ON THE BEHAVIOR OF QUASICONFORMAL MAPPINGS AT A POINT(1)

BY

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1. **Introduction.** A sense-preserving homeomorphism $w(z)$ of the domain G in the z-plane onto the domain $w(G)$ of the w-plane is called quasiconformal (more precisely, K-quasiconformal) if there exists a constant $K \geqq 1$ such that for every quadrilateral Q, $\bar{Q} \subset G$, the modulus $M(Q)$ of Q, and the modulus $M(w(Q))$ of the image quadrilateral $w(Q)$, satisfy

$$M(w(Q)) \leqq KM(Q).$$

If $w(z)$ is quasiconformal in G, then the Jacobian $J(z) \neq 0$ a.e. in G; also the complex dilatation

$$\kappa(z) = \frac{w_{\bar{z}}}{w_z} = \frac{w_x + iw_y}{w_x - iw_y}$$

exists a.e. in G, and $\sup_{z \in G} |\kappa(z)| < 1$. (Cf., for instance, [3].) Conversely, it is well known that if $\mu(z)$ is any complex-valued measurable function, specified a.e. in G, with $\sup_{z \in G} |\mu(z)| < 1$, then there exists a quasiconformal mapping $w(z)$ with complex dilatation $\mu(z)$ a.e. (Cf., the exposition in [4, p. 115], or, for a direct proof, [5].) Moreover, $w_1(z)$ and $w_2(z)$ are two such mappings if and only if $w_2(w_1^{-1}(z))$ is conformal.

If $\kappa(z_0)$ exists, then the ("real") dilatation at z_0 is defined as

$$D(z_0) = \frac{1 + |\kappa(z_0)|}{1 - |\kappa(z_0)|}.$$

This is the ratio of major to minor axis of the infinitesimal ellipse onto which $w(z)$ maps an infinitesimal circle centered at z_0, providing $w_z(z_0)$ exists and is different from 0. If $w(z)$ is K-quasiconformal, then $D(z) \leqq K$ a.e. in G.

The present work is motivated by the following consideration, evident from the above. If $\kappa(z)$ is specified a.e. in a neighborhood of a particular point z_0 of G, then whether or not $\kappa(z_0)$ exists, and what the value of $\kappa(z_0)$ is, if it does exist, is completely determined. The problem is to explore the qualitative properties of $\kappa(z)$

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in a neighborhood of z_0 which cause $\kappa(z_0)$ to exist and have some particular value. The important special case of this question to which we shall restrict ourselves here is the following. Suppose $w(z)$ is quasiconformal for $|z| < 1$, $w(0) = 0$. It is desired to explore the properties of $D(z)$, or, more generally, $\kappa(z)$, which affect the existence of $\lim_{z\to 0}(w(z)/z)$.

The chief result in this direction that is well known is the following. Let

$$\tilde{D}(r) = \frac{1}{2\pi}\int_0^{2\pi} D(re^{i\theta})\, d\theta, \quad 0 < r < 1.$$

Then the following holds ([2], [6], [7], [8]).

THEOREM A. *If $w(z)$ is quasiconformal for $|z| < 1$, $w(0) = 0$, and if*

$$\int_0^1 \frac{\tilde{D}(r) - 1}{r}\, dr < \infty \tag{1.1}$$

then

$$\lim_{z\to 0} \frac{w(z)}{z} \tag{1.2}$$

exists and is different from 0.

Historically, the first complete proof of this result, using the modern definition of quasiconformality, appears to be Lehto's [6]. The proof that the hypothesis of the theorem implies convergence of the modulus, that is,

$$|w(z)| = c|z| + o(|z|) \qquad (c \neq 0), \text{ as } z \to 0\,, \tag{1.3}$$

is due to Teichmüller [7] and Wittich [8], using a somewhat narrower definition of quasiconformality than the present one. The very interesting result that the same hypothesis implies that

$$\lim_{z\to 0}\left[\arg \frac{w(z)}{z}\right] \text{ exists} \tag{1.4}$$

is due to Belinskiĭ [2] and Lehto [6].

Contributions to the following two questions will be made in the sections that follow.

(a) To what extent is (1.1) necessary for (1.2)?

(b) How can one separate those properties of $\kappa(z)$ which imply (1.3) from those properties which imply (1.4)?

Our approach will be to take advantage of knowledge provided by the specification of the complex dilatation $\kappa(z)$ (instead of merely the real dilatation $D(z)$), so as not to discard the data contained in $\arg \kappa(z)$(2). In this connection, we define the notion of ac(α) (asymptotically conformal on a logarithmic spiral of inclination α).

(2) We wish to point out that this idea is also exploited in the work of Andreian Cazacu [1].

