ON THE MARX CONJECTURE FOR STARLIKE FUNCTIONS(1)

BY
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Let S^* denote the class of functions $f(z) = z + a_2 z^2 + \cdots$ which map the unit disk $|z| < 1$ conformally onto a domain starlike with respect to the origin. An important example is the Koebe function $k(z) = z(1-z)^{-2}$, which maps the disk onto the entire plane slit along the negative real axis from $-1/4$ to $-\infty$. In 1932, A. Marx [3] observed that for every $f(z) \in S^*$, $f(z)/z$ is *subordinate* to $k(z)/z$ in the sense that for each fixed $r < 1$, the image of the disk $|z| \leqq r$ under $f(z)/z$ is contained in the image under $k(z)/z$. Marx conjectured that a similar statement could be made for derivatives; namely, that for every $f(z) \in S^*$, $f'(z)$ is subordinate to $k'(z)$. Since $f(z) \in S^*$ implies $f(\alpha z)/\alpha \in S^*$ for $|\alpha| \leqq 1$, an equivalent form of the conjecture is as follows. *For each fixed* $z_0, |z_0| < 1$, *the set of values* $f'(z_0)$ *for all* $f \in S^*$, *is precisely the set of values* $k'(z)$ *for all z in the disk* $|z| \leqq |z_0|$.

Marx verified this conjecture for $|z_0| \leqq 2 - 3^{1/2} = 0.267\cdots$. R. M. Robinson [4] improved the constant to $(5 - 17^{1/2})/2 = 0.438\cdots$, and later [5] made a further improvement to 0.6. More recently, J. A. Hummel [2] attacked the problem as an application of his variational method within S^*, but was able to obtain only a partial result previously found by Robinson.

In the present paper, we increase the constant to $r_0 = 0.736\cdots$, the exact value of r_0^2 being a solution of the cubic equation $x^3 + 3x^2 + 11x = 7$. Our method is essentially the same as Robinson's in [5], but we establish the stronger result by a more detailed analysis. The constant seems to be the best obtainable by this method, although it is not best possible (see §4). We prove that for each fixed $z_0, |z_0| < r_0$, and for each fixed ψ, $0 \leqq \psi < 2\pi$, the extremal problem

$$\max_{f \in S^*} \operatorname{Re}\{e^{i\psi} \log f'(z_0)\} \tag{1}$$

is solved by a function mapping $|z| < 1$ onto the exterior of *one* radial slit; that is, by some rotation $e^{-i\phi}k(e^{i\phi}z)$ of the Koebe function. (Robinson and Hummel proved the extremal map has at most two radial slits, $|z_0| < 1$.) Later (§3) we do a calculation to show that the function $\log k'(z)$ is convex in

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$|z| < R_0 = 0.886\cdots$, where the exact (largest) value of R_0^2 is a solution of the quintic equation (10). In particular, $\log k'(z)$ maps each disk $|z| \leqq r < r_0$ onto a convex region. From these two results the Marx conjecture is easily deduced.

Indeed, for fixed $z_0, |z_0| < r_0$, let $R(z_0)$ denote the set of all numbers $\log f'(z_0), f \in S^*$; and let $K(z_0)$ denote the set of all numbers $\log k'(z)$, $|z| \leqq |z_0|$. It is clear that $K(z_0) \subset R(z_0)$. The solution to problem (1) shows that each supporting line of $R(z_0)$ meets $R(z_0)$ at a point which is also in $K(z_0)$. Hence $R(z_0)$ is contained in the convex hull of $K(z_0)$; that is, $R(z_0) \subset K(z_0)$. Therefore, $R(z_0) = K(z_0)$, which is the Marx conjecture.

Having proved the conjecture for $|z_0| < r_0$, it is a simple matter to extend it to $|z_0| \leqq r_0$. Indeed, if for some z_0 of modulus r_0 there were a function $f \in S^*$ for which $\log f'(z_0) \notin K(z_0)$, then (since $K(z_0)$ is closed) it would follow by continuity that $\log f'(z_1) \notin K(z_0) \supset K(z_1)$ for some $z_1, |z_1| < r_0$. This is impossible.

1. **Preliminaries.** In considering the extremal problem (1), it suffices to take $z_0 = r$, $0 < r < 1$. Robinson [5] proved that an extremal function must have the form

$$f(z) = z \prod_{\nu=1}^{n} (1 - ze^{i\phi_\nu})^{-2_\nu}, \tag{2}$$

where $a_\nu > 0$, $a_1 + a_2 + \cdots + a_n = 1$, and the $e^{i\phi_\nu}$ are distinct. This also results from a general theorem of Hummel. For the particular problem (1), Robinson and Hummel both showed $n \leqq 2$, but this knowledge does not simplify our argument. For $f(z)$ given by (2), we calculate

$$\log f'(r) = \log \sum_{\nu=1}^{n} a_\nu \frac{1 + re^{i\phi_\nu}}{1 - re^{i\phi_\nu}} - 2 \sum_{\nu=1}^{n} a_\nu \log(1 - re^{i\phi_\nu}). \tag{3}$$

We shall have need of the following lemma. (Compare Robinson [5, Theorem 1].)

LEMMA. *Let $F(z_1, z_2, \cdots, z_n)$ be an analytic function of the n complex variables $z_\nu, |z_\nu| \leqq 1$. Among all systems of points z_ν with $|z_1| = |z_2| = \cdots = |z_n| = 1$, let* $\mathrm{Re}\{F\}$ *attain its maximum at $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then*

$$\alpha_\nu \frac{\partial F}{\partial z_\nu}(\alpha_1, \alpha_2, \cdots, \alpha_n) \geqq 0, \qquad \nu = 1, 2, \cdots, n. \tag{4}$$

Proof. Let $\partial F(\alpha_1, \alpha_2, \cdots, \alpha_n)/\partial z_\nu = A_\nu + iB_\nu$. By the maximum principle, the α_ν also maximize $\mathrm{Re}\{F\}$ in $|z_\nu| \leqq 1$. Hence, for any vector $\xi + i\eta$ which points from α_ν toward the interior of the unit circle,

$$\mathrm{Re}\{(A_\nu + iB_\nu)(\xi + i\eta)\} = A_\nu \xi - B_\nu \eta \leqq 0.$$

But these vectors $\xi + i\eta$ are characterized by $a_\nu \xi + b_\nu \eta < 0$, where $\alpha_\nu = a_\nu + ib_\nu$.

The conclusion is that $A_\nu + iB_\nu = \lambda_\nu(a_\nu - ib_\nu)$ for some real $\lambda_\nu \geqq 0$, which is equivalent to (4).

It should be remarked that the vanishing of the partial derivative of $\mathrm{Re}\{F\}$ with respect to θ_ν $(z_\nu = e^{i\theta_\nu})$ tells us that the expression (4) is real. The non-negativity comes from the maximum property.

2. **Solution of the extremal problem.** Let us fix attention on some solution to problem (1), for $z_0 = r$. For such an extremal function (2), $\log f'(r)$ has the structure (3). In particular, among all functions having the same n and the same weights a_ν as the extremal function, the expression $\mathrm{Re}\{e^{i\psi}\log f'(r)\}$ is maximized by the numbers $e^{i\phi_\nu}$ which occur in the extremal function. We are now in a position to apply the lemma, with

$$F(z_1, \cdots, z_n) = e^{i\psi}\left[\log \sum_{\nu=1}^{n} a_\nu \Phi(z_\nu) - 2\sum_{\nu=1}^{n} a_\nu \log(1 - rz_\nu)\right];$$

$$\Phi(z) = (1 + rz)/(1 - rz).$$

Setting $\zeta = \sum a_\nu \Phi(z_\nu)$, we compute

$$z_\nu \frac{\partial F}{\partial z_\nu} = 2ra_\nu e^{i\psi} z_\nu [\zeta^{-1}(1 - rz_\nu)^{-2} + (1 - rz_\nu)^{-1}]. \tag{5}$$

According to the lemma, each of the expressions (5), $\nu = 1, \cdots, n$, is real and non-negative for $z_\nu = e^{i\phi_\nu}$. From this we wish to conclude $n = 1$. It suffices to prove that for every fixed ζ inside or on the circle $\zeta = \Phi(e^{i\theta})$, $0 \leqq \theta \leqq 2\pi$, the function

$$G(z) = z[\zeta^{-1}(1 - rz)^{-2} + (1 - rz)^{-1}]$$

is *starlike* in $|z| \leqq 1$. This is true, as we shall show, for $r < r_0$, but false for $r > r_0$.

A short calculation leads to the expression

$$\frac{zG'(z)}{G(z)} = 1 + \frac{2rz}{1 - rz} - \frac{\zeta_1 rz}{1 - \zeta_1 rz}, \tag{6}$$

where $\zeta_1 = \zeta/(1 + \zeta)$ is some fixed number in the closed disk with center at $1/2$ and radius $r/2$. Our strategy is to choose ζ_1, as a function of $z = e^{i\theta}$, to *minimize* the real part of (6); then to determine the largest r for which this minimum is non-negative for all θ. Equivalently, for fixed $z = e^{i\theta}$, we seek to maximize the real part of $w = \zeta_1 rz/(1 - \zeta_1 rz)$ for ζ_1 on the circle with center $1/2$ and radius $r/2$. A bit of manipulation gives

$$\frac{w - rz/(2 - rz)}{w + 1} = (\zeta_1 - 1/2)2rz/(2 - rz).$$

This shows that the image of the given circle in the ζ_1-plane is the circle $|(w - p)/(w - q)| = k$, where

$$p = \frac{re^{i\theta}}{2 - re^{i\theta}}, \qquad q = -1, \qquad k = \frac{r^2}{|2 - re^{i\theta}|}.$$

It is not difficult to show (see, e.g., [6, pp. 191–192]) that this is the circle with center $w_0 = (p - k^2 q)/(1 - k^2)$ and radius $\rho = k|p - q|/(1 - k^2)$. Hence the maximum value of $\mathrm{Re}\{w\}$ on this circle is attained at $w_0 + \rho$. Replacing the last term in (6) by $-(w_0 + \rho)$ and setting $x = \cos\theta$, one calculates $H(x) = \mathrm{Re}\{e^{i\theta}G'(e^{i\theta})/G(e^{i\theta})\}$ to be

$$\begin{aligned} H(x) &= 1 + \frac{2r(x - r)}{1 + r^2 - 2rx} - \frac{r(r + r^3 + 2x)}{4 + r^2 - r^4 - 4rx} \\ &= 2h(x)[1 + r^2 - 2rx]^{-1}[4(1 - rx) + r^2(1 - r^2)]^{-1}, \end{aligned}$$

where

$$h(x) = 2(1 - r^2 - r^4) + r(-3 + 2r^2 + r^4)x + 2r^2x^2.$$

Our task is to find the largest value of r for which $h(x) \geqq 0$ throughout the interval $-1 \leqq x \leqq 1$. The minimum of $h(x)$ is easily seen to occur at $x_0 = (3 - 2r^2 - r^4)/4r$, a number which for $r^2 \geqq 1/2$ satisfies $-1 \leqq x_0 \leqq 1$. One computes

$$8h(x_0) = (1 + s)(7 - 11s - 3s^2 - s^3), \qquad s = r^2.$$

The cubic equation

$$s^3 + 3s^2 + 11s - 7 = 0 \tag{7}$$

has a unique solution $s = r_0^2$ in the interval $0 < s < 1$, the value of which is computed most conveniently by successive approximations (Newton's method). We find $r_0 = 0.736\cdots$. Since $r_0^2 \geqq 1/2$, we have proved that $G(z)$ is starlike in $|z| \leqq 1$ for the parameter r in the range $0 \leqq r < r_0$. Hence for $|z_0| < r_0$, the extremal problem (1) is solved by some rotation of the Koebe function. The argument fails for $r > r_0$, since for no such r is $G(z)$ starlike in $|z| \leqq 1$ for all ζ.

3. **Radius of convexity of** $\log k'(z)$. The proof can now be completed by verifying that $\log k'(z)$ is convex in $|z| < r_0$. We shall do so by calculating the exact radius of convexity. Set $g(z) = \log k'(z)$; then

$$1 + \frac{zg''(z)}{g'(z)} = \frac{2(1 + z + z^2)}{(1 - z^2)(2 + z)}. \tag{8}$$

The radius of convexity of $g(z)$ is the largest value of ρ for which the real part of (8) is positive in $|z| < \rho$. A short calculation gives

$$\begin{aligned} &(1/2)|(1 - z^2)(2 + z)|^2 \,\mathrm{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} \\ &\quad = (2 + r^2 - 2r^4) + (3r - r^3 - r^5)\cos\theta - r^4\cos 2\theta - r^3\cos 3\theta, \end{aligned}$$

where $z = re^{i\theta}$. Now set $x = \cos\theta$, so that $\cos 2\theta = 2x^2 - 1$ and $\cos 3\theta = 4x^3 - 3x$. The problem reduces to finding the largest value of r for which the cubic polynomial

$$P(x) = (2 + r^2 - r^4) + (3r + 2r^3 - r^5)x - 2r^4x^2 - 4r^3x^3$$

is non-negative throughout the interval $-1 \leqq x \leqq 1$.

Observe first that $P(1) = (2 + 3r + r^2)(1 - r^3) > 0$, so only the relative minimum of $P(x)$ needs to be considered. Straight forward differentiation shows this relative minimum occurs at

$$x_0 = -(r/6)[1 + (9r^{-4} + 6r^{-2} - 2)^{1/2}].$$

Note that $-1 \leqq x_0 \leqq 1$ for $r^2 \geqq 1/2$. Another calculation leads to

$$54P(x_0) = 108 + 27r^2 - 72r^4 + 7r^6 - 2(9 + 6r^2 - 2r^4)^{3/2}.$$

The condition $P(x_0) = 0$ is therefore equivalent to $s = r^2$ being a solution of the sixth-degree equation

$$(108 + 27s - 72s^2 + 7s^3)^2 = 4(9 + 6s - 2s^2)^3. \tag{9}$$

After expansion, simplification, and division by $(s + 1)$, (9) reduces to

$$s^5 - 17s^4 + 91s^3 - 99s^2 - 108s + 108 = 0. \tag{10}$$

The quintic equation (10) has a unique solution $s = R_0^2$ in the interval $0 < s < 1$, since the derivative of the given polynomial is negative throughout this range. Using an automatic computer this time, we found

$$R_0 = 0.886\cdots.$$

This is the radius of convexity of $\log k'(z)$. Since $R_0 > r_0$, the Marx conjecture is proved for $|z_0| < r_0$. Hence, as noted in §1, it is true for $|z_0| \leqq r_0$.

We mention without proof that $\log k'(z)$ is starlike in the entire circle $|z| < 1$.

4. **Remarks.** R. M. Robinson has kindly pointed out to me that the constant r_0 is not best possible; that is, the Marx conjecture is true in a disk larger than $|z_0| \leqq r_0$. The proof is presented here with his permission.

We have observed that for any fixed $r < R_0$, the proof of the Marx conjecture for $|z_0| \leqq r$ can be reduced to showing that for each ψ the expression $\mathrm{Re}\{e^{i\psi}\log f'(r)\}$ is maximized by a function (2) for which $n = 1$; that is, by some rotation of the Koebe function. In §2 we reduced a proof of this latter proposition to the following statement. *If* $z_\nu = e^{i\phi_\nu}$, $\nu = 1, 2, \cdots, n$, *are distinct numbers such that* (in notation previously used) *all the points* $G(z_\nu)$ *lie on the same ray, where* $\zeta = \sum a_\nu \Phi(z_\nu)$, *then* $n = 1$. This we verified for $r < r_0$ by a proof that for *every* value of the parameter ζ inside and on the circle

$$C\colon \zeta = \Phi(e^{i\theta}), \qquad 0 \leqq \theta < 2\pi,$$

$G(z)$ is starlike in $|z| \leqq 1$. Although G no longer has this starlikeness property for $r > r_0$, the italicized statement can nevertheless be proved for r slightly greater than r_0 by a continuity argument.

For each fixed z on $|z| = 1$, there is a unique ζ on C which minimizes $\mathrm{Re}\{zG'(z)/G(z)\}$. With this choice of ζ (as a function of z), there are two points z_0 and $\bar{z}_0$ which minimize $\mathrm{Re}\{zG'(z)/G(z)\}$. Let ζ_0 correspond to z_0; then $\bar{\zeta}_0$ corresponds to $\bar{z}_0$. As r increases, the minimum of $\mathrm{Re}\{zG'(z)/G(z)\}$ (taken over z and ζ) decreases monotonically to zero at $r = r_0$. For r slightly greater than r_0, it can happen that $\mathrm{Re}\{zG'(z)/G(z)\} < 0$ only for z near z_0 and ζ near ζ_0, or for z near $\bar{z}_0$ and ζ near $\bar{\zeta}_0$.

Now suppose that for each $r > r_0$ there are $n = n(r) > 1$ distinct points $z_1, \cdots, z_n$ on the unit circle such that $G(z_1), \cdots, G(z_n)$ lie on a ray. The parameter ζ occurring in G is understood to be $\zeta = \sum a_\nu \Phi(z_\nu)$. It is clear geometrically that for r slightly greater than r_0, either all the points $z_1, \cdots, z_n$ are near z_0 and ζ is near ζ_0, or all the points $z_1, \cdots, z_n$ are near $\bar{z}_0$ and ζ is near $\bar{\zeta}_0$. But for each r, ζ is a weighted average of the points $\Phi(z_\nu)$. Therefore, by taking limits as $r \searrow r_0$, it follows that $\zeta_0 = \Phi(z_0)$ for $r = r_0$.

To conclude the proof that $n = 1$ for all r in some neighborhood of r_0, we show $\zeta_0 \neq \Phi(z_0)$, which is contradiction. By construction, $\mathrm{Re}\{z_0 G'(z_0)/G(z_0)\} = 0$ for $\zeta = \zeta_0$ and $r = r_0$. On the other hand, if $\zeta = \Phi(z)$, a direct calculation from (6) leads to the simple expression

$$\frac{zG'(z)}{G(z)} = \frac{2(1 + rz)}{2 - rz(1 + rz)}. \tag{11}$$

With $z = e^{i\theta}$ and $x = \cos\theta$, the real part of (11) is found to be a positive multiple of

$$\begin{aligned} 2 + r(1 - r^2)\cos\theta - 2r^2\cos^2\theta &\geqq 2 - r(1 - r^2) - 2r^2 \\ &= (2 - r)(1 - r^2) > 0, \qquad r < 1. \end{aligned}$$

Therefore, $\mathrm{Re}\{zG'(z)/G(z)\} > 0$ on $|z| = 1$ for *all* r $(0 < r < 1)$ if $\zeta = \Phi(z)$. This shows $\zeta_0 \neq \Phi(z_0)$ for $r = r_0$, and finishes the proof.

Since r_0 is not best possible, it is natural to ask whether some modification of the method might lead to an improved result. One such modification would be to map $|z| < 1$ conformally onto $|w| < 1$ and to apply the lemma not directly to $F(z_1, \cdots, z_n)$, but to the induced function of $w_1, \cdots, w_n$. Robinson [5] used this idea. However, Professor Robinson has recently communicated to me the following proof that *every* such mapping leads to the same bound r_0.

For the function $F(z_1, \cdots, z_n)$, we found $z_\nu \partial F/\partial z_\nu = 2ra_\nu e^{i\psi} G(z_\nu)$, and we proved $n = 1$ (for $r < r_0$) by showing $G(z)$ is starlike; hence $z\partial F/\partial z \geqq 0$ for only one value of z on $|z| = 1$. But under a conformal mapping $z = e^{it}(w - \alpha)/(1 - \bar{\alpha}w)$ of $|w| < 1$ onto $|z| < 1$,

$$z_\nu \frac{\partial F}{\partial z_\nu} = (1 - |\alpha|^2)^{-1} \; |w_\nu - \alpha|^2 \, w_\nu \frac{\partial F}{\partial w_\nu}, \qquad |w_\nu| = 1.$$

Hence $w_\nu \partial F / \partial w_\nu \geqq 0$ can happen for only one value of $w_\nu, |w_\nu| = 1$.

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