ON SPHERICAL BESSEL FUNCTIONS

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In [1] solutions were obtained for certain second order linear differential equations with polynomial coefficients in terms of generalized Rodrigues' formulas and iterated indefinite integrals. The purpose of this paper is to apply these results to the Bessel equation. Since the results of [1] are soon to be published [2], only special cases which apply directly to the present situation will be discussed here (see also [3]).

For the present situation it suffices to consider the differential equation

$$zy'' + (az + 2m)y' + amy = 0, \tag{1}$$

where z is a complex variable, $a \neq 0$ is a complex constant, and m is an integer.

THEOREM 1. *If m is a positive integer then* (1) *is satisfied by*

$$y_1(z) = \frac{d^{m-1}}{dz^{m-1}}\{z^{-m}e^{-az}\} \tag{2}$$

and

$$y_2(z) = \frac{d^{m-1}}{dz^{m-1}}\left\{z^{-m}e^{-az}\int_{z_0}^{z} t^{2m-2}e^{at}\,dt\right\}, \tag{3}$$

where z_0 is any point (finite or infinite) for which the indicated integration and differentiation is possible.

Proof. First note that since the integrand in (3) is holomorphic in the finite plane the path of integration is arbitrary. Theorem 1 follows from differentiating m times the first order equation

$$zw' + (az + m)w = kz^{m-1} \tag{4}$$

and the particular solutions

$$w_1(z) = z^{-m}e^{-az}, \quad w_2(z) = z^{-m}e^{-az}\int_{z_0}^{z} t^{2m-2}e^{at}\,dt$$

corresponding respectively to $k = 0$ and $k = 1$.

THEOREM 2. *If m is a nonpositive integer then every solution $y(z)$ of* (1) *has the property*

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$$y^{(-m+1)} = cz^{-m}e^{-az} \tag{5}$$

for some constant c.

Proof. Differentiation of (1) $-m$ times shows that $y^{(-m+1)}$ satisfies (4) when $k = 0$.

Many interesting results can be obtained by applying the preceding theorems to a transformed Bessel equation. Throughout the following discussion n indicates a nonnegative integer, and all nonintegral powers of complex numbers are assumed to have principal values with arguments θ taken in $0 \leqq \theta < 2\pi$.

In the Bessel equation

$$z^2u'' + zu' + [z^2 - (n + 1/2)^2]u = 0, \tag{6}$$

let

$$u = z^{n+1/2}e^{iz}v(z). \tag{7}$$

This gives

$$zv'' + [2iz + 2(n + 1)]v' + 2i(n + 1)v = 0. \tag{8}$$

If, instead of (7), the transformation is

$$u = z^{-n-1/2}e^{iz}w(z), \tag{9}$$

the resulting differential equation is

$$zw'' + [2iz - 2n]w' - 2inw = 0. \tag{10}$$

Both (8) and (10) are of the form (1) so that Theorems 1 and 2 give the following result immediately.

THEOREM 3. *The Bessel equation* (6) *is satisfied by*

$$u_1(z) = e^{iz}z^{n+1/2}\frac{d^n}{dz^n}\{z^{-n-1}e^{-2iz}\}, \tag{11}$$

$$u_2(z) = e^{iz}z^{n+1/2}\frac{d^n}{dz^n}\left\{z^{-n-1}e^{-2iz}\int_0^z t^{2n}e^{2it}\,dt\right\}, \tag{12}$$

$$u_3(z) = e^{iz}z^{n+1/2}\frac{d^n}{dz^n}\left\{z^{-n-1}e^{-2iz}\int_{i\infty}^z t^{2n}e^{2it}\,dt\right\}, \tag{13}$$

and every solution $u(z)$ of (6) *has the property that for some constant c*

$$\frac{d^{n+1}}{dz^{n+1}}\{z^{n+1/2}e^{-iz}u(z)\} = cz^ne^{-2iz}. \tag{14}$$

In (12) *and* (13) *the path of integration is arbitrary.*

It is well known (e.g. [4]) that two linearly independent solutions of (7) are $J_{n+1/2}(z)$ and $J_{-n-1/2}(z)$, where

$$J_u(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{2r+u}}{r!\,\Gamma(u+r+1)}. \tag{15}$$