A function $w(z)$, quasiconformal for $|z| < 1$, will be said to be ac(α) at $z = 0$ if

$$\int_0^1 \int_0^{2\pi} \left| \frac{|D_{\alpha+\theta} w(re^{i\theta})|^2}{J(re^{i\theta})} - 1 \right| \frac{d\theta dr}{r} < \infty, \tag{1.5}$$

where D_β is the directional derivative of w in the direction β. Since

$$\frac{|D_\beta w(z)|^2}{J(z)} = \frac{|1 + e^{-2i\beta}\kappa(z)|^2}{1 - |\kappa(z)|^2},$$

the condition (1.5) depends only on κ. If $w(z)$ is K-quasiconformal, then

$$\frac{1}{K} \leqq \frac{1}{D(z)} \leqq \frac{|1 + e^{-2i\beta}\kappa(z)|^2}{1 - |\kappa(z)|^2} \leqq D(z) \leqq K. \tag{1.6}$$

Hence

$$\left| \frac{|1 + e^{-2i\beta}\kappa(z)|^2}{1 - |\kappa(z)|^2} - 1 \right| \leqq [D(z) - 1].$$

Therefore, we see that (1.1) implies that $w(z)$ is ac(θ) for *all* θ, $0 \leqq \theta \leqq 2\pi$. In view of this fact, and also that if the integral (1.5) equalled *zero*, for say, $\alpha = 0$, and $\alpha = \pi/2$, then $w(z)$ would be *conformal* for $|z| < 1$, it appears natural to explore the consequences of assuming that $w(z)$ is simultaneously ac(0) and ac($\pi/2$). This is done in §§ 3 and 5.

2. Lemmas on moduli of quadrilaterals and rings.

LEMMA 2.1. *Suppose the function $w(z)$, quasiconformal in the rectangle $R = \{(x, y) \mid 0 \leqq x \leqq a, 0 \leqq y \leqq b\}$ has the following properties:*

(i) $|w(x + ib) - w(x)| \geqq b' > 0$, *for all* x, $0 \leqq x \leqq a$.

(ii) $|w(a + iy) - w(iy)| \geqq a' > 0$, *for all* y, $0 \leqq y \leqq b$.

(iii) *Area* $(w(R)) \leqq a'b'$.

Then, if $\kappa(x + iy)$ is the complex dilatation of w at $z = x + iy$,

$$\left[\frac{1}{a} \int_0^a \frac{dx}{\frac{1}{b}\int_0^b \frac{|1-\kappa|^2}{1-|\kappa|^2} dy} \right] \frac{a}{b} \leqq \frac{a'}{b'} \leqq \left[\frac{1}{b} \int_0^b \frac{dy}{\frac{1}{a}\int_0^a \frac{|1+\kappa|^2}{1-|\kappa|^2} dx} \right]^{-1} \frac{a}{b}. \tag{2.1}$$

Proof. For those x (that is, a.a. x), $0 \leqq x \leqq a$, for which $w(x + iy)$ is absolutely continuous in y,

$$b' \leqq \int_0^b |w_y(x + iy)|\, dy.$$

Now

$$\frac{1}{K} \leqq \frac{|w_y|^2}{J(z)} = \frac{|1 - \kappa|^2}{1 - |\kappa|^2} \leqq K, \qquad \text{a.e. in } R.$$

Hence, by Fubini's theorem, $\int_0^b(|w_y|^2/J)\,dy$ exists for a.a. x. Similarly, $\int_0^b J\,dy$ exists for a.a. x. Therefore, by Schwarz's inequality,

$$b'^2 \leqq \int_0^b \frac{|1-\kappa|^2}{1-|\kappa|^2}\,dy \int_0^b J\,dy \qquad \text{(a.a. } x\text{)}.$$

Solving for $\int_0^b J\,dy$, then integrating over $0 \leqq x \leqq b$, yields the left side of (2.1). The right side follows on interchanging the roles of x and y.

An inequality, weaker than (2.1), and to be used later, is obtained, by Schwarz's inequality, if the right side of (2.1) is replaced by

$$\frac{a'}{b'} \leqq \left[\frac{1}{ab}\iint_R \frac{|1+\kappa|^2}{1-|\kappa|^2}\,dxdy\right]\frac{a}{b}. \tag{2.2}$$

For the remainder of the section we assume that $w(z)$ $(z = re^{i\theta})$ is quasiconformal in the annulus $\mathfrak{A} = \{r_1 < r < r_2\}$, and has complex dilatation $\kappa = \kappa(re^{i\theta})$. We map $\mathfrak{A}$, slit along an arbitrary radius, $\theta = \beta$, onto a rectangle R in the $\sigma = \log(z/r_1)$ plane. The two sides of (2.1), evaluated in the σ plane, become, respectively, when expressed as integrals in the z plane,

$$(2\pi)^{-1}I(r_1,r_2) = \int_{r_1}^{r_2} \frac{dr/r}{\displaystyle\int_0^{2\pi} \frac{|1-e^{-2i\theta}\kappa|^2}{1-|\kappa|^2}\,d\theta},$$

and

$$(2\pi)^{-1}J(r_1,r_2) = \left[\int_0^{2\pi} \frac{d\theta}{\displaystyle\int_{r_1}^{r_2} \frac{|1+e^{-2i\theta}\kappa|^2}{1-|\kappa|^2}\,\frac{dr}{r}}\right]^{-1}.$$

The image, $\mathfrak{R} = w(\mathfrak{A})$, of $\mathfrak{A}$ under w, is a certain ring domain (doubly connected domain), whose modulus we denote by $M(r_1,r_2)$. Let $\hat{\mathfrak{A}}$ denote $\mathfrak{A}$ slit along the radius referred to above, and let $\hat{\mathfrak{R}} = w(\hat{\mathfrak{A}})$.

