A CHARACTERIZATION OF Δ_2-SETS

BY

B. SCARPELLINI(1)

A Δ_2-set m is defined as follows: it is constructible and $Od^{\epsilon}m$ is isomorphic to an analytic well-ordering W of the natural numbers which is expressible in both two-quantifier forms. J. R. Shoenfield has proved in [1] that a set of natural numbers is a Δ_2-set if and only if it is analytic and expressible in both two-quantifier forms. In this paper two characterizations of Δ_2-sets are given.

Let formulas of set theory be formulas of predicate-calculus with identity and the single predicate-variable $\in$. A finite set S of closed formulas of set theory will be called a finite system of axioms. Then we have the result: m_0 is a Δ_2-set if and only if there is a finite system of axioms S with the properties (a) S has exactly one transitive model m, (b) m is denumerable and $m_0 \in m$. Another characterization of Δ_2-sets is obtained by giving a family K of functions defined by transfinite induction such that (a) for $f \in K$, every x belonging to the range of f is a Δ_2-set, (b) corresponding to every Δ_2-set x there is an f in K whose range contains x.

We use the terminology and notation of [2], [3] and [4]. The power of a set m is denoted by $|m|$.

A. We start with some preliminaries. A lemma is needed, which because of its simplicity is stated without proof (see for instance [5]). In what follows, $D(f)$, where f is a function, will denote the domain of f and $W(f)$ the range of f. We also need

DEFINITION 1. A set m is called *simple* if for $a, b \in m$,

$$(x)(x \in m \cdot \longrightarrow \cdot x \in a \equiv x \in b) . \equiv . a = b .$$

Then we have

LEMMA 1. *Let m be nonvoid and simple. Then there is a function g_m with $D(g_m) = m$ such that, for any $x \in m$, $g_m(x) = \{g_m(s) \mid s \in x \cap m\}$. The set $W(m)$ (to be denoted by m^*) is transitive. If m_0 is transitive and $m_0 \subseteq m$, then $g_m(x) = x$ for every $x \in m_0$. The function g_m is uniquely determined by this property.*

A transitive m_0 with $m_0 \subset m$, $m_0 \in m$, we call a transitive part of m. Thus, transitive parts are invariant under the isomorphism g_m.

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B. Let m be infinite and let $R_1, R_2, \cdots, R_s$ be a system of $(n_1, n_2, \cdots, n_s)$-place relations over m. If m has a well-ordering and $m_0 \subseteq m$ then by the Loewenheim-Skolem theorem m has an elementary subsystem $\langle m' \mid R_1', \cdots, R_s' \rangle$ such that $m_0 \subseteq m'$ and $|m'| = |m_0| + \chi_0$, where R_i' is the restriction of R_i to m'. This fact will be used in what follows. It is also convenient to introduce the notation $p[m]$ with the meaning: if $p(x_1, \cdots, x_n)$ is a formula of set theory with the free variables $x_1, \cdots, x_n$, then $p[m]$ denotes the n-ary relation defined by p in the model m. In general, if R is an n-ary relation, we write $R(x_1, \cdots, x_n)$ to mean that $\langle x_1, \cdots, x_n \rangle \in R$.

LEMMA 2. *Let m, m_0 be transitive and $m_0 \subseteq m$. There exists a transitive set m_1 with $m_0 \subseteq m_1$ such that for elements $a_1, \cdots, a_s$ of m_0 the equivalence $p[m_1](a_1, \cdots, a_s) \equiv p[m](a_1, \cdots, a_s)$ holds. If m_0 is infinite, then m_1 has the same power as m_0.*

Proof. By the Loewenheim-Skolem theorem there is a set m' with $m_0 \in m'$ such that for elements $a_1, \cdots, a_s$ in m' we have $p[m](a_1, \cdots, a_s) \equiv p[m'](a_1, \cdots, a_s)$. For $a, b \in m'$,

$$(x)(x \in m' \cdot \longrightarrow \cdot x \in a \equiv x \in b) . \equiv . (x)(x \in m \cdot \longrightarrow \cdot x \in a \equiv x \in b) . \equiv . a = b$$

as m is transitive. So m' is simple. For $m_1 = (m')^*$ and $a_1, \cdots, a_s \in m_0$, it follows as a consequence of Lemma 1 that

$$p[m](a_1, \cdots, a_s) \equiv p[m'](a_1, \cdots, a_s) \equiv p[m_1](g_{m'}(a_1), \cdots, g_{m'}(a_s)) \equiv p[m_1](a_1, \cdots, a_s),$$

as m_0 is a transitive part of m' and hence invariant with respect to $g_{m'}$. If m_0 is infinite, the set m_1 has the same power as m_0, by the Loewenheim-Skolem theorem.

