doctoral dissertation, Univ. of California, Berkeley, Calif., 1961. Also Nederl. Akad. Wetensch. Proc. Ser. A **64** (1961), 477–495 = Indag. Math. **23**.

18. G. Kreisel, *Some uses of metamathematics*, British J. Philos. Sci. **7** (1956), 161–173.

19. J. Łoś, *Quelques remarques, théorèmes, et problèmes sur les classes définissables d'algèbres*, Mathematical Interpretations of Formal Systems, pp. 98–113, Nort-Holland, Amsterdam, 1955.

20. R. C. Lyndon, *Existential Horn sentences*, Proc. Amer. Math. Soc. **10** (1959), 995–998.

21. ———, *Properties preserved under homomorphism*, Pacific J. Math. **9** (1959), 143–154.

22. J. C. C. McKinsey, *The decision problem for some classes of sentences without quantifiers*, J. Symbolic Logic **8** (1943), 61-76.

23. M. Morley and R. Vaught, *Homogeneous universal models*, Math. Scand. **11** (1962), 37–57.

24. A. Oberschelp, *Über die Axiome produkt-abgeschlossener arithemetischer Klassen*, Arch. Math. Logik Grundlagenforsch. **4** (1958), 95–123.

25. A. D. Taimonov, *A class of models closed under direct products*, Izv. Akad. Nauk SSSR Ser. Mat. **24** (1960), 493–510. (Russian)

26. A. Tarski, *Contributions to the theory of models*. I, II, Nederl. Akad. Wetensch. Proc. Ser. A **57** (1954), 572–581; 582–588 = Indag. Math. **16**.

27. A. Tarski and R. Vaught, *Artihmetical extensions of relational systems*, Compositio Math. **13** (1958), 81–102.

28. R. Vaught, *Models of complete theories*, Bull. Amer. Math. Soc. **69** (1963), 299–313.

29. J. Weinstein, *Sentences preserved under direct products*, Abstract, Notices Amer. Math. Soc. (to appear).

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