LEMMA 2.2. *The modulus of $\mathfrak{R}$ satisfies the inequality*

$$I(r_1,r_2) \leqq M(r_1,r_2) \leqq J(r_1,r_2). \tag{2.3}$$

Proof. Since the left and right sides of (2.3) depend on $w(z)$ only to the extent of depending on κ, we can assume (if necessary, after a conformal transformation) that $\mathfrak{R}$ is the annulus

$$1 < |w| < e^{M(r_1,r_2)}.$$

By introducing $\eta = \log w$ we map $\hat{\mathfrak{R}}$ onto a quadrilateral Q in the η plane bounded by two vertical segments of height 2π, and distance $M(r_1,r_2)$ apart. Since $\eta(w(z(\sigma)))$ has period $2\pi i$, Lemma 2.1 applies with $a = \log(r_2/r_1)$, $b = b' = 2\pi$, $a' = M(r_1,r_2)$. This yields (2.3).

The above estimate is closely related to a similar estimate for extremal length. If Γ is the family of locally rectifiable arcs in $\mathfrak{R}$ which join the boundary components of $\mathfrak{R}$, then the extremal length $\lambda(\Gamma)$ of Γ is, of course, $(2\pi)^{-1}M(r_1,r_2)$. Consider, however, the subset $\Gamma_\beta \subset \Gamma$ which consists of the curves of Γ lying entirely in $\hat{\mathfrak{R}}$. By the monotonicity property of extremal lengths, we have, clearly,

$$2\pi\lambda(\Gamma_\beta) \geqq 2\pi\lambda(\Gamma) = M(r_1,r_2) \geqq I(r_1,r_2).$$

But $2\pi\lambda(\Gamma_\beta)$ also turns out to satisfy the same bound that we have obtained for $M(r_1,r_2)$ from *above*:

$$I(r_1,r_2) \leqq 2\pi\lambda(\Gamma_\beta) \leqq J(r_1,r_2). \tag{2.4}$$

The proof of (2.4) is as follows. We map Q conformally onto a rectangle of dimensions a'', b'', corresponding to the sides of length a, b of R. Then $\lambda(\Gamma_\beta) = a''/b''$. We obtain (2.4) on noting that the mapping from R to R' satisfies the hypotheses of Lemma 2.1.

3. **Convergence of the modulus.** According to a result of Teichmüller [7], if $w(0) = 0$, and if there exists a function $\phi(r)$, $\lim_{r\to 0}\phi(r) = 0$, such that

$$\left|M(r_1,r_2) - \log\frac{r_2}{r_1}\right| \leqq \phi(r_2), \quad \text{whenever } r_1 < r_2, \tag{3.1}$$

then (1.3) holds. Hence, by Lemma 2.2, the hypothesis $w(0) = 0$,

$$\lim_{(r_1,r_2)\to(0,0)} \left[I(r_1,r_2) - \log\frac{r_2}{r_1}\right] = \lim_{(r_1,r_2)\to(0,0)} \left[J(r_1,r_2) - \log\frac{r_2}{r_1}\right] = 0 \tag{3.2}$$

implies (1.3). In fact, as may be verified by direct computation, for the special case when $w(z)$ is of the form

$$w(re^{i\theta}) = f(r)e^{ig(\theta)},$$

the relation (3.2) is both necessary and sufficient for (1.3). This is a consequence of the fact that both sides of (2.1) become equalities for functions of the form $F(x) + iG(y)$. For more general $w(z)$, however, (1.3) may hold, in spite of the fact that (3.2) is not satisfied. An example is

$$w(re^{i\theta}) = \begin{cases} re^{i(\theta - \log r)}, & 0 < r < 1, \\ 0, & r = 0. \end{cases}$$

In this case, $e^{-2i\theta}\,\kappa(re^{i\theta}) = (1+2i)^{-1}$ $(z \neq 0)$, $J(r_1,r_2) = 2\log(r_2/r_1)$, so that the second limit in (3.2) fails to exist.

The following sufficient condition for convergence of the modulus is somewhat weaker, but more intuitive than (3.2.)