THEOREM 4.

$$J_{n+1/2}(z) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{2^n}{(2n)!} u_2(z). \tag{16}$$

Proof. From (12),

$$\begin{aligned} u_2(z) &= [z^{n+1/2} + o(z^{n+1/2})] \frac{d^n}{dz^n} \{[z^{-n-1} + o(z^{-n-1})] \\ &\qquad \cdot [z^{2n+1}/(2n+1) + o(z^{2n+1})]\} \\ &= [z^{n+1/2} + o(z^{n+1/2})] \frac{d^n}{dz^n} [z^n/(2n+1) + o(z^n)] \\ &= n!\, z^{n+1/2}/(2n+1) + o(z^{n+1/2}). \end{aligned}$$

So

$$\lim_{z \to 0} z^{-n-1/2} u_2(z) = n!/(2n+1).$$

Then $u_2(z)$ is well defined at $z = 0$ and satisfies (6) so there is a constant A such that $u_2(z) = AJ_{n+1/2}(z)$. Multiplying both members of this equality by $z^{-n-1/2}$ and letting $z \to 0$ gives, with the aid of (15), $n!/(2n+1) = A2^{-n-1/2}/\Gamma(n+3/2)$. So $A = n!\,2^{n-1/2}\,\Gamma(n+1/2)$ and (16) follows by using the duplication formula for the gamma function.

THEOREM 5.

$$J_{-n-1/2}(z) - (-1)^{n+1/2} J_{n+1/2}(z) = \sqrt{\left(\frac{2}{\pi}\right)} 2^{-n} u_1(z). \tag{17}$$

Proof. From (11),

$$\begin{aligned} u_1(z) &= e^{iz} z^{n+1/2} \frac{d^n}{dz^n} \{z^{-n-1} + o(z^{-n-1})\} \\ &= e^{iz} z^{n+1/2} [(-1)^n (2n)!\, z^{-2n-1}/n! + o(z^{-2n-1})] \\ &= (-1)^n (2n)!\, z^{-n-1/2}/n! + o(z^{-n-1/2}). \end{aligned}$$

So

$$\lim_{z \to 0} z^{n+1/2} u_1(z) = (-1)^n (2n)!/n!.$$

Since $u_1(z)$ satisfies (6) there exist constants A and B such that

$$u_1(z) = AJ_{n+1/2}(z) + BJ_{-n-1/2}(z).$$

Multiplying both members of this equality by $z^{n+1/2}$ and letting $z \to 0$ gives $(-1)^n(2n)!/n! = B2^{n+1/2}/\Gamma(1/2 - n)$. Using the relation $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$ and the duplication formula for the gamma function gives $B = 2^n\sqrt{(\pi/2)}$, so

$$u_1(z) = AJ_{n+1/2}(z) + \sqrt{\left(\frac{\pi}{2}\right)}\, 2^n J_{-n-1/2}(z)\,. \tag{18}$$

To evaluate A multiply both members of (18) by $z^{-n-1/2}$ and then equate constant terms in the Laurent series. Since $z^{-n-1/2}J_{-n-1/2}(z)$ is an odd function there is no constant term in its Laurent series. Also

$$z^{-n-1/2}J_{n+1/2}(z) = 2^{-n-1/2}/\Gamma(n + 3/2) + o(1)$$

so using the right member of (18), the constant term in the Laurent series for $z^{-n-1/2}u_1(z)$ is $A2^{-n-1/2}/\Gamma(n + 3/2)$. Using (11),

$$z^{-n-1/2}u_1(z) = e^{iz}\sum_{j=0}^{n} \frac{(-2i)^j(2n-j)!(-1)^n z^{j-2n-1}}{j!(n-j)!}$$
$$+ e^{iz}\sum_{j=n}^{\infty} \frac{(-2i)^{j+n+1}\, z^{j-n}}{(j-n)!(j+n+1)!}.$$

