Let $S^*$ denote the class of functions $f(z) = z + a_2z^2 + \cdots$ which map the unit disk $|z| < 1$ conformally onto a domain starlike with respect to the origin. An important example is the Koebe function $k(z) = z(1 - z)^{-2}$, which maps the disk onto the entire plane slit along the negative real axis from $-1$ to $-\infty$. In 1932, A. Marx [3] observed that for every $f(z) \in S^*$, $f(z)/z$ is subordinate to $k(z)/z$ in the sense that for each fixed $r < 1$, the image of the disk $|z| \leq r$ under $f(z)/z$ is contained in the image under $k(z)/z$. Marx conjectured that a similar statement could be made for derivatives; namely, that for every $f(z) \in S^*$, $f'(z)$ is subordinate to $k'(z)$. Since $f(z) \in S^*$ implies $f(az)/a \in S^*$ for $|a| < 1$, an equivalent form of the conjecture is as follows. For each fixed $z_0$, $|z_0| < 1$, the set of values $f'(z_0)$ for all $f \in S^*$, is precisely the set of values $k'(z)$ for all $z$ in the disk $|z| \leq |z_0|$.

Marx verified this conjecture for $|z_0| \leq 2 - 3^{1/2} = 0.267 \ldots$. R. M. Robinson [4] improved the constant to $(5 - 17^{1/2})/2 = 0.438 \ldots$, and later [5] made a further improvement to 0.6. More recently, J. A. Hummel [2] attacked the problem as an application of his variational method within $S^*$, but was able to obtain only a partial result previously found by Robinson.

In the present paper, we increase the constant to $r_0 = 0.736 \ldots$, the exact value of $r_0^2$ being a solution of the cubic equation $x^3 + 3x^2 + 11x = 7$. Our method is essentially the same as Robinson's in [5], but we establish the stronger result by a more detailed analysis. The constant seems to be the best obtainable by this method, although it is not best possible (see §4). We prove that for each fixed $z_0$, $|z_0| < r_0$, and for each fixed $\psi$, $0 \leq \psi < 2\pi$, the extremal problem

$$\max_{f \in S^*} \text{Re}\{e^{i\psi}\log f'(z_0)\}$$

is solved by a function mapping $|z| < 1$ onto the exterior of one radial slit; that is, by some rotation $e^{-i\psi}k(e^{i\psi}z)$ of the Koebe function. (Robinson and Hummel proved the extremal map has at most two radial slits, $|z_0| < 1$.) Later (§3) we do a calculation to show that the function $\log k'(z)$ is convex in
\[ |z| < R_0 = 0.886 \ldots \], where the exact (largest) value of \( R_0^2 \) is a solution of the quintic equation (10). In particular, \( \log k'(z) \) maps each disk \( |z| \leq r < r_0 \) onto a convex region. From these two results the Marx conjecture is easily deduced.

Indeed, for fixed \( z_0, |z_0| < r_0 \), let \( R(z_0) \) denote the set of all numbers \( \log f'(z_0), f \in S^*; \) and let \( K(z_0) \) denote the set of all numbers \( \log k'(z), |z| \leq |z_0| \). It is clear that \( K(z_0) \subseteq R(z_0) \). The solution to problem (1) shows that each supporting line of \( R(z_0) \) meets \( R(z_0) \) at a point which is also in \( K(z_0) \). Hence \( R(z_0) \) is contained in the convex hull of \( K(z_0) \); that is, \( R(z_0) \subseteq K(z_0) \). Therefore, \( R(z_0) = K(z_0) \), which is the Marx conjecture.

Having proved the conjecture for \( |z_0| < r_0 \), it is a simple matter to extend it to \( |z_0| \leq r_0 \). Indeed, if for some \( z_0 \) of modulus \( r_0 \) there were a function \( f \in S^* \) for which \( \log f'(z_0) \notin K(z_0) \), then (since \( K(z_0) \) is closed) it would follow by continuity that \( \log f'(z_1) \notin K(z_1) \) for some \( z_1, |z_1| < r_0 \). This is impossible.