THEOREM 3.1. *Suppose $w(z)$ is quasiconformal for $|z|<1$, $w(0)=0$, and $w(z)$ is* ac(0) *and* ac($\pi/2$) *at $z=0$. Then convergence of the modulus takes place, i.e.,* (1.3) *holds.*

Proof. In accordance with (2.2),

$$
\begin{aligned}
J(r_1,r_2)-\log\frac{r_2}{r_1} &\leqq J'(r_1,r_2)-\log\frac{r_2}{r_1} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_{r_1}^{r_2}\left[\frac{|1+e^{-2i\theta}\kappa|^2}{1-|\kappa|^2}-1\right]\frac{drd\theta}{r} \\
&\leqq \frac{1}{2\pi}\int_0^{2\pi}\int_0^{r_2}\left|\frac{|1+e^{-2i\theta}\kappa|^2}{1-|\kappa|^2}-1\right|\frac{drd\theta}{r}.
\end{aligned}
\tag{3.3}
$$

Let

$$
A(r)=\frac{1}{2\pi}\int_0^{2\pi}\frac{|1-e^{-2i\theta}\kappa|^2}{1-|\kappa|^2}d\theta.
$$

In view of (1.6),

$$
\begin{aligned}
I(r_1,r_2)-\log\frac{r_2}{r_1} &= \int_{r_1}^{r_2}\frac{1-A(r)}{A(r)}\frac{dr}{r} \\
&= \frac{1}{2\pi}\int_{r_1}^{r_2}\int_0^{2\pi}\left[\frac{|1-e^{-2i\theta}\kappa|^2}{1-|\kappa|^2}-1\right]\frac{d\theta dr}{rA(r)} \\
&\geqq -\frac{K}{2\pi}\int_0^{r_2}\int_0^{2\pi}\left|\frac{|1-e^{-2i\theta}\kappa|^2}{1-|\kappa|^2}-1\right|\frac{d\theta dr}{r}.
\end{aligned}
\tag{3.4}
$$

Therefore (3.1) is satisfied, with $\phi(r_2)$ the difference of the rightmost terms of (3.3) and (3.4).

4. **A class of mappings with given** $D(z)$. Let $\mathscr{F}$ be the class of measurable, real-valued, bounded functions, with lower bound at least 1, defined for $|z|<1$.

It is known [6, Theorem 4](3) that there exist $w(z)$, quasiconformal for $|z|<1$ for which (1.3) and (1.4) hold simultaneously, but for which

$$
\operatorname{ess}\overline{\lim_{z\to 0}}\,D(z)>1, \qquad \int_0^1\frac{\tilde{D}(r)-1}{r}dr=\infty.
$$

More generally, it would be of interest to characterize the subclass $\mathscr{F}_D$ of $\mathscr{F}$ which has the property that if $H\in\mathscr{F}_D$, then there exists a quasiconformal mapping

(3) Simpler examples also exist, and can in fact be provided by specializing the construction in the proof of Theorem 4.2.

of $|z| < 1$ with real dilatation H a.e. such that (1.3) and (1.4) hold simultaneously. Our conjecture, which is partially confirmed by the two theorems which follow, is that $\mathscr{F}_D = \mathscr{F}$.

THEOREM 4.1. *If $H \in \mathscr{F}$, and H satisfies a Hölder condition at $z = 0$ then there exists a quasiconformal mapping $w(z)$ with real dilatation H a.e., for which* (1.3) *holds.*

Proof. Given $H(z)$ ($|z| < 1$), we shall specify the complex dilatation $\kappa(z)$.

Let S_0, S_1 be disjoint sets with union $(0,1)$, to be described in more detail later. If $r \in S_k$, $(k = 0,1)$, we define

$$\kappa(re^{i\theta}) = (-1)^k e^{2i\theta} \frac{H(re^{i\theta}) - 1}{H(re^{i\theta}) + 1}, \qquad 0 \leqq \theta < 2\pi. \tag{4.1}$$

Let

$$\tilde{H}(r) = \frac{1}{2\pi} \int_0^{2\pi} H(re^{i\theta})\, d\theta, \quad \tilde{h}(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{H(re^{i\theta})},$$

and

$$v_k(r) = \begin{cases} \tilde{h}(r), & r \in S_k \\ \tilde{H}(r), & r \in S_{1-k} \end{cases} \qquad (k = 0,1).$$

This gives, after a short computation,

$$J'(r_1, r_2) - \log \frac{r_2}{r_1} = \int_{r_1}^{r_2} \frac{v_1(r) - 1}{r}\, dr,$$

and

$$J'(r_1, r_2) - I(r_1, r_2) = \int_{r_1}^{r_2} \frac{\tilde{H}(r)\tilde{h}(r) - 1}{r v_0(r)}\, dr.$$

Since the integrand is nonnegative,

$$J'(r_1, r_2) - I(r_1, r_2) \leqq \sup H(z) \int_{r_1}^{r_2} [\tilde{H}(r)\tilde{h}(r) - 1] \frac{dr}{r}.$$

Hence, by §3, a sufficient condition for (1.3) is that the two quantities

$$A = \lim_{r \to 0} \int_r^1 \frac{v_1(t) - 1}{t}\, dt \text{ and } B = \int_0^1 [\tilde{H}(r)\tilde{h}(r) - 1] \frac{dr}{r} \tag{4.2}$$

exist.