In the following, $\mathfrak{F}_i$ $(i = 1, \cdots, 8)$, J, K_1, K_2, and F denote the functions defined in [3, Definitions 9.1, 9.21, 9.24, 9.3]. By the methods developed, e.g. in [6], one can easily prove

LEMMA 3. *There is a formula $A_0(x_1, x_2, x_3, x_4, x_5, y)$ such that for any nonvoid transitive m and any elements $O_0, O_0', J_0, F_0, v, \xi$ of m, $A_0[m](O_0, O_0', J_0, F_0, v, \xi)$ if and only if O_0 is an ordinal greater than zero, $O_0' = 9 \times O_0^2$, $J_0 = J^{-1} \restriction O_0$, $F_0 = F \restriction O_0$ and $F_0 ` v = \xi$.*

As a consequence of Lemma 3 we have:

THEOREM 1. *If m_0 is infinite, transitive and constructible and if $a \in m_0$ then $|Od ` a| \leq |m_0|$.*

Proof. First, one easily shows that there exists a constructible transitive set m with $m_0 \subseteq m$ such that $(Ex_1\ x_2\ x_3\ x_4\ x_5) A_0[m](x_1, x_2, x_3, x_4, x_5, a)$

holds with A_0 as in Lemma 3. By Lemma 2 there is a transitive set m_1 with $m_0 \subseteq m_1 \subseteq m$ and $|m_0| = |m_1|$ such that $(Ex_1 \cdots x_5) A_0 [m_1](x_1, \cdots, x_5, a)$.

By Lemma 3, $Od^{\prime}a \in m_1$ and consequently $|Od^{\prime}a| \leqq |m_1| = |m_0|$.

REMARK. Lemmas 1–3 and Theorem 1 can be formalized without difficulty in the system of Gödel-Bernays with the aid of the methods of [5], [6]. If we add the axiom $V = L$, we obtain a new proof of the statement $P(F''\omega_\alpha) \subseteq F''\omega_{\alpha+1}$ which implies the continuum hypothesis. For this purpose it is sufficient to show: $x \subseteq F''\omega_\alpha \to |Od^{\prime}x| \leqq \omega_\alpha$. But this is an immediate consequence of Theorem 1, if we put there $m_0 = x \cup \{x\} \cup F''\omega_\alpha$. As was pointed out to the author by the referee, the above variant of Gödel's proof was observed by D. Scott three years ago and given by him in a seminar a year ago.

C. As shown in [1] the following statement holds:

I. If γ_0 is a constructible function, then

$$(\alpha)(E\beta)(x) A(\bar{\alpha}, \bar{\beta}, \bar{\gamma}_0) \equiv (\alpha)_L (E\beta)_L (x) A(\bar{\alpha}, \bar{\beta}, \bar{\gamma}_0).$$

Here, A is a recursive relation and the index L on the right side indicates that the quantifiers are restricted to constructive functions.

Functions which are recursive in $\Sigma_k^1 \cap \Pi_k^1$-predicates (or equivalently whose graphs are $\Sigma_k^1 \cap \Pi_k^1$-predicates) are called $\Sigma_k^1 \cap \Pi_k^1$-functions. By I, the notion of $\Sigma_2^1 \cap \Pi_2^1$-function is absolute and by [1] the set of $\Sigma_2^1 \cap \Pi_2^1$-functions coincide with the set of Δ_2-functions (and similarly for predicates). In [4] it was shown, that under the assumption $V = L$ we have:

II. The set of $\Sigma_{k+1}^1 \cap \Pi_{k+1}^1$-functions is a basis for Π_k^1-predicates. From I and II we immediately obtain

LEMMA 4. *If there is a function γ_0 with $(E\alpha)(\beta)(Ex) A(\bar{\alpha}(x), \bar{\beta}(x), \bar{\gamma}_0(x))$ then there exists a Δ_2-function γ_1 with $(E\alpha)(\beta)(Ex) A(\bar{\alpha}(x), \bar{\beta}(x), \bar{\gamma}_1(x))$.*

Proof. From $(E\gamma)(E\alpha)(\beta)(Ex) A(\bar{\alpha}(x), \bar{\beta}(x), \bar{\gamma}(x))$ and I, we obtain $(E\gamma)_L (E\alpha)_L (\beta)_L (Ex) A(\bar{\alpha}(x), \bar{\beta}(x), \bar{\gamma}(x))$. By II there are Δ_2-functions γ_1, α_1 with $(\beta)_L (Ex) A(\bar{\alpha}_1(x), \bar{\beta}(x), \bar{\gamma}_1(x))$. This implies $(E\alpha)_L (\beta)_L (Ex) A(\bar{\alpha}(x), \bar{\beta}(x), \bar{\gamma}_1(x))$ and consequently by I also $(E\alpha)(\beta)(Ex) A(\bar{\alpha}(x), \bar{\beta}(x), \bar{\gamma}_1(x))$.