1. Preliminaries. In considering the extremal problem (1), it suffices to take \( z_0 = r, 0 < r < 1 \). Robinson [5] proved that an extremal function must have the form

\[ f(z) = z \prod_{v=1}^{n} (1 - ze^{i\phi_v})^{-2v}, \]

where \( a_v > 0, a_1 + a_2 + \cdots + a_n = 1 \), and the \( e^{i\phi_v} \) are distinct. This also results from a general theorem of Hummel. For the particular problem (1), Robinson and Hummel both showed \( n \leq 2 \), but this knowledge does not simplify our argument. For \( f(z) \) given by (2), we calculate

\[ \log f'(r) = \log \sum_{v=1}^{n} a_v \left( \frac{1 + re^{i\phi_v}}{1 - re^{i\phi_v}} \right) - 2 \sum_{v=1}^{n} a_v \log(1 - re^{i\phi_v}). \]

We shall have need of the following lemma. (Compare Robinson [5, Theorem 1].)

**Lemma.** Let \( F(z_1, z_2, \ldots, z_n) \) be an analytic function of the \( n \) complex variables \( z_v, |z_v| \leq 1 \). Among all systems of points \( z_v \) with \( |z_1| = |z_2| = \cdots = |z_n| = 1 \), let \( \text{Re}\{F\} \) attain its maximum at \( \alpha_1, \alpha_2, \ldots, \alpha_n \). Then

\[ \alpha_v \frac{\partial F}{\partial z_v}(\alpha_1, \alpha_2, \ldots, \alpha_n) \geq 0, \quad v = 1, 2, \ldots, n. \]

**Proof.** Let \( \partial F(\alpha_1, \alpha_2, \ldots, \alpha_n)/\partial z_v = A_v + iB_v \). By the maximum principle, the \( \alpha_v \) also maximize \( \text{Re}\{F\} \) in \( |z_v| \leq 1 \). Hence, for any vector \( \xi + i\eta \) which points from \( \alpha_v \) toward the interior of the unit circle,

\[ \text{Re}\{(A_v + iB_v)(\xi + i\eta)\} = A_v \xi - B_v \eta \leq 0. \]

But these vectors \( \xi + i\eta \) are characterized by \( a_v \xi + b_v \eta < 0 \), where \( \alpha_v = a_v + ib_v \).
The conclusion is that $A_v + iB_v = \lambda_v(a_v - ib_v)$ for some real $\lambda_v \geq 0$, which is equivalent to (4).

It should be remarked that the vanishing of the partial derivative of $\text{Re}\{F\}$ with respect to $\theta_v (z_v = e^{i\theta_v})$ tells us that the expression (4) is real. The non-negativity comes from the maximum property.

2. Solution of the extremal problem. Let us fix attention on some solution to problem (1), for $z_0 = r$. For such an extremal function (2), $\log f'(r)$ has the structure (3). In particular, among all functions having the same $n$ and the same weights $a_v$ as the extremal function, the expression $\text{Re}\{e^{i\Phi} \log f'(r)\}$ is maximized by the numbers $e^{i\Phi_v}$ which occur in the extremal function. We are now in a position to apply the lemma, with

\[ F(z_1, \cdots, z_n) = e^{i\Phi} \left[ \log \sum_{v=1}^{n} a_v \Phi(z_v) - 2 \sum_{v=1}^{n} a_v (1 - rz_v) \right]; \]
\[ \Phi(z) = (1 + rz)/(1 - rz). \]

Setting $C = \sum a_v \Phi(z_v)$, we compute

\[ z_v \frac{\partial F}{\partial z_v} = 2r a_v e^{i\Phi} z_v \left[ \zeta^{-1} (1 - rz_v)^{-2} + (1 - rz_v)^{-1} \right]. \]

According to the lemma, each of the expressions (5), $v = 1, \cdots, n$, is real and non-negative for $z_v = e^{i\Phi_v}$. From this we wish to conclude $n = 1$. It suffices to prove that for every fixed $\zeta$ inside or on the circle $\zeta = \Phi(e^{i\theta})$, $0 \leq \theta \leq 2\pi$, the function

\[ G(z) = z^{\zeta^{-1} (1 - rz)^{-2} + (1 - rz)^{-1}} \]

is starlike in $|z| \leq 1$. This is true, as we shall show, for $r < r_0$, but false for $r > r_0$.