The second term on the right has only one constant term while the first term has constants for each term in the series for e^{iz} having powers of z from $n+1$ to $2n+1$ inclusively. Removing these constants and equating to the result obtained using the right member of (18) gives

$$\frac{A2^{-n-1/2}}{\Gamma(n+3/2)} = \sum_{j=0}^{n} \frac{i(-2)^j}{j!(n-j)!(2n-j+1)} + \frac{i(-1)^{n+1}2^{2n+1}n!}{(2n+1)!}$$
$$= \frac{i2^{2n+1}}{n!}\int_{-1/2}^{0} x^n(1+x)^n\,dx + \frac{i(-1)^{n+1}2^{2n+1}\,n!}{(2n+1)!}.$$

Now in the integral put $x = -\sin^2(u/2)$, $0 \leqq u \leqq \pi/2$. Then

$$\frac{A2^{-n-1/2}}{\Gamma(n+3/2)} = \frac{(-1)^{n+1/2}}{n!}\int_0^{\pi/2}(\sin u)^{2n+1}du - \frac{(-1)^{n+1/2}2^{2n+1}n!}{(2n+1)!}$$
$$= \frac{\sqrt{(\pi)}(-1)^{n+1/2}}{2\Gamma(n+3/2)} - \frac{(-1)^{n+1/2}2^{2n+1}n!}{(2n+1)!}.$$

Thus

$$A = (-2)^{n+1/2}\left[\frac{\sqrt{(\pi)}}{2} - \frac{2^{2n+1}\,\Gamma(n+1)\Gamma(n+3/2)}{\Gamma(2n+2)}\right]$$

and, applying the duplication formula for the gamma function,

$$A = -(-1)^{n+1/2}2^n\sqrt{(\pi/2)}.$$

Substitution of this result into (18) completes the proof.

THEOREM 6.

$$J_{n+1/2}(z) - (-1)^{n+1/2}J_{-n-1/2}(z) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{2^{n+1}}{(2n)!}u_3(z). \tag{19}$$

Proof. Since the path of integration is arbitrary in (12) and (13) it is evident that

$$u_3(z) = u_2(z) + u_1(z)\int_{i\infty}^{0} t^{2n}e^{2it}\,dt. \tag{20}$$

But

$$\begin{aligned}\int_{i\infty}^{0} t^{2n}e^{2it}dt &= i(-1)^{n+1}\int_{0}^{\infty} u^{2n}e^{-2u}du \\ &= -(-1)^{n+1/2}2^{-2n-1}(2n)\,!.\end{aligned}$$

When this result along with (16) and (17) is substituted into (20) the conclusion is immediate.

Since $J_u(-z) = (-1)^u J_u(z)$, (17) and (19) can be written in the following form.

COROLLARY.

$$J_{-n-1/2}(z) - J_{n+1/2}(-z) = \sqrt{\left(\frac{2}{\pi}\right)}2^{-n}u_1(z), \tag{21}$$

$$J_{n+1/2}(z) + J_{-n-1/2}(-z) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{2^{n+1}}{(2n)!}u_3(z). \tag{22}$$

The result of Theorem 5 is sufficient to give the following Rodrigues type formulas for spherical Bessel functions.

THEOREM 7.

$$J_{n+1/2}(z) = 2\sqrt{\left(\frac{2}{\pi z}\right)}\frac{d^n}{da^n}\left\{\left(\frac{-z}{2a}\right)^{n+1}\sin(z-2a)\right\}_{a=z}, \tag{23}$$

$$J_{-n-1/2}(z) = 2\sqrt{\left(\frac{2}{\pi z}\right)}\frac{d^n}{da^n}\left\{\left(\frac{z}{2a}\right)^{n+1}\cos(z-2a)\right\}_{a=z}. \tag{24}$$

Proof. Both $J_{-n-1/2}(z)$ and $J_{n+1/2}(z)$ are real for real positive z, so suppose $z = x$ is real and positive and equate real and imaginary parts in (17). This gives (23) and (24) for $z = x$. Since $z^{1/2}J_{n+1/2}(z)$ is holomorphic for all z and $z^{1/2}J_{-n-1/2}(z)$ is holomorphic except at $z = 0$ the identity principle for holomorphic functions [5] assures the validity of (23) and (24) for all z except $z = 0$ in (24).