The fact that H is not merely in $\mathscr{F}$, but also satisfies a Hölder condition at $z = 0$, as postulated in the hypothesis, is used solely to insure the existence of B.

The assumption $|H(z)-H(0)| \leqq C_1|z|^h$ $(h>0)$, implies $|H(r)-H(0)| \leqq C_1 r^h$, $|\tilde{h}(r)-1/H(0)| \leqq C_2 r^h$. In view of the identity

$$\tilde{H}(r)\tilde{h}(r)-1=[\tilde{H}(r)-c][\tilde{h}(r)-c^{-1}]+c^{-1}[\tilde{H}(r)-c]+c[\tilde{h}(r)-c^{-1}] \tag{4.3}$$

valid for any $c \neq 0$, the existence of B follows. (Put $c=H(0)$.)

Next we show that S_0 can be chosen so as to force A to exist.

Since B exists, the integrals

$$\int_0^1 \frac{\tilde{H}(r)-1}{r}\,dr, \qquad \int_0^1 \frac{\tilde{h}(r)-1}{r}\,dr \tag{4.4}$$

converge and diverge together. (Consider (4.3), with c set equal to 1.) If they converge, A exists for any choice of S_0. If both integrals (4.4) diverge, and $\sum_{k=0}^{\infty}(-1)^k a_k$ $(0<a_{k+1}<a_k)$ is an arbitrary alternating conditionally convergent series, let $\{r_k\}$, $1=r_0>r_1>\cdots$, be determined, successively, in such a manner that

$$\int_{r_{k+1}}^{r_k} \frac{\tilde{H}(r)-1}{r}\,dr=a_k, \quad k=0,2,4,\cdots, \qquad \int_{r_{k+1}}^{r_k} \frac{1-\tilde{h}(r)}{r}\,dr=a_k, \quad k=1,3,5,\cdots.$$

Evidently, $\lim_{k\to\infty} r_k=0$. Defining S_0 as the union of the intervals (r_{k+1},r_k), $k=0,2,4,\cdots$, we obtain $A=\sum_{k=0}^{\infty}(-1)^k a_k$. This completes the proof.

THEOREM 4.2. *Suppose $F(r)$ is an arbitrary bounded measurable function, $F(r)\geqq 1$ $(0<r<1)$. There exists a quasiconformal mapping $w(z)$ $(|z|<1)$, with real dilatation $D(z)$ satisfying*

$$D(re^{i\theta})=F(r) \quad a.e.$$

for which (1.3) *and* (1.4) *hold simultaneously.*

Proof. We define $H(re^{i\theta})=F(r)$ $(0<r<1,\ 0\leqq\theta<2\pi)$, and construct $\kappa(z)$, $v_1(r)$, and A as in the previous proof. As may be verified, directly, the function

$$w(re^{i\theta})=\exp\left[-\int_r^1 \frac{v_1(t)}{t}\,dt+i\theta\right]$$

defines a quasiconformal mapping of $|z|<1$ with real dilatation $D(re^{i\theta})=F(r)$ a.e., and

$$\lim_{z\to 0}\frac{w(z)}{z}=e^{-A}.$$

5. **Behavior of** $\arg[w(z)/z]$. We have seen that if $w(0)=0$, and $w(z)$ is ac(0) and ac$(\pi/2)$ then (1.3) holds. We will now see that the same hypotheses do *not* imply (1.4). On the contrary, $\arg w(re^{i\beta})$ may go to infinity as $r\to 0$, for fixed β.

In this section we will obtain a sharp estimate of the worst rate at which this can occur.

First we need an estimate of the effect of "obstacles" on extremal length. The situation we need to consider is the following. Suppose A is the annulus $\{\rho < |z| < \rho e^{\alpha}\}$ $(\alpha > 0)$. Let $\Omega \subset \bar{A}$ be a connected set (the "obstacle") on which there is a single-valued, continuous determination of $\arg z$. We wish to compare the following two measures of the size of Ω.

(i) $h(\Omega)$, the diameter of the range of $\arg z$ on Ω,

(ii) $\lambda(\Gamma_\Omega)$, the extremal length of the family Γ_Ω of curves in A, joining $\{|z| = \rho\}$ to $\{|z| = \rho e^{\alpha}\}$, without intersecting Ω.