DEFINITION 2. We call a binary number-theoretic function α a *founded ε-function* if (a) $(x, y)(\alpha(x, y) = 1 \,.\vee .\, \alpha(x, y) = 0)$, (b) there does not exist an infinite sequence $i_1, i_2, \cdots, i_n, \cdots$ of numbers such that $\alpha(i_2, i_1) = 1, \alpha(i_3, i_2) = 1, \cdots$, (c) $(x)(\alpha(x, i) = 1 \equiv \alpha(x, k) = 1) \cdot\!\longrightarrow\!\cdot\, i = k$. Functions for which only (a) hold will be called characteristic functions.

Let ω denote the set of natural numbers. As shown in [5, Theorem 3], the following statement holds.

LEMMA 5. *Corresponding to a founded ε-function α, there exists exactly one transitive set m_α and one biunique function g_α with $D(g_\alpha) = \omega$ and*

$W(g_\alpha) = m_\alpha$ such that $\alpha(i, k) = 1 \equiv g_\alpha(i) \in g_\alpha(k)$. If α is constructible, then so are g_α and m_α.

In the proof of the following theorem only the essential steps are given; the necessary details are easily supplied by using the methods of [4].

THEOREM 2. *If α is a founded ε-function belonging to $\Sigma_2^1 \cap \Pi_2^1$, then the set m_α of Lemma 5 is a Δ_2-set.*

Proof. 1. By hypothesis there are two recursive predicates P, Q with

$$(x, y)(\alpha(x, y) = 1 \equiv (E\psi)(\phi)(Et)\, P(\bar{\psi}(t), \bar{\phi}(t), x, y)),$$

$$(x, y)(\alpha(x, y) = 1 \equiv (\psi)(E\phi)(t)\, Q(\bar{\psi}(t), \bar{\phi}(t), x, y)).$$

The predicate R which applies to ξ if and only if

$$(x, y)(\xi(x, y) = 1 \,.\vee.\, \xi(x, y) = 0)$$

$$.\wedge.\,(x, y)((\psi)(E\phi)(s)\, Q(x, y, \bar{\psi}(s), \bar{\phi}(s), x, y) \cdot\!\longrightarrow\!\cdot\, \xi(x, y) = 1)$$

$$.\wedge.\,(x, y)(\xi(x, y) = 1 \cdot\!\longrightarrow\!\cdot\, (E\psi)(\phi)(Es)\, P(\bar{\psi}(s), \bar{\phi}(s), x, y))$$

has the form $(E\phi)(\psi)(Ex)R_1(\bar{\phi}(x), \bar{\psi}(x), \bar{\xi}(x))$ with R_1 recursive. R is satisfied by exactly one ξ, namely α.

2. Let S be the relation defined as follows: $S(k, \lambda, \mu, \phi)$ if and only if λ, μ are founded ε-functions such that

(a) $(x, y)(\lambda(x, k) = 1 \,.\wedge.\, \lambda(y, x) = 1 \cdot\!\longrightarrow\!\cdot\, \lambda(y, k) = 1)$,
(b) $D(\phi) = \omega$, $W(\phi) = \{x \mid \lambda(x, k) = 1\}$ and

$$(x, y)(\phi(x) = \phi(y) \equiv x = y \,.\wedge.\, \mu(x, y) = 1 \equiv \lambda(\phi(x), \phi(y)) = 1).$$

One shows without difficulty that S is a Π_1^1-relation. Furthermore, it is easy to see that $S(k, \lambda, \mu, \phi)$ implies $g_\lambda(k) = m_\mu$ with g_λ, m_μ as in Lemma 5.

3. Let $A_0(x_1, x_2, \cdots, x_5, y)$ be the formula of Lemma 3. Define the relation G as follows: $G(\lambda, i, k)$ if and only if λ is a founded ε-function such that

$$(Ex_1, \cdots, x_4)\, A_0\,[m_\lambda]\,(x_1, \cdots, x_4, g_\lambda(i), g_\lambda(k)).$$

Also G can be shown to be a Π_1^1-relation.