A short calculation leads to the expression

\[ \frac{zG'(z)}{G(z)} = 1 + \frac{2rz}{1 - rz} - \frac{\zeta_1 rz}{1 - \zeta_1 rz}, \]

where $\zeta_1 = \zeta/(1 + \zeta)$ is some fixed number in the closed disk with center at $1/2$ and radius $r/2$. Our strategy is to choose $\zeta_1$, as a function of $z = e^{i\theta}$, to minimize the real part of (6); then to determine the largest $r$ for which this minimum is non-negative for all $\theta$. Equivalently, for fixed $z = e^{i\theta}$, we seek to maximize the real part of $w = \zeta_1 rz/(1 - \zeta_1 rz)$ for $\zeta_1$ on the circle with center $1/2$ and radius $r/2$. A bit of manipulation gives

\[ \frac{w - rz/(2 - rz)}{w + 1} = (\zeta_1 - 1/2) 2rz/(2 - rz). \]

This shows that the image of the given circle in the $\zeta_1$-plane is the circle $|(w - p)/(w - q)| = k$, where
It is not difficult to show (see, e.g., [6, pp. 191-192]) that this is the circle with
center \( w_0 = (p - k^2 q)/(1 - k^2) \) and radius \( \rho = k |p - q|/(1 - k^2) \). Hence the
maximum value of \( \Re{w} \) on this circle is attained at \( w_0 + \rho \). Replacing the
last term in (6) by \(- (w_0 + \rho)\) and setting \( x = \cos \theta \), one calculates
\( H(x) = \Re{e^{i\theta} G'(e^{i\theta})/G(e^{i\theta})} \) to be
\[
H(x) = 1 + \frac{2r(x - r)}{1 + r^2 - 2rx} - \frac{r(r + r^3 + 2x)}{4 + r^2 - r^4 - 4rx}
\]
where
\[
h(x) = 2(1 - r^2 - r^4) + r(-3 + 2r^2 + r^4)x + 2r^2x^2.
\]
Our task is to find the largest value of \( r \) for which \( h(x) \geq 0 \) throughout the
interval \(-1 \leq x \leq 1\). The minimum of \( h(x) \) is easily seen to occur at
\( x_0 = (3 - 2r^2 - r^4)/4r \), a number which for \( r^2 \leq 1 \) satisfies \(-1 \leq x_0 \leq 1\).
One computes
\[
8h(x_0) = (1 + s)(7 - 11s - 3s^2 - s^3), \quad s = r^2.
\]
The cubic equation
\[
(7) \quad s^3 + 3s^2 + 11s - 7 = 0
\]
has a unique solution \( s = r_0^2 \) in the interval \( 0 < s < 1 \), the value of which is
computed most conveniently by successive approximations (Newton’s method).
We find \( r_0 = 0.736 \ldots \). Since \( r_0^2 \geq 1/2 \), we have proved that \( G(z) \) is starlike in
\( |z| \leq 1 \) for the parameter \( r \) in the range \( 0 \leq r < r_0 \). Hence for \( |z_0| < r_0 \), the
extremal problem (1) is solved by some rotation of the Koebe function. The
argument fails for \( r > r_0 \), since for no such \( r \) is \( G(z) \) starlike in \( |z| \leq 1 \) for all \( \zeta \).