LEMMA 5.1. *Suppose* $2\pi\lambda(\Gamma_\Omega) \leqq \alpha + \delta$ *for some* $\delta > 0$. *Then*(4)

$$h(\Omega) \leqq F(\alpha, \delta) = \begin{cases} \text{solution of (5.3) for } h \text{ as a function of } \alpha, \delta, \text{ if } 0 \leqq h(\Omega) \leqq 6\pi. \\ 3\sqrt{(\alpha\delta)}, \quad \text{if} \quad h(\Omega) \geqq 6\pi. \end{cases}$$

Proof. Choose θ_0 so that the range of $\arg z$ on Ω covers the interval $(\theta_0, \theta_0 + h)$. Suppose, first, that $h \leqq 6\pi$. Consider the sector

$$S = \left\{ z \,\middle|\, \theta_0 + \frac{h}{3} \leqq \arg z \leqq \theta_0 + \frac{2h}{3}, \quad \rho \leqq |z| \leqq \rho e^{\alpha} \right\}.$$

For any $\gamma \in \Gamma_\Omega$ we have

$$\int_\gamma d|\log z| \geqq \alpha. \tag{5.1}$$

If $\gamma \cap S \neq \varnothing$ we obtain a sharper estimate, because in this case γ must detour through an angle of at least $h/3$ in order to intersect Ω; that is,

$$\int_\gamma d|\log z| \geqq \sqrt{(\alpha^2 + (h/3)^2)}, \quad \text{if} \quad \gamma \cap S \neq \varnothing. \tag{5.2}$$

Let us introduce the following metric in A.

$$\rho^*(z) = \begin{cases} \dfrac{1}{|z|}, & z \in A - S, \\ \left(1 + \dfrac{h^2}{9\alpha^2}\right)^{-1/2} \cdot \dfrac{1}{|z|}, & z \in S. \end{cases}$$

By (5.1), (5.2),

$$\inf_{\gamma \in \Gamma_\Omega} \int_\gamma \rho^*(z)|dz| \geqq \alpha.$$

(4) We shall need Lemma 5.1 only for the case $h(\Omega) \geqq 6\pi$, but have not restricted ourselves to this case to suggest that the problem of finding the best inequality for arbitrary h is of some interest. It may be conjectured that the extremal Ω is an arc of a spiral, depending continuously on $h(\Omega)$.

On the other hand,

$$\iint_A [\rho^*(z)]^2 |dz|^2 = \iint_{A-S} \frac{dr d\theta}{r} + \left(1 + \frac{h^2}{9\alpha^2}\right)^{-1} \iint_S \frac{dr d\theta}{r}$$

$$= \left(2\pi - \frac{h}{3}\right)\alpha + \left(1 + \frac{h^2}{9\alpha^2}\right)^{-1} \frac{h}{3}\alpha$$

$$= 2\pi\alpha - \frac{h\alpha}{3}\left[1 - \left(1 + \frac{h^2}{9\alpha^2}\right)^{-1}\right].$$

Therefore,

$$\frac{1}{\lambda} \leqq \frac{2\pi}{\alpha} - \frac{h}{3\alpha}\left[1 - \left(1 + \frac{h^2}{9\alpha^2}\right)^{-1}\right].$$

Hence, by hypothesis,

$$(5.3) \quad 2\pi\alpha(\alpha + \delta)^{-1} \leqq 2\pi - \frac{h}{3}\left[1 - \left(1 + \frac{h^2}{9\alpha^2}\right)^{-1}\right], \quad \text{if}^{(5)}\ h \leqq 6\pi.$$

If $h \geqq 6\pi$, we take $\rho^*(z) = 1/|z|$ in A. As a crude estimate, for any $\gamma \in \Gamma_\Omega$,

$$\int_\gamma d|\log z| \geqq \sqrt{(\alpha^2 + (h/3)^2)}.$$

Obviously,

$$\iint_A [\rho^*(z)]^2 |dz|^2 = 2\pi\alpha.$$

Therefore, if $h \geqq 6\pi$,

$$\frac{\alpha + \delta}{2\pi} \geqq \lambda \geqq \frac{\alpha^2 + (h/3)^2}{2\pi\alpha} = \frac{\alpha}{2\pi} + \frac{h^2}{18\pi\alpha}.$$

This completes the proof.

THEOREM 5.1. *Suppose $w(z)$ is quasiconformal for $|z| < 1$, $w(0) = 0$, and $w(z)$ is* ac(0) *and* ac($\pi/2$) *at $z = 0$. Then, for every fixed $\theta, 0 \leqq \theta < 2\pi$,*

$$\arg w(re^{i\theta}) = o(\sqrt{(-\log r)}), \qquad \textit{as } r \to 0.$$

Proof. As in §2, we consider $w(z)$ in the annulus $\mathfrak{A}(r_1, r_2) = \{r_1 < |z| < r_2\}$, $0 < r_1 < r_2 < 1$, and let $\mathfrak{R}(r_1, r_2) = w(\mathfrak{A}(r_1, r_2))$, $M(r_1, r_2) =$ modulus of $\mathfrak{R}(r_1, r_2)$.

