ON THE LEFSCHETZ NUMBER AND THE EULER CLASS

BY

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Let M be a compact connected orientable differentiable n-manifold (with empty boundary) and let $H^*(M)$ denote its singular cohomology group (integer coefficients). Thom [3] defined an element $X \in H^n(M)$ called the *Euler class* of M which has the property of being equal to the Euler characteristic of M times the fundamental class of M. Fadell [1] extended the definition of Euler class to compact connected orientable topological (i.e., not necessarily triangulated) manifolds. Let M be such an n-manifold, let $f: M \to M$ be a map, and let G be a commutative ring with unit. By modifying Milnor's development of the Euler class [2], we will define an endomorphism of $H^n(M; G)$ (singular cohomology with G coefficients) which maps the fundamental class of M to an element $L_f(G) \in H^n(M; G)$ which, when $G \cong Z$ (the integers), is equal to $(-1)^n$ times the Lefschetz number of f times the fundamental class of M. In particular, when f is the identity map, $L_f(Z)$ is Fadell's generalized Euler class. We shall obtain Thom's result in this more general setting. A simple proof of the Lefschetz Fixed-Point Theorem for this category of spaces also arises naturally from our development.

1. **Preliminaries.** Let (E, p, B) and (E_0, p_0, B) be Hurewicz fibre spaces over the same base space B. Fadell [1] defined (E_0, p_0, B) to be a *fibre subspace* of (E, p, B) provided $E_0 \subset E$, $p_0 = p \mid E_0$, and (E, p, B) admits a lifting function λ with the additional property that if $e_0 \in E_0$ and $w \in B^I$ such that $p(e_0) = w(0)$, then $\lambda(e_0, w) \in (E_0)^I$. When (E_0, p_0, B) is a fibre subspace of (E, p, B) Fadell calls $\mathfrak{F} = (E, E_0, p, B)$ a *fibred pair*. When $\mathfrak{F} = (E, E_0, p, B)$ is a fibred pair, the *fibre* over some $b \in B$ is the pair (F, F_0) where $F = p^{-1}(b)$ and $F_0 = F \cap E_0$.

A *generalized n-plane bundle* (n-gpb), $n \geqq 2$, is a map $p: E \to B$ together with a map $s: B \to E$ such that $ps = \mathrm{id}: B \to B$ and, if $E_0 = E - s(B)$, then

(1) (E, E_0, p, B) is a fibred pair with fibre (F, F_0),

(2) there is a homotopy $H: F \times I \to F$ such that $H(F \times [0,1)) \subset F_0$ and $H(F \times 1) = F \cap s(B)$,

Received by the editors September 9, 1963 and, in revised form, January 9, 1964.

(1) This paper is part of the author's doctoral dissertation written at the University of Wisconsin under the supervision of Professor Edward Fadell.

This research was supported by the National Science Foundation and the Air Force Office of Scientific Research under grant AFOSR 90-63.

(3) F_0 is arcwise connected and, when $n \geqq 3$, $\pi_1(F_0) = 0$,

(4) $H_*(F, F_0) \cong H_*(E^n, E^n - 0)$, where E^n denotes Euclidean n-dimensional space and 0 is the origin of E^n.

This definition is slightly more general than that used by Fadell [1].

THEOREM 1.1 (FADELL). *Let* $\mathfrak{F} = (E, E_0, p, B; F, F_0)$ *be a fibred pair such that*

(1) *B is arcwise connected,*

(2) *$\pi_1(B, b)$ acts trivially on* $H^*(F, F_0) = H^*(p^{-1}(b), p_0^{-1}(b))$,

(3) $H^*(F, F_0) \cong H^*(E^n, E^n - 0)$;

then there exist natural "Thom" isomorphisms

$$\phi : H^q(B; H^n(F, F_0)) \to H^{n+q}(E, E_0),$$

where ordinary coefficients appear on the left. The inclusion map $i : (F, F_0) \to (E, E_0)$ *induces an isomorphism* $i^* : H^n(E, E_0) \to H^n(F, F_0)$. *If, in addition,* $\mathfrak{F}$ *is an* n-gpb, *then*

$$\begin{array}{ccccc} H^q(B; H^n(F, F_0)) & = & H^q(B) & \xrightarrow{\ \phi\ } & H^{n+q}(E, E_0) \\ & & & {\scriptstyle p^*}\searrow \quad \nearrow{\scriptstyle \cup \mathfrak{U}} & \\ & & & H^q(E) & \end{array}$$

is commutative, where $i^*(\mathfrak{U})$ *is a generator of* $H^n(F, F_0) \cong Z$ (*the integers*).