4. The function α is constructible and so are the sets m_α, $m_\alpha \cup \{m_\alpha\}$. By Theorem 1 the cardinality of $Od'm_\alpha$ is not greater than that of $m_\alpha \cup \{m_\alpha\}$; that is $Od'm_\alpha$ is denumerable. Hence there is denumerable transitive set m_1 and an ordinal $\nu \in m_1$ such that $(Ex_1, \cdots, x_4)\, A_0\,[m_1]\,(x_1, \cdots, x_4, \nu, m_\alpha)$. Let M be the predicate defined by the equivalence $M(\xi) \equiv (Ex, y, \mu)(G(x, y, \xi) \wedge S(y, \xi, \mu) \wedge R(\xi))$.

M is a Σ_2^1-predicate by 1., 2., 3.,. Since m_1 is denumerable, there is at least one founded ε-function λ with $m_\lambda = m_1$.

Obviously $M(\lambda)$ holds as a consequence of 1., 2., 3., and hence $(Ex)\,M(x)$.

By Lemma 4 there is a Δ_2-function β satisfying Definition 2 with $M(\beta)$. From $M(\beta)$ it follows that there is a number i with $(Ex_1,\cdots,x_4)A_0[m_\beta](x_1,\cdots,x_4,g_\beta(i),m_\alpha)$. Let i_0 be the smallest such number. Obviously $Od'm_\alpha$ is order-isomorphic to $\{\langle x, y\rangle \mid \beta(x, i_0) = 1 \wedge \beta(y, i_0) = 1 \wedge \beta(x, y) = 1\}$. By standard techniques it now easily follows that $Od'm_\alpha$ is order-isomorphic to a Δ_2-well-ordering of ω. q.e.d.

Here and later on we shall use a predicate Sat which is defined as follows: $\mathrm{Sat}(n, \lambda)$ if and only if (a) λ is a founded ε-function, (b) n is the Gödel number of a closed formula A of set theory which is true in the model $\langle \varepsilon', \omega\rangle$ with $\varepsilon' = \{\langle x, y\rangle \mid \lambda(x, y) = 1\}$. Obviously, $\mathrm{Sat}(n, \lambda)$ if and only if A is true in the model $\langle \varepsilon^*, m_\lambda\rangle$ with $\varepsilon^* = \{\langle x, y\rangle \mid x \in m_\lambda,\ y \in m_\lambda,\ x \in y\}$. Again by the methods of [4] one shows without difficulty that Sat is in Σ^1_2. An easy consequence of Theorem 2 is:

COROLLARY 1. *If A is a finite system of axioms such that* (a) *A has exactly one transitive model m and* (b) *m is denumerable, then m is a Δ_2-set.*

Proof. Let n be the Gödel number of A. From (a), (b) it follows that there exists at least one founded ε-function λ with $\mathrm{Sat}(n, \lambda)$. Obviously by (a), for any founded ε-function λ' with $\mathrm{Sat}(n, \lambda')$, $m_{\lambda'} = m$. By Lemma 4, there is a Δ_2-function μ which satisfies Definition 2 such that $\mathrm{Sat}(n, \mu)$. Again $m_\mu = m$, by (b). Hence, Theorem 2 applies, so m is a Δ_2-set.

With Corollary 1 the first half of the statement in the introduction is proved: if for a given set m_0 there exists a finite set of axioms A and a transitive set m satisfying conditions (a), (b) of Corollary 1 and such that $m_0 \in m$, then by Corollary 1 m_0 is obviously a Δ_2-set.

REMARK. It would suffice to replace (b) in Corollary 2 by (b′) which states that m can be well-ordered.

D. The statement given in the introduction is proved if we can show that to every Δ_2-set m_0 there exists a finite system of axioms A which satisfies (a), (b) of Corollary 1 and whose only transitive model m contains m_0 as element. This will be proved in this section. Let $\alpha_1, \cdots, \alpha_n$ be $s_1, \cdots, s_n$-place number-theoretic functions, and let $W(\alpha_1, \cdots, \alpha_n)$ be the set of $(\alpha_1, \cdots, \alpha_n)$-well-orderings (that is, the set of Gödel numbers e of $(\alpha_1, \cdots, \alpha_n)$-recursive functions such that the relation $\{\langle x, y\rangle \mid (e)(x, y) = 0\}$ is a well-ordering). Let, furthermore, $\mathrm{Rec}(A)$ be the set of A-recursive functions and P a binary $(\alpha_1, \cdots, \alpha_n)$-recursive relation. As proved in [2]:

III. If $(\phi)(\phi \in \mathrm{Rec}(W(\alpha_1, \cdots, \alpha_n)) \cdot \longrightarrow \cdot (Et)\, P(\bar\phi(t), t))$ then $(\phi)\,(Et)\,P(\bar\phi(t), t)$.
A consequence of III, is:

LEMMA 5. *If R is a recursive relation, then for any function α*

$$(\phi)\,(\phi \in \mathrm{Rec}(W(\alpha)) \cdot \longrightarrow \cdot (Et)\, R(\bar\alpha(t), \bar\phi(t)))\,.\equiv.\,(\phi)\,(Et)\,R(\bar\alpha(t), \bar\phi(t))\,.$$

Proof. The implication from right to left is obvious. The implication from left to right is implied by III, and the fact that the relation $\{\langle x, y\rangle \mid R(\bar{\alpha}(x\, y)\}$ is α-recursive.

In the proof of Theorem 3 will again perform only the principal steps. The effective construction of certain formulas and the verification of their properties will be surpassed and can easily be supplied by standard methods (e.g., [4], [6]). $L(\alpha,\beta)$ will denote the set of Gödel numbers of α, β recursive linear orderings and for $f \in L(\alpha,\beta)$, $d(f)$ denotes the set $\{x \mid (Ey)(\{f\}^{\alpha,\beta}(x,y) = 0. \vee . \{f\}^{\alpha,\beta}(y,x) = 0)\}$. If $f \in L(\alpha, \beta)$ or if ξ is the characteristic function of a linear ordering, $\leqq_f$ and $\leqq_\xi$ will denote the corresponding order-relation.

THEOREM 3. *If m_0 is a Δ_2-set, then there exists a finite system of axioms A such that*: (a) *A has exactly one transitive model m*, (b) *m is denumerable and $m_0 \in m$.*

Proof. We proceed by steps.

1. Since m_0 is a Δ_2-set, we conclude as in the proof of Theorem 2 (no.1), that there is a characteristic function α_0 and a recursive relation P such that (a) α_0 is the only element in the set $\{\xi \mid (E\psi)(\phi)(Et)P(\bar{\xi}(t), \bar{\psi}(t), \bar{\phi}(t))\}$, (b) $Od^{\prime}m_0$ is order-isomorphic to $\{\langle x, y\rangle \mid \alpha_0(x, y) = 1\}$.

2. Let m be transitive and O_0, O_0', J_0, F_0 be elements of m such that $(Ex_1, x_2)\, A_0\, [m]\, (O_0, O_0', J_0, F_0, x_1, x_2)$ with A_0 as in Lemma 3. For any set m', let $h(m')$ be the smallest transitive set containing m' as subset (the transitive hull of m'). Consider the formula $x=y. \vee . x \in y. \vee . x \in \bigcup y. \vee . x \in \bigcup\bigcup y$ (to be denoted by $H_4(x, y)$). Since O_0 is an ordinal and $F \upharpoonright O_0 = F_0$ (Lemma 3), $\{x \mid H_4[m]\,(x, F_0)\} = h(\{F_0\})$. Similarly there are formulas $H_i(x, y)$ $(i=1,2,3)$ with $\{x \mid H_1[m](x,O_0)\} = h(\{O_0\})$, $\{x \mid H_2[m](x,O_0')\} = h(\{O_0'\})$, $\{x \mid H_3[m](x,J_0)\} = h(\{J_0\})$. Consequently the formula

$$(Ex_5, x_6)\, A_0\, (x_1, \cdots, x_4, x_5, x_6) . \wedge . (s)\, (H_1(s, x_1) \vee \cdots \vee H_4(s, x_4))$$

(to be denoted by $A_1(x_1, \cdots, x_4)$) has the property:

$$A_1[m]\,(O_0, O_0', J_0, F_0) \cdot\!\longrightarrow\cdot\, m = h(\{O_0\} \cup \{O_0'\} \cup \{J_0\} \cup \{F_0\}) .$$

Obviously, for any ordinal $\nu > 0$, there exists exactly one transitive set m with $\nu \in m$ and $(Ex_1, x_2, x_3)\, A_1\, [m]\, (\nu, x_1, x_2, x_3)$, namely,

$$m = h(\{\nu\} \cup \{9 \times \nu^2\} \cup \{J^{-1} \upharpoonright \nu\} \cup \{F \upharpoonright \nu\})$$

(to be denoted in the following by h_ν). Furthermore, for transitive m, $m = h_\nu$ for some $\nu > 0$ if and only if $(Ex_1, \cdots, x_4)\, A_1\, [m]\, (x_1, \cdots, x_4)$.