THEOREM 8.

$$(25)\qquad J_{n+1/2}(z) = \sqrt{\left(\frac{2z}{\pi}\right)}\frac{(2z)^n}{(2n)!}e^{iz}\frac{d^n}{dz^n}\left\{z^n\int_0^1 (1-u)^{2n}e^{-2izu}\,du\right\},$$

$$(26)\quad J_{n+1/2}(z)\cos wz = \sqrt{\left(\frac{2z}{\pi}\right)}\frac{(2z)^n}{(2n)!}\frac{d^n}{dz^n}\left\{z^n\int_0^1 (1-u)^{2n}\cos(1+w-2u)z\,du\right\},$$

$$(27)\quad J_{n+1/2}(z)\sin wz = \sqrt{\left(\frac{2z}{\pi}\right)}\frac{(2z)^n}{(2n)!}\frac{d^n}{dz^n}\left\{z^n\int_0^1 (1-u)^{2n}\sin(1+w-2u)z\,du\right\}.$$

Proof. To obtain (25) combine (16) and (12) and put $t = (1-u)z$ in the integral. Suppose then that z and w are real and positive, multiply (25) by e^{iwz} and equate real and imaginary parts. This gives (26) and (27) for real positive z and w. Repeated application of the identity principle gives (26) and (27) for all complex z and w.

Some interesting integral identities can be obtained from (25)–(27) by multiplying both members of each equation by $z^{-n-1/2}$ and appealing to Cauchy's formula for repeated integrals, viz.

$$(28)\qquad \int_0^z\int_0^t\cdots\int_0^t f(t)\,(dt)^{p+1} = \frac{1}{p!}\int_0^z (z-t)^p f(t)\,dt$$

$$= \frac{z^{p+1}}{p!}\int_0^1 (1-u)^p f(uz)\,du.$$

For example, (25) and (28) yield, for $n \geqq 1$,

$$\int_0^1 (1-u)^{n-1}e^{-iuz}(uz)^{-n-1/2}\,J_{n+1/2}(uz)\,du$$

$$= \sqrt{\left(\frac{2}{\pi}\right)}\frac{2^n(n-1)!}{(2n)!}\int_0^1 (1-u)^{2n}e^{-2iuz}\,du.$$

Comparable results follow from (26) and (27).

The results obtained thus far are all dependent on (11)–(13) in Theorem 3. Numerous other interesting identities follow from (14).

THEOREM 8.

$$(29)\qquad \frac{d^{n+1}}{dz^{n+1}}\{z^{n+1/2}e^{-iz}J_{n+1/2}(z)\} = \sqrt{\left(\frac{2}{\pi}\right)}(2z)^n e^{-2iz},$$

$$(30)\qquad \frac{d^{n+1}}{dz^{n+1}}\{z^{n+1/2}\cos z\,J_{n+1/2}(z)\} = \sqrt{\left(\frac{2}{\pi}\right)}(2z)^n\cos 2z,$$

$$(31)\qquad \frac{d^{n+1}}{dz^{n+1}}\{z^{n+1/2}\sin z\,J_{n+1/2}(z)\} = \sqrt{\left(\frac{2}{\pi}\right)}(2z)^n\sin 2z.$$

Proof. To obtain (29) put $u(z) = J_{n+1/2}(z)$ in (14) and equate coefficients of z^n. This gives

$$c = z^{-n}\frac{d^{n+1}}{dz^{n+1}}\left\{\frac{(z^2/2)^{n+1/2}}{\Gamma(n+3/2)}\right\} = \frac{(2n+1)!\,2^{-n-1/2}}{n!\,\Gamma(n+3/2)} = \sqrt{\left(\frac{2}{\pi}\right)}2^n$$

and verifies (29). If z is real, (30) and (31) follow from (29) by equating real and imaginary parts. Their validity for all complex z follows from the identity principle.