For fixed β, $0 \leqq \beta < 2\pi$, let

$$\Omega(r_1, r_2) = \{w \mid w = w(z),\ \arg z = \beta,\ r_1 < |z| < r_2\}.$$

Also, let $\lambda(r_1, r_2)$ be the extremal length of the family of curves in $\mathfrak{R}(r_1, r_2)$ joining the boundary components of $\mathfrak{R}(r_1, r_2)$, and not intersecting $\Omega(r_1, r_2)$. The quantity

(5) Equation (5.3) implies that $h(\Omega) = O(\delta^{1/3})$, as $\delta \to 0$ (α fixed).

$\lambda(r_1, r_2)$ is clearly identical with the quantity denoted by $\lambda(\Gamma_\beta)$ in §2, and satisfies (2.4).

Furthermore, let

$$r_1'(r_1, r_2) = \inf_{w \in \Omega(r_1, r_2)} |w|, \qquad r_2'(r_1, r_2) = \sup_{w \in \Omega(r_1, r_2)} |w|,$$

$$\rho_1(r_1, r_2) = \inf\{|z| \,|\, |w(z)| = r_1'(r_1, r_2)\}, \quad \rho_2(r_1, r_2) = \sup\{|z| \,|\, |w(z)| = r_2'(r_1, r_2)\}.$$

We also denote by $\mathfrak{A}'(r_1, r_2)$ the annulus $\{r_1'(r_1, r_2) < |w| < r_2'(r_1, r_2)\}$, and by $\lambda'(r_1, r_2)$ the extremal length of the family of curves in $\mathfrak{A}'(r_1, r_2)$ joining the boundary components of $\mathfrak{A}'(r_1, r_2)$ without intersecting $\Omega(r_1, r_2)$. Note that, by Lemma 5.1,

$$\begin{aligned} &|\arg w(r_2 e^{i\beta}) - \arg w(r_1 e^{i\beta})| \\ &\leqq h(\Omega) \leqq \max\left\{6\pi, 3\sqrt{\left(\left(\log\frac{r_2'}{r_1'}\left[2\pi\lambda'(r_1, r_2) - \log\frac{r_2'}{r_1'}\right]\right)\right)}\right\}. \end{aligned} \tag{5.4}$$

Since $\Omega(r_1, r_2) \subset \mathfrak{A}'(r_1, r_2) \subset \mathfrak{R}(\rho_1, \rho_2)$, we conclude that $\rho_1 \leqq r_1$, $\rho_2 \geqq r_2$, $\Omega(\rho_1, \rho_2) \supset \Omega(r_1, r_2)$. Hence every curve in $\mathfrak{R}(\rho_1, \rho_2)$, joining the boundary components of $\mathfrak{R}(\rho_1, \rho_2)$ without touching $\Omega(\rho_1, \rho_2)$, contains an arc in $\mathfrak{A}'(r_1, r_2)$ which joins the boundary components of $\mathfrak{A}'(r_1, r_2)$ without touching $\Omega(r_1, r_2)$. It follows that

$$\lambda(\rho_1, \rho_2) \geqq \lambda'(r_1, r_2),$$

and, therefore, by (2.4),

$$2\pi\lambda'(r_1, r_2) \leqq J(\rho_1, \rho_2). \tag{5.5}$$

If r_2 is small, then, by Theorem 3.1, the image of $\{|z| = \rho_k\}$ under $w(z)$ is very close to the circle $\{|w| = r_k'\}$, $k = 1, 2$. More precisely, given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$\log\frac{r_2'}{r_1'} \geqq M(\rho_1, \rho_2) - \varepsilon, \quad \text{whenever } 0 < r_1 < r_2 \leqq \delta(\varepsilon).$$

Therefore, by (2.3), and (5.5),

$$2\pi\lambda'(r_1, r_2) - \log\frac{r_2'}{r_1'} \leqq \varepsilon + J(\rho_1, \rho_2) - I(\rho_1, \rho_2), \quad \text{if } 0 < r_1 < r_2 \leqq \delta(\varepsilon).$$

Hence (consider (3.3) minus (3.4)), there exists a $\delta_1(\varepsilon) > 0$, such that

$$2\pi\lambda'(r_1, r_2) - \log\frac{r_2'}{r_1'} \leqq 2\varepsilon, \quad \text{if } 0 < r_1 < r_2 < \delta_1(\varepsilon). \tag{5.6}$$

Again, by Theorem 3.1, there exists a $\delta_2(\varepsilon) > 0$, such that

$$\log\frac{r_2'}{r_1'} \leqq \log\frac{r_2}{r_1} + \varepsilon, \qquad \text{if} \quad 0 < r_1 < r_2 < \delta_2(\varepsilon). \tag{5.7}$$

Substituting (5.6), (5.7) into (5.4), we find that

$$\left|\arg w(r_2 e^{i\beta}) - \arg w(r_1 e^{i\beta})\right| \leqq \max\left\{6\pi, 3\sqrt{\left(2\varepsilon\left(\varepsilon + \log\frac{r_2}{r_1}\right)\right)}\right\}, \qquad 0 < r_1 < r_2 < \delta'(\varepsilon).$$

Letting $r_1 \to 0$, for fixed r_2, we obtain the desired conclusion.