3. Consider the class C of ordinals such that $\nu \in C$ if and only if (a) $F''\nu$ contains ω (the set of natural numbers), ω^2 and the sets

$$\{\langle x_1, x_2, x_3\rangle \mid x_i \in \omega \wedge x_1 + x_2 = x_3\}, \{\langle x_1, {}_2 x_3\rangle \mid x_i \in \omega \wedge x_1 x_2 = x_3\}$$

as members, (b) if α,β are one- or two-place number-theoretic functions or subsets of ω and $\alpha, \beta \in F''\nu$, then $\mathrm{Rec}(\alpha, \beta) \subseteq F''\nu$. The class C is nonvoid.

4. Let D be the subclass of C containing exactly those ordinals of C for which the following holds: $F''\nu$ contains number-theoretic functions α, β and functions G, H having the properties

(a) $D(H) \subseteq L(\alpha, \beta)$ and $D(H) \in F''\nu$,

(b) for $f \in D(H)$, $H'f$ is a function with $D(H'f) = d(f)$, $W(H'f) \in \nu$ and $(x, y)(x \leqq_f y \wedge y \nleqq_f x \,.\equiv.\, (H'f)'x \in (H'f)'y)$,

(c) for $f \in L(\alpha, \beta) - D(H)$, there is in $F''\nu$ a number-theoretic function ξ with $(x)(\xi(x+1) \leqq_f \xi(x) \,.\wedge.\, \xi(x) \nleqq_f \xi(x+1))$,

(d) for all number-theoretic functions ξ in $F''\nu$, $(Et)\, R(\bar{\alpha}(t), \bar{\beta}(t), \bar{\xi}(t))$,

(e) $D(G) = \omega$, $W(G) \in \nu$ and α is the characteristic function of a linear ordering such that $(x, y)(x \leqq_\alpha y \wedge y \nleqq_\alpha x \,.\equiv.\, G'x \in G'y)$.

By taking ordinals of sufficiently high cardinality, it is easy to see that the class D is nonvoid. Let $m = h_\nu$ for $\nu \in D$, and let α, β, G, H be the elements of m having the properties (a) – (e). From (a), (b), (c) it follows that $D(H)$ is the set of (α, β)-recursive well-orderings. Since $F''\nu$ is closed under recursive operations, $\mathrm{Rec}(D(H))$ is a subset of $F''\nu$, and hence by (d)

$$(\xi)(\xi \in \mathrm{Rec}(D(H)) \rightarrow (Et)\, R(\bar{\alpha}(t), \bar{\beta}(t), \bar{\xi}(t)) .$$

This, together with Lemma 5, implies $(\phi)(Et)\, R(\bar{\alpha}(t), \bar{\beta}(t), \bar{\phi}(t))$ and hence $(E\psi)(\phi)(Et)\, R(\bar{\alpha}(t), \bar{\psi}(t), \bar{\phi}(t))$. Consequently, α is the characteristic function α_0 considered in no. 1. By (e), $W(G)$ is an ordinal order-isomorphic with

$$\{\langle x, y\rangle \mid \alpha_0(x, y) = 1\},$$

that is, $W(G) = Od'm_0$. Hence, again by Lemma 3, for every $\nu \in D$, m_0 is an element of h_ν.

5. By standard methods one constructs without difficulty a closed formula A_2 and a formula $A_3(x)$ having x as its only free variable such that (a) $A_2[h_\nu]$ if and only if $\nu \in D$, (b) for $\mu \in h_\nu$, $A_3[h_\nu](\mu)$ if and only if $\mu \in D$. It is now easy to see that the formula $(Ex_1, \cdots, x_4) A_1(x_1, \cdots, x_4) \,.\wedge.\, A_2 \,.\wedge.\, \neg (Ex) A_3(x)$ has exactly one transitive model, namely h_δ, where δ is the smallest ordinal in D. Furthermore, the theorem of Loewenheim-Skolem implies that h_δ is denumerable (h_δ can be well-ordered). Since $m_0 \in h_\delta$ (no. 4), the theorem is proved.