COROLLARY.

$$J_{n+1/2}(z) = \frac{(2z)^{n+1/2}e^{iz}}{\sqrt{(\pi)}n!}\int_0^1 t^n(1-t)^n e^{-2izt}\,dt, \tag{32}$$

$$J_{n+1/2}(z)\cos wz = \frac{(2z)^{n+1/2}}{\sqrt{(\pi)}n!}\int_0^1 t^n(1-t)^n\cos(1+w-2t)z\,dt, \tag{33}$$

$$J_{n+1/2}(z)\sin wz = \frac{(2z)^{n+1/2}}{\sqrt{(\pi)}n!}\int_0^1 t^n(1-t)^n\sin(1+w-2t)z\,dt. \tag{34}$$

Proof. To obtain (32) apply (28) to (29). For real w and z multiply (32) by e^{iwz} and equate real and imaginary parts. This gives (33) and (34) for real z and w and, as in previous proofs, the identity principle assures their validity for all complex z and w

THEOREM 9.

$$\frac{d^{n+1}}{dz^{n+1}}\{z^{n+1/2}e^{-iz}J_{-n-1/2}(z)\} = -i\sqrt{\left(\frac{2}{\pi}\right)}(-2z)^n e^{-2iz}, \tag{35}$$

$$\frac{d^{n+1}}{dz^{n+1}}\{z^{n+1/2}\cos z\,J_{-n-1/2}(z)\} = -\sqrt{\left(\frac{2}{\pi}\right)}(-2z)^n\sin 2z, \tag{36}$$

$$\frac{d^{n+1}}{dz^{n+1}}\{z^{n+1/2}\sin z\,J_{-n-1/2}(z)\} = \sqrt{\left(\frac{2}{\pi}\right)}(-2z)^n\cos 2z. \tag{37}$$

Proof. To obtain (35) put $u(z) = J_{-n-1/2}(z)$ in (14) and equate coefficients of z^n. This gives

$$c = z^{-n}\frac{d^{n+1}}{dz^{n+1}}\sum_{r=0}^{n}\frac{2^{n+1/2}(-1)^r(z/2)^{2r}(-iz)^{2n+1-2r}}{r!\,\Gamma(r+1-n-1/2)(2n+1-2r)!}.$$

Performing the differentiation and using the relation

$$\Gamma(r-n+1/2)\Gamma(n-r+1/2) = \pi(-1)^{r-n}$$

yields

$$c = -\frac{i}{\pi}\frac{(2n+1)!}{n!}2^{n+1/2}\sum_{r=0}^{n}\frac{(-1)^r\Gamma(n-r+1/2)2^{-2r}}{r!\,(2n-2r+1)\Gamma(2n-2r+1)}.$$

The gamma duplication formula puts the sum in a recognizable form

$$c = -\sqrt{\left(\frac{2}{\pi}\right)}\frac{i(2n+1)!}{n!\,n!\,2^n}\int_0^1 (u^2-1)^n\,du = i(-1)^{n+1}2^{n+1/2}/\sqrt{\pi},$$

and (35) is verified. (36) and (37) follow by taking z real in (35), equating real and imaginary parts, and using the identity principle.

Identities corresponding to (32)–(34) do not follow from (35)–(37) by applying (28) since in the multiple integral the left members of (35)–(37) do not vanish at the lower limit after each integration. However, (30) and (31) with (36) and (37) give the following result.