THEOREM 5.2. *The estimate of Theorem* 5.1 *is sharp in the sense that, given* $\varepsilon > 0$, β, *there exists a* $w(z)$, *quasiconformal for* $|z| < 1$, *and* ac(0), *and* ac($\pi/2$), *such that* $\arg w(re^{i\beta})$ *is not* $o[(-\log r)^{1/2-\varepsilon}]$, *as* $r \to 0$.

Proof. We can choose $w(z)$ independently of β. Namely, let

$$w(re^{i\theta}) = \begin{cases} re^{i(\theta+\psi(r))}, & 0 < r < 1, \\ 0, & r = 0, \end{cases}$$

where

$$\psi(r) = \left[\log\frac{2}{r}\right]^{1/2-\varepsilon}.$$

We find that

$$e^{-2i\theta}\kappa(re^{i\theta}) = \frac{ir\psi'(r)}{2 + ir\psi'(r)}, \qquad 0 < r < 1. \tag{5.8}$$

Since $r\psi'(r)$ is bounded, $\sup_{|z|<1}|\kappa(z)| < 1$.

By (5.8),

$$\begin{aligned} \frac{|1 + e^{-2i\theta}\kappa(re^{i\theta})|^2}{1 - |\kappa(re^{i\theta})|^2} &= 1 + r^2[\psi'(r)]^2 \\ \frac{|1 - e^{-2i\theta}\kappa(re^{i\theta})|^2}{1 - |\kappa(re^{i\theta})|^2} &= 1 \end{aligned} \qquad (0 < r < 1).$$

Thus, $w(z)$ is trivially ac($\pi/2$). The fact that $w(z)$ is ac(0) is due to the fact that $\int_0^1 r[\psi'(r)]^2 dr$ converges.

As a concluding consideration, let us suppose that the hypothesis of Theorem 5.1 holds, and that, in addition, $w(z)$ is asymptotically conformal in a *third* direction at $z = 0$; that is, assume that for some β_0 (not a multiple of $\pi/2$),

$$w(z) \text{ is ac}(0), \text{ac}(\pi/2), \text{ and ac}(\beta_0). \tag{5.9}$$

It is natural to ask whether we can now prove that (1.2) holds. The answer is *affirmative*. This is a trivial consequence of the following observation: *Condition* (5.9) *implies condition* (1.1). To see this, we note that

$$\left|\frac{|1+e^{-2i\beta}\kappa|^2}{1-|\kappa|^2}-1\right| = \frac{2}{1-|\kappa|^2}\left|\,|\kappa|^2+\mathrm{Re}(e^{-2i\beta}\kappa)\right|.$$

Therefore, (5.9) implies that

$$\int_0^1\int_0^{2\pi}\left|\,|\kappa|^2+\mathrm{Re}(e^{-2i\theta}\kappa)\right|\frac{d\theta dr}{r}<\infty, \tag{5.10a}$$

$$\int_0^1\int_0^{2\pi}\left|\,|\kappa|^2-\mathrm{Re}(e^{-2i\theta}\kappa)\right|\frac{d\theta dr}{r}<\infty, \tag{5.10b}$$

and

$$\int_0^1\int_0^{2\pi}\left|\,|\kappa|^2+\mathrm{Re}(e^{-2i\theta-2i\beta_0}\kappa)\right|\frac{d\theta dr}{r}<\infty. \tag{5.10c}$$

Hence,

$$\int_0^1\int_0^{2\pi}|\kappa|^2\,\frac{d\theta dr}{r}<\infty, \tag{5.11}$$

and

$$\int_0^1\int_0^{2\pi}|\mathrm{Re}(e^{-2i\theta}\kappa)|\frac{d\theta dr}{r}<\infty,\quad \int_0^1\int_0^{2\pi}|\mathrm{Re}(e^{-2i\theta-2i\beta_0}\kappa)|\,\frac{d\theta dr}{r}<\infty. \tag{5.12}$$

For any complex number Z, and real t, we have

$$Z\sin t = -ie^{it}\mathrm{Re}(Z)+i\mathrm{Re}(e^{-it}Z),$$

so that

$$|Z||\sin t|\leqq|\mathrm{Re}(Z)|+|\mathrm{Re}(e^{-it}Z)|.$$

Applying this with $Z=e^{-2i\theta}\kappa(re^{i\theta})$, $t=2\beta_0$, we deduce, using (5.12), that

$$\int_0^1\int_0^{2\pi}|\kappa|\,\frac{d\theta dr}{r}<\infty.$$

This is equivalent to (1.1).

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