Let M be the set of Gödel numbers of those closed formulas which admit at least one denumerable transitive model. The methods applied in the proof of Theorem 3 can be used to prove

THEOREM 4. *$x \in M$ is a complete Σ_2^1-predicate.*

Proof. 1. Consider the predicate Sat introduced in C (following Theorem 2). Obviously, $n \in M \equiv (E\lambda)$ Sat (n, λ). Since Sat is a Π_1^1-predicate, $x \in M$ is a Σ_2^1-predicate.

2. Let P be a predicate with $P(n) \equiv (E\phi)(\psi)(Et)P_1(n, \bar{\phi}(t), \bar{\psi}(t))$ with P_1 recursive, and let C be the same class of ordinals as defined in no. 3 in the proof of Theorem 3. Now for a given natural number n, let D_n be the subclass of C containing exactly those ordinals ν of C for which the following holds: $F''\nu$ contains a number-theoretic function α and functions G, H having the properties

(a) $D(H) \subseteq L(\alpha)$ and $D(H) \in F''\nu$,

(b) for $f \in D(H)$, $H'f$ is a function with $D(H'f) = d(f)$, $W(H'f) \in \nu$ and $(x, y)(x \leqq_f y \,.\, \wedge \,.\, y \nleqq_f x \,.\, \equiv \,.\, (H'f)'x \in (H'f)'y)$,

(c) for $f \in L(\alpha) - D(H)$, there is in $F''\nu$ a number-theoretic function ξ with $(x)(\xi(x + 1) \leqq_f \xi(x) \,.\, \wedge \,.\, \xi(x) \nleqq_f \xi(x + 1))$,

(d) for all number-theoretic functions ζ in $F''\nu$, $(Et)P_1(\bar{\alpha}(t), \bar{\zeta}(t))$. Using the same arguments as in no. 4, proof of Theorem 4, we conclude that D_n is nonvoid if and only if $P(n)$. Furthermore, as in no. 5 of the proof, a closed formula G_n can be given effectively which has the property: for a transitive set m, $G_n[m]$ if and only if D_n is nonvoid and $m = h_\nu$ where ν is the smallest ordinal in D_n. Clearly, the function g which attributes to every n the Gödel number $g(n)$ of G_n is recursive. Since obviously $g(n) \in M \,.\, \equiv \,.\, P(n)$, the theorem follows.

E. In this section, another characterization of the Δ_2-sets is given. In order to avoid a mere repetition of the arguments given in previous sections, we will state the definitions and theorems and give only some indications. We start with some notations and definitions. Let m be a transitive set in what follows. A list $m_1, m_2, \cdots, m_k$ of elements of m and of formulas $G_i(x_0, x_1, \cdots, x_k, y)$ $(i = 1, \cdots, s)$ whose only free variables are $x_0, x_1, \cdots, x_k, y$ will be called an *m-list*.

DEFINITION 3. For a given m-list $m_1, \cdots, m_k$, $G_1, \cdots, G_s$ one can define by transfinite induction functions $F_1, \cdots, F_s$ as follows:

$$F_i'0 = i \qquad (i = 1, \cdots, s),$$

and

$$F_i'\nu = \{x \mid G_i[m_\nu](m, m_1, \cdots, m_k, x) \wedge x \in m_\nu'\}$$

for $\nu > 0$. Here, m_ν and m_ν' are given by

(a) $m_\nu = m \cup \{m\} \cup_k (\{F_k \upharpoonright \nu\} \cup (F_k \upharpoonright \nu) \cup (\bigcup F_k \upharpoonright \nu) \cup (\bigcup\bigcup F_k \upharpoonright \nu))$,

(b) $m_\nu' = m \cup \{m\} \cup_k ((F_k \upharpoonright \nu) \cup (\bigcup F_k \upharpoonright \nu) \cup (\bigcup\bigcup F_k \upharpoonright \nu))$.

The functions so defined are called the functions *associated* with the given m-list.

It is an immediate consequence of Definition 3 that the sets m_ν, m_ν' considered there are transitive.

DEFINITION 4. An m-list is said to be *terminating* if there is an ordinal $\mu > 0$ such that the associated functions $F_1, \cdots, F_s$ have the following properties: (a) for $\lambda < \mu$, $\bigcup_i F_i'\lambda \neq 0$, (b) $\bigcup_i F_i'\mu = 0$.

We say that the given m-list *terminates* at μ, and call the set m_μ given by Definition 3, (a) the *characteristic model* of the list.

DEFINITION 5. A set m' is called *m-definable*, if there is an ordinal $\nu > 0$ and an m-list terminating at ν such that the associated functions F_i $(i = 1, \cdots, s)$ satisfy the condition $m' \in \bigcup_i F_i'' \nu$. We also say that m' is *m-defined* by the given list. If $m = 0$, m' is simply called *definable*.

In the following lemma, some properties of the concepts defined above are described.

LEMMA 7. 1. *If an m-list is terminating, then the ordinal at which it terminates is uniquely determined.*

2. *To any given m-list $m_1, \cdots, m_k, G_1, \cdots, G_s$ one can find effectively a formula $G(x_0, x_1, \cdots, x_k)$ with the property: for a transitive set m' with $m \cup \{m\} \in m'$, $G[m'](m, m_1, \cdots, m_k)$ if and only if the m-list terminates at an ordinal μ and m' is the characteristic model m_μ. If especially $m = 0$, then G can be chosen so as to be closed.*

Proof. Part 1 is an immediate consequence of Definitions 3, 4. The proof of Part 2 is based on the same standard techniques used in A – D and developed e.g. in [6].

DEFINITION 7. The formula G in Lemma 7 associated with the given m-list $m_1, \cdots, m_k, G_1, \cdots, G_s$ will be called the *characteristic formula* of the given list.

The connection between the concepts just defined and the considerations in A – D is given by

THEOREM 5. *A set m is definable if and only if it is a Δ_2-set.*

Proof. Let m be definable by a given 0-list whose characteristic formula and characteristic model are G and m_μ, respectively. Obviously, m_μ is denumerable. Since m_μ is the unique transitive model of G by Lemma 7 and $m \in m_\mu$, m is a Δ_2-set by Theorem 2. The proof of the converse statement is tedious but does not involve anything new; for this reason we content ourselves with some indications. One can show that there is a special 0-list whose first associated function F_1 has the properties: (a) to every ordinal ν there is an ordinal $h(\nu)$ not greater than ν such that $F_1''\nu = F \restriction h(\nu)$, (b) $W(F_1) = F$, (c) if $\nu_1 \leqq \nu_2$ then $h(\nu_1) \leqq h(\nu_2)$. It is not difficult now to enlarge this special 0-list to a second one which contains among its associated functions the function F_1 and which terminates at a certain ordinal μ. The ordinal μ will be the smallest for which a certain set of conditions are satisfied, which are quite similar to (a)–(e) in no. 4, proof of Theorem 4; m will then be a member of $W(F_1''\mu)$

Since an 0-list is completely determined by its characteristic formula (Lemma 7), we can equally characterize the given 0-list by the Gödel number of its characteristic formula. Let M' be the set of Gödel numbers whose corresponding 0-lists are terminating. By reasoning which parallels the proof of Theorem 4 we obtain:

THEOREM 6. *The predicate* $x \in M'$ *is a complete* Σ_2^1*-predicate.*

REMARKS AND PROBLEMS. The notion of m-definability given here is in a loose sense somewhat analogous to the notion of computation in the theory of relative recursiveness: while in the latter case the process of computation terminates after a finite number of steps or continues a denumerable number of times, the process of transfinite induction connected with an m-list terminates at a certain ordinal μ or runs through the class of all ordinals. In this connection, it is worth noting that the following statement can be proved: if m, m_1 are transitive, and if m_1 is m-definable, then any m_1-definable set is also m-definable. It is also interesting that several schemes of transfinite induction which one could choose as alternatives to that one given by Definition 3 turn out to be equivalent with the latter.

We close by considering two problems. Assume for simplicity the axiom of choice. If the transitive hull $h(m)$ of m is a Δ_2-set, we cannot necessarily conclude that m is also a Δ_2-set, as simple examples show. However, m is a Δ_2-set if and only if $h(m \cup \{m\})$ is Δ_2, as is easily seen. One can ask if the result obtained in A – D can be sharpened in the following way: m is Δ_2 if and only if there is a formula G whose only transitive model is $h(m \cup \{m\})$. Another question arises in connection with the considerations of E. As a consequence of Lemma 7, if m' is m-definable, there are elements $m_1, \cdots, m_k$ in m and a formula $G(x_0, x_1, \cdots, x_k)$ with the property: there is exactly one transitive set m'' having $m, m_1, \cdots, m_k$ as members such that $G[m''](m, m_1, \cdots, m_k)$. The obvious question is: does the converse hold? If $m = 0$, then the answer is affirmative as a consequence of Corollary 1 and Theorem 5.

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BATTELLE INSTITUTE,
GENEVA, SWITZERLAND