THEOREM 10.

$$z^{n+1/2}\{\sin z\, J_{n+1/2}(z) + (-1)^n \cos z\, J_{-n-1/2}(z)\} = F_n(z), \tag{38}$$

$$z^{n+1/2}\{\cos z\, J_{n+1/2}(z) - (-1)^n \sin z\, J_{-n-1/2}(z)\} = G_n(z), \tag{39}$$

where $F_n(z)$ and $G_n(z)$ are polynomials of degree $\leqq n$. Furthermore $F_n(z)=F_n(-z)$, $G_n(z)=-G_n(-z)$, $F_n(0)=2^{n+1/2}\Gamma(n+1/2)/\pi$ and $F_n(z)$ and $G_n(z)$ have real coefficients.

Proof. (38) follows immediately from (31) and (36) as does (39) from (30) and (37). Using (15) it is obvious that $F_n(z)$ is even and $G_n(z)$ is odd. Also

$$\begin{aligned} F_n(0) &= \lim_{z\to 0}(-1)^n z^{n+1/2} J_{-n-1/2}(z) = (-1)^n 2^{n+1/2}/\Gamma(1/2-n) \\ &= 2^{n+1/2}\,\Gamma(n+1/2)/\pi. \end{aligned}$$

Since the left members of (38) and (39) are sums and products of series with real coefficients then $F_n(z)$ and $G_n(z)$ have real coefficients.

From (38) and (39) the following results are immediately evident.

COROLLARY.

$$\cos z\, F_n(z) - \sin z\, G_n(z) = (-1)^n z^{n+1/2} J_{-n-1/2}(z), \tag{40}$$

$$\sin z\, F_n(z) + \cos z\, G_n(z) = z^{n+1/2} J_{n+1/2}(z), \tag{41}$$

$$F_n^2(z) + G_n^2(z) = z^{2n+1}\{J^2_{-n-1/2}(z) + J^2_{n+1/2}(z)\}. \tag{42}$$

The two results (12) and (13) of Theorem 3 provide some interesting relations between spherical Bessel functions and Prym's functions [5],

$$P(z,w) = \int_0^z e^{-t} t^{w-1}\,dt; \qquad \operatorname{Re} w > 0, \tag{43}$$

$$Q(z,w) = \int_z^\infty e^{-t} t^{w-1}\,dt. \tag{44}$$

In both (43) and (44), $z + |z| \neq 0$. In (43) the path of integation is any path lying in the region $z + |z| \neq 0$, while in (44) the path of integration is any path connecting z and 1 and lying in the region $z + |z| \neq 0$ followed by the real integral along the infinite ray from 1 to $+\infty$. When $w - 1$ is a positive integer the restriction $z + |z| \neq 0$ can be removed. Also [5] $Q(z, w)$ is an integral function of w and holomorphic as regards z when $z + |z| \neq 0$ (or for all z if $w-1$ is a positive integer) and

$$z^{-w}e^{z}Q(z,w) = \int_0^\infty e^{-zt}(1+t)^{w-1}\,dt; \qquad \operatorname{Re} z > 0. \tag{45}$$

THEOREM 11.

$$e^{z/2}\left(\frac{iz}{2}\right)^{-n-1/2} J_{n+1/2}\left(\frac{iz}{2}\right) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{2^n}{n!}\,\frac{d^n}{dz^n}\{z^{-n-1}e^{z}P(z,2n+1)\}. \tag{46}$$

Proof. Combine (12) and (16), put $t = iv/2$ in the integral, and then put $z = iw/2$ and simplify. This gives (46) with z replaced by w.

THEOREM 12.

$$\begin{aligned} &e^{z}P(z,2n+1) \\ &= \frac{\sqrt{(\pi)}\,(2n)!}{(n-1)!}\left(\frac{z}{i}\right)^{n+1/2}\int_0^1 (1-t)^{n-1}t^{-n-1/2}e^{tz/2}J_{n+1/2}\left(\frac{itz}{2}\right)dt. \end{aligned} \tag{47}$$

Proof. Apply (28) to (46). Since $z^{-n-1}e^{z}P(z,2n+1) = z^n/(2n+1) + o(z^n)$ the right member of (46) vanishes at the lower limit after each integration.