3. Radius of convexity of \( \log k'(z) \). The proof can now be completed by
verifying that \( \log k'(z) \) is convex in \( |z| < r_0 \). We shall do so by calculating the
exact radius of convexity. Set \( g(z) = \log k'(z) \); then
\[
(8) \quad 1 + \frac{zg''(z)}{g'(z)} = \frac{2(1 + z + z^2)}{(1 - z^2)(2 + z)}.
\]
The radius of convexity of \( g(z) \) is the largest value of \( \rho \) for which the real part
of (8) is positive in \( |z| < \rho \). A short calculation gives
\[
(1/2)|1 - z^2|(2 + z)|^2 \Re \left( 1 + \frac{zg''(z)}{g'(z)} \right)
= (2 + r^2 - 2r^4) + (3r - r^3 - r^5) \cos \theta - r^4 \cos 2\theta - r^3 \cos 3\theta,
\]
where \( z = re^{i\theta} \). Now set \( x = \cos \theta \), so that \( \cos 2\theta = 2x^2 - 1 \) and \( \cos 3\theta = 4x^3 - 3x \). The problem reduces to finding the largest value of \( r \) for which the cubic polynomial

\[
P(x) = (2 + r^2 - r^4) + (3r + 2r^3 - r^5)x - 2r^4x^2 - 4r^3x^3
\]

is non-negative throughout the interval \(-1 \leq x \leq 1\).

Observe first that \( P(1) = (2 + 3r + r^2)(1 - r^3) > 0 \), so only the relative minimum of \( P(x) \) needs to be considered. Straight forward differentiation shows this relative minimum occurs at

\[
x_0 = -\left(\frac{r}{6}\right)[1 + (9r^{-4} + 6r^{-2} - 2)^{1/2}].
\]

Note that \(-1 \leq x_0 \leq 1 \) for \( r^2 \geq 1/2 \). Another calculation leads to

\[
54P(x_0) = 108 + 27r^2 - 72r^4 + 7r^6 - 2(9 + 6r^2 - 2r^4)^{3/2}.
\]

The condition \( P(x_0) = 0 \) is therefore equivalent to \( s = r^2 \) being a solution of the sixth-degree equation

\[
(9) \quad (108 + 27s - 72s^2 + 7s^3)^2 = 4(9 + 6s - 2s^2)^{3/2}.
\]

After expansion, simplification, and division by \( (s + 1) \), \( (9) \) reduces to

\[
(10) \quad s^5 - 17s^4 + 91s^3 - 99s^2 - 108s + 108 = 0.
\]

The quintic equation \( (10) \) has a unique solution \( s = R_0^2 \) in the interval \( 0 < s < 1 \), since the derivative of the given polynomial is negative throughout this range. Using an automatic computer this time, we found

\[
R_0 = 0.886 \ldots.
\]

This is the radius of convexity of \( \log k'(z) \). Since \( R_0 > r_0 \), the Marx conjecture is proved for \( |z_0| < r_0 \). Hence, as noted in §1, it is true for \( |z_0| \leq r_0 \).

We mention without proof that \( \log k'(z) \) is starlike in the entire circle \(|z| < 1\).

### 4. Remarks.

R. M. Robinson has kindly pointed out to me that the constant \( r_0 \) is not best possible; that is, the Marx conjecture is true in a disk larger than \( |z_0| \leq r_0 \). The proof is presented here with his permission.

We have observed that for any fixed \( r < R_0 \), the proof of the Marx conjecture for \( |z_0| \leq r \) can be reduced to showing that for each \( \psi \) the expression \( \Re\{e^{i\psi}\log f'(r)\} \) is maximized by a function \( (2) \) for which \( n = 1 \); that is, by some rotation of the Koebe function. In §2 we reduced a proof of this latter proposition to the following statement. If \( z_v = e^{i\theta_v}, v = 1, 2, \ldots, n, \) are distinct numbers such that (in notation previously used) all the points \( G(z_v) \) lie on the same ray, where \( \zeta = \sum a_v \Phi(z_v) \), then \( n = 1 \). This we verified for \( r < r_0 \) by a proof that for every value of the parameter \( \zeta \) inside and on the circle

\[
C: \zeta = \Phi(e^{i\theta}), \quad 0 \leq \theta < 2\pi,
\]
G(z) is starlike in \(|z| \leq 1\). Although G no longer has this starlikeness property for \(r > r_0\), the italicized statement can nevertheless be proved for \(r\) slightly greater than \(r_0\) by a continuity argument.