Let M be a compact connected topological n-manifold. Define $p : M \times M \to M$ by $p(x, y) = x$ and let $\Delta = \{(x, y) \in M \times M \mid x = y\}$. Fadell proved in [1] that $\mathfrak{F} = (M \times M,\ M \times M - \Delta,\ p,\ M)$ is a locally trivial fibred pair with fibre $(p^{-1}(x), p_0^{-1}(x)) = (M, M - x)$ where $p_0 = p \mid M \times M - \Delta$. The lifting function of $\mathfrak{F}$ induces an action of $\pi_1(M; x)$ on $H^*(M, M - x)$. Fadell defined M to be *orientable* if this action is trivial and showed that in that case the inclusion-induced homomorphism $k^* : H^n(M, M - x) \to H^n(M)$ is an isomorphism. Let G be a commutative ring with unit. If M is orientable, it is G-orientable, so

$$k^* : H^n(M, M - x; G) \to H^n(M; G)$$

is an isomorphism. Choosing a generator $\bar{\mu} \in H^n(M, M - x; G)$, we obtain a generator $\mu \in H^n(M; G)$ by letting $\mu = k^*(\bar{\mu})$.

The fibred pair $\mathfrak{F} = (M \times M, M \times M - \Delta, p, M)$ satisfies the hypotheses of the first part of Theorem 1.1 when M is an orientable n-manifold. Therefore, the inclusion i: $(M, M - x) \to (M \times M, M \times M - \Delta)$ induces an isomorphism

$$i^* :\ H^n(M \times M, M \times M - \Delta; G) \to H^n(M, M - x; G).$$

Given a map f: $M \to M$ where M is a compact orientable n-manifold, let $j : (M \times M) \to (M \times M, M \times M - \Delta)$ be inclusion, let $\bar{f}$: $M \times M \to M \times M$ be given by $\bar{f}(y,z) = (y,f(z))$ and let $d\colon M \to M \times M$ be the diagonal map $d(y) = (y,y)$. Diagram (1) defines a homomorphism λ_G and we define the *Lefschetz class* of $f\colon M \to M$ (for cohomology with coefficient ring G), $L_f(G) \in H^n(M;G)$, by $\lambda_G(\mu) = L_f(G)$. We write $L_f(Z) = L_f$.

$$
(1)\qquad
\begin{array}{ccc}
H^n(M;G) & \xrightarrow{\quad\lambda_G\quad} & H^n(M;G) \\
k^*\uparrow\approx & & \uparrow d^* \\
H^n(M, M - x; G) & & H^n(M \times M; G) \\
i^*\uparrow\approx & & \uparrow \bar{f}^* \\
H^n(M \times M, M \times M - \Delta; G) & \xrightarrow[j^*]{\quad\quad} & H^n(M \times M; G)
\end{array}
$$

If $f\colon M \to M$ is fixed-point free then $\bar{f}d(M) \subset M \times M - \Delta$. Let $e\colon M \times M - \Delta \to M \times M$ be inclusion, then in this case $\bar{f}d = e\bar{f}d$ and $\lambda_G = d^*\bar{f}^*e^*j^*i^{*-1}k^{*-}$. From the exact cohomology sequence of the pair $(M \times M, M \times M - \Delta)$, we know that

$$e^*j^* = 0\colon\ H^n(M \times M, M \times M - \Delta; G) \to H^n(M \times M - \Delta; G),$$

which proves:

THEOREM 1.2. *Let M be a compact orientable manifold and let G be a commutative ring with unit. If $f\colon M \to M$ is fixed-point free, then $L_f(G) = 0$.*

2. **The main theorem.** Let Q denote the rationals. We will examine the relationship between the Lefschetz class $L_f(Q)$ of a map f and the Lefschetz number Λ_f. The reader may easily verify

LEMMA 2.1. *If a_{ij}, b_{ij} and c_{ij} are elements of a field, $i,j = 1,\cdots,m$, then*

$$\sum_{i,j=1}^{m} a_{ij} \sum_{k=1}^{m} b_{ik}c_{jk} = \sum_{i,j=1}^{m} c_{ij} \sum_{k=1}^{m} b_{kj}a_{ki}.$$

The proof of the following theorem is based on a modification of the proof of Theorem 16 of [2].