The remaining results are concerned with the Bessel polynomials $y_n(z)$, [6], where

$$y_n\left(\frac{1}{ir}\right) = \left(\frac{\pi r}{2}\right)^{1/2} e^{ir}[i^{-n-1}J_{n+1/2}(r) + i^{n}J_{-n-1/2}(r)] \tag{48}$$

or, by some manipulation,

$$y_n\left(\frac{1}{ir}\right) = i^n\left(\frac{\pi r}{2}\right)^{1/2} e^{ir}[J_{-n-1/2}(r) - (-1)^{n+1/2}J_{n+1/2}(r)]. \tag{49}$$

Then from (17),

$$y_n\left(\frac{1}{ir}\right) = i^n r^{1/2} e^{ir} 2^{-n} u_1(r).$$

Using (11) and putting $r = -iz$ gives

THEOREM 13.

$$y_n\left(\frac{1}{z}\right) = \left(-\frac{1}{2}\right)^n z^{n+1}e^{2z}\,\frac{d^n}{dz^n}\{z^{-n-1}e^{-2z}\}. \tag{50}$$

THEOREM 14.

$$y_n\left(\frac{1}{z}\right) = (-1)^n \frac{(2z)^{n+1}}{(2n)!} \frac{d^n}{dz^n}\left\{ z^{-n-1} e^{2z} \int_z^\infty t^{2n} e^{-2t}\, dt \right\}, \tag{51}$$

where the integral goes to ∞ *along the real positive axis.*

Proof. From the relation $J_u(-z) = (-1)^u J_u(z)$ and (19), (49) yields

$$y_n\left(\frac{1}{ir}\right) = -i^n \sqrt{(r)}\, e^{ir} \frac{2^{n+1}}{(2n)!} u_3(-r).$$

Using (13) an putting $r = -iz$ gives, after simplification,

$$y_n\left(\frac{1}{z}\right) = i\, \frac{(2z)^{n+1}}{(2n)!} \frac{d^n}{dz^n}\left\{ z^{-n-1} e^{2z} \int_{i\infty}^{iz} t^{2n} e^{2it}\, dt \right\}.$$

Now put $t = iu$ in the integral and (51) follows.

THEOREM 15.

$$y_n\left(\frac{1}{2z}\right) = \frac{-(-z)^{n+1}}{(2n)!} \frac{d^n}{dz^n}\{z^{-n-1} e^z Q(z, 2n+1)\}. \tag{52}$$

Proof. Put $t = u/2$ in (51) and then replace z by $2z$. Using (45), (52) takes the following alternate form.

COROLLARY. *If* $\operatorname{Re} z > 0$

$$y_n\left(\frac{1}{2z}\right) = \frac{-(-z)^{n+1}}{(2n)!} \frac{d^n}{dz^n}\left\{ z^n \int_0^\infty e^{-zt} (1+t)^{2n}\, dt \right\} \tag{53}$$

In conclusion notice that from among the above results several unusual identities for $\tan wz$ or $\tan 2z$ can be obtained. For example divide (27) or (34) by either (26) or (33) and divide (31) or (36) by either (30) or (37).

REFERENCES

1. James M. Horner, *Generalizations of the formulas of Rodrigues and Schläfli*, M. A. Thesis, University of Alabama, University, Ala., August, 1962.

2. ———, *Generalizations of the formulas of Rodrigues and Schläfli*, Amer. Math. Monthly **71** (1964), 870–876. This paper contains those results of [1] above which are concerned with second order equations.

3. ———, *A note on the derivation of Rodrigues' formulae*, Amer. Math. Monthly **70** (1963), 81–82.

4. Earl D. Rainville, *Special functions*, Macmillan, New York, 1960.

5. Giovanni Sansone and Johan Gerratson, *Lectures on the theory of functions of a complex variable*, Noordhoff, Groningen, 1960.

6. H. L. Krall and Orrin Frink, *A new class of orthogonal polynomials: the Bessel polynomials*, Trans. Amer. Math. Soc. **65** (1949), 100–115.

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