For each fixed \(z\) on \(|z| = 1\), there is a unique \(\zeta\) on \(C\) which minimizes \(\text{Re}\{zG'(z)/G(z)\}\). With this choice of \(\zeta\) (as a function of \(z\)), there are two points \(z_0\) and \(\bar{z}_0\) which minimize \(\text{Re}\{zG'(z)/G(z)\}\). Let \(\zeta_0\) correspond to \(z_0\); then \(\zeta_0\) corresponds to \(\bar{z}_0\). As \(r\) increases, the minimum of \(\text{Re}\{zG'(z)/G(z)\}\) (taken over \(z\) and \(\zeta\)) decreases monotonically to zero at \(r = r_0\). For \(r\) slightly greater than \(r_0\), it can happen that \(\text{Re}\{zG'(z)/G(z)\} < 0\) only for \(z\) near \(z_0\) and \(\zeta\) near \(\zeta_0\), or for \(z\) near \(\bar{z}_0\) and \(\zeta\) near \(\bar{\zeta}_0\).

Now suppose that for each \(r > r_0\) there are \(n = n(r) > 1\) distinct points \(z_1, \ldots, z_n\) on the unit circle such that \(G(z_1), \ldots, G(z_n)\) lie on a ray. The parameter \(\zeta\) occurring in \(G\) is understood to be \(\zeta = \sum a_i \Phi(z_i)\). It is clear geometrically that for \(r\) slightly greater than \(r_0\), either all the points \(z_1, \ldots, z_n\) are near \(z_0\) and \(\zeta\) is near \(\zeta_0\), or all the points \(z_1, \ldots, z_n\) are near \(\bar{z}_0\) and \(\zeta\) is near \(\bar{\zeta}_0\). But for each \(r\), \(\zeta\) is a weighted average of the points \(\Phi(z_i)\). Therefore, by taking limits as \(r \searrow r_0\), it follows that \(\zeta_0 = \Phi(z_0)\) for \(r = r_0\).

To conclude the proof that \(n = 1\) for all \(r\) in some neighborhood of \(r_0\), we show \(\zeta_0 \neq \Phi(z_0)\), which is contradiction. By construction, \(\text{Re}\{z_0G'(z_0)/G(z_0)\} = 0\) for \(\zeta = \zeta_0\) and \(r = r_0\). On the other hand, if \(\zeta = \Phi(z)\), a direct calculation from (6) leads to the simple expression

\[
\frac{zG'(z)}{G(z)} = \frac{2(1 + rz)}{2 - rz(1 + rz)}.
\]

With \(z = e^{i\theta}\) and \(x = \cos \theta\), the real part of (11) is found to be a positive multiple of

\[
2 + r(1 - r^2) \cos \theta - 2r^2 \cos^2 \theta \geq 2 - r(1 - r^2) - 2r^2
\]

\[
= (2 - r)(1 - r^2) > 0, \quad r < 1.
\]

Therefore, \(\text{Re}\{zG'(z)/G(z)\} > 0\) on \(|z| = 1\) for all \(r\) (0 < \(r < 1\)) if \(\zeta = \Phi(z)\). This shows \(\zeta_0 \neq \Phi(z_0)\) for \(r = r_0\), and finishes the proof.

Since \(r_0\) is not best possible, it is natural to ask whether some modification of the method might lead to an improved result. One such modification would be to map \(|z| < 1\) conformally onto \(|w| < 1\) and to apply the lemma not directly to \(F(z_1, \ldots, z_n)\), but to the induced function of \(w_1, \ldots, w_n\). Robinson [5] used this idea. However, Professor Robinson has recently communicated to me the following proof that every such mapping leads to the same bound \(r_0\).

For the function \(F(z_1, \ldots, z_n)\), we found \(z_0 \partial F/\partial z = 2ra, e