THEOREM 2.2. *If M is a compact orientable n-manifold, $f\colon M \to M$ is a map, and $\mu \in H^n(M;Q)$ is the generator such that $\lambda_Q(\mu) = L_f(Q)$, then $L_f(Q) = (-1)^n \Lambda_f \cdot \mu$.*

Proof. Let $j^*i^{*-1}k^{*-1}(\mu) = U \in H^n(M \times M;\ Q)$ and extend μ to a basis $\alpha_1,\cdots,\alpha_N$ for $H^*(M,Q)$. If $\alpha_j \in H_q(M;Q)$ we say that the *dimension* of α_j is q, written $\dim(\alpha_j) = q$. By the Künneth formula we may write $U = \sum_{i,j=1}^{N} c_{ij}(\alpha_i \otimes \alpha_j)$ where $c_{ij} = 0$ if $\dim(\alpha_i) + \dim(\alpha_j) \neq n$. Define $y_{ij} \in Q$ by $\alpha_i \cup \alpha_j = y_{ij} \cdot \mu$ if $\dim(\alpha_i) + \dim(\alpha_j) = n$ and $y_{ij} = 0$ otherwise. Let f_{ij} be the ijth entry in the matrix representing $f^*: H^*(M;Q)\to H^*(M;Q)$ with respect to this basis. Note that $f_{ij}=0$ if $\dim(\alpha_i)\neq\dim(\alpha_j)$. Let $C=[c_{ij}]$, $Y=[y_{ij}]$ and $F=[f_{ij}]$ be the $N\times N$ matrices thus defined. Now

$$\begin{aligned} L_f(Q) &= d^* f^*U = d^*f^* \sum_{i,j=1}^{N} c_{ij}(\alpha_i\otimes\alpha_j) \\ &= \sum_{i,j=1}^{N} c_{ij} \sum_{k=1}^{N} f_{jk}(\alpha_i\cup\alpha_k) \\ &= \left(\sum_{i,j=1}^{N} c_{ij}\sum_{k=1}^{N} y_{ik}f_{jk}\right)\cdot\mu. \end{aligned}$$

Let $Y^T=[y^T_{ij}]$ be the transpose of Y, then applying 2.1,

$$\begin{aligned} L_f(Q) &= \left(\sum_{i,j=1}^{N} f_{ij}\sum_{k=1}^{N} y^T_{jk}c_{ki}\right)\cdot\mu \\ &= \left(\sum_{i,j=1}^{N} f_{ij}\,(y^Tc)_{ji}\right)\cdot\mu = \left(\sum_{i=1}^{N} a_{ii}\right)\cdot\mu, \end{aligned}$$

where $Y^TC=[(y^Tc)_{ij}]$ and $[a_{ij}]=A=FY^TC$. Renumber the basis elements of $H^*(M;Q)$ of dimension q as $\alpha_{q_1},\cdots,\alpha_{q_r}$, then by duality the basis elements of dimension $n-q$ are $\alpha_{(n-q)_1},\cdots,\alpha_{(n-q)_r}$. By the skew-symmetry of cup product, $y^T_{jk}=(-1)^e y_{jk}$, where $e=(\dim\alpha_j)(\dim\alpha_k)$, thus

$$\begin{aligned} \sum_{i=1}^{r} a_{q_i,q_i} &= \sum_{i=1}^{r}\sum_{j=1}^{N} f_{q_i,j}\sum_{k=1}^{N} y^T_{jk}c_{k,q_i} \\ &= \sum_{,j=1}^{r} f_{q_i,q_j}\sum_{k=1}^{r} y^T_{q_j,(n-q)_k}c_{(n-q)_k,q_i} \\ &= (-1)^{q(n-q)}\sum_{i,j=1}^{r} f_{q_i,q_j}\sum_{k=1}^{r} y_{q_j,(n-q)_k}c_{(n-q)_k,q_i}. \end{aligned}$$

Let $Y_q=[y_{q_i,(n-q)_j}]$, $C_q=[c_{(n-q)_i,q_j}]$, $i,j=1,\cdots,r$. By [2, p. 50], $C_qY_q=(-1)^{n(n-q)}E$, where E is the $r\times r$ identity matrix. Hence $Y_qC_q=(-1)^{n(n-q)}E$ and

$$\sum_{i=1}^{r} a_{q_i,q_i} = (-1)^{q(n-q)}\sum_{j=1}^{r}(-1)^{n(n-q)}f_{q_j,q_j} = (-1)^n\sum_{j=1}^{r}(-1)^q f_{q_j,q_j}.$$

Therefore

$$L_f(Q)=\left(\sum_{i=1}^{N} a_{ii}\right)\cdot\mu=\left((-1)^n\sum_{k=1}^{N}(-1)^{\dim(\alpha_k)}f_{kk}\right)\cdot\mu = (-1)^n\Lambda_f\cdot\mu.$$

3. The Lefschetz number.

THEOREM 3.1. *Let f: $M\to M$ be a map of a compact connected orientable n-manifold into itself; then the Lefschetz number of f is an integer.*

Proof. Let $i\colon Z \to Q$ be inclusion, then i induces a monomorphism $\bar{\imath}\colon H^n(M) \to H^n(M;\, Q)$. Let μ be a generator of $H^n(M)$, then the image of μ under $\bar{\imath}$, which we also call μ, is a generator of $H^n(M;\, Q)$. The restriction of λ_Q to $\bar{\imath}(H^n(M)) \subset H^n(M;\, Q)$ is just λ_Z so

$$L_f(Q) = \lambda_Q(\mu) = \bar{\imath}(\lambda_Z(\mu)) \in H^n(M),$$

which implies that $\Lambda_f \cdot \mu = (-1)^n L_f(Q)$ is in $\bar{\imath}(H^n(M))$ and $\Lambda_f \in Z$.

Putting together 2.2, 1.2, and 3.1 we obtain:

THEOREM 3.2 (LEFSCHETZ FIXED-POINT THEOREM). *If M is a compact connected orientable n-manifold and $f\colon M \to M$ is a map, then the Lefschetz number of f, Λ_f, is an integer such that if $\Lambda_f \neq 0$, then f has a fixed point.*

4. **The Euler class.** Let $\mathfrak{F} = (E, E_0, p, B; F, F_0)$ be an n-gpb where B is arcwise connected and $\pi_1(B; x)$ acts trivially on $H^*(F, F_0)$, then by Theorem 1.1 there exist isomorphisms $\phi\colon H^i(B) \to H^{n+i}(E, E_0)$ with $\phi(z) = p^*(z) \cup \mathfrak{U}$. In [1], the *Euler class* of $\mathfrak{F}$, $X(\mathfrak{F}) \in H^n(B)$, is defined by $X(\mathfrak{F}) = \phi^{-1}(\mathfrak{U} \cup \mathfrak{U})$. When M is a manifold, let

$$T_0 = \{\alpha \in M^I \mid \alpha(t) = \alpha(0) \text{ if and only if } t = 0\},$$

let T be the union of T_0 and the constant paths on M, and give T the compact-open topology. Define $q\colon T \to M$ by $q(\alpha) = \alpha(0)$, then by [1], $\mathfrak{F} = (T, T_0, q, M; F, F_0)$ is an n-gpb and the *Euler class* of M, $X(M) \in H^n(M)$, is defined by $X(M) = X(\mathfrak{F})$. The following result is based on results and techniques in [1] and [2]. We content ourselves with a sketch of the proof.

THEOREM 4.1. *Let M be a compact connected orientable n-manifold, then for an appropriately chosen orientation of M, $X(M) = L_f$ where $f = \mathrm{id}\colon M \to M$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} (T, T_0) & \xrightarrow{\quad\gamma\quad} & (M \times M, M \times M - \Delta), \\ & {\scriptstyle q}\searrow \qquad \swarrow{\scriptstyle p} & \\ & M & \end{array}$$

where for $\alpha \in T$, $\gamma(\alpha) = (\alpha(0), \alpha(1))$. Then γ induces isomorphisms

$$\gamma^*\colon H^*(M \times M, M \times M - \Delta) \to H^*(T, T_0),$$

and when we restrict γ to the fibre we also have isomorphisms

$$\gamma^*\colon H^*(M, M - x) \to H^*(F, F_0).$$

Choose an orientation $\bar{\mu} \in H^n(M, M - x)$ and let $U = \gamma^*(\bar{\mu})$, then $i_1^*(\mathfrak{U}) = U$ where $i_1^*\colon H^n(T, T_0) \to H^n(F, F_0)$ is the isomorphism induced by inclusion. Let

$s\colon M \to T$ be the canonical cross-section, i.e., $s(y)$ is the constant path at y for all $y \in M$. Then, if $j_1\colon\ T \to (T, T_0)$ is the inclusion, $X(M) = s^* j_1^*\ (\mathfrak{U})$. Diagram (2) completes the proof.

$$
\begin{array}{ccc}
H^n(F, F_0) & \xleftarrow[\approx]{\ \gamma^*\ } & H\ (M, M - x) \\
{\scriptstyle i_1^*}\uparrow{\scriptstyle \approx} & & {\scriptstyle \approx}\uparrow{\scriptstyle i^*} \\
H^n(T, T_0) & \xleftarrow[\gamma^*]{\ \approx\ } & H^n(M \times M, M \times M - \Delta) \\
{\scriptstyle j_1^*}\downarrow & & \downarrow{\scriptstyle j^*} \\
H^n(T) & \xleftarrow{\ \gamma^*\ } & H^n(M \times M) \\
 & {\scriptstyle s^*}\searrow \qquad \swarrow{\scriptstyle d} & \\
 & H^n(M) &
\end{array} \tag{2}
$$

THEOREM 4.2. *If M is a compact connected orientable n-manifold, then $X(M) = \chi(M) \cdot \mu$ where $\chi(M)$ is the Euler characteristic of M.*

Proof. By 2.2 and 3.1, $L_f = (-1)^n \Lambda_f \cdot \mu$ so since $\Lambda_{\mathrm{id}} = \chi(M)$, the result follows for n even. When n is odd, $\chi(M) = 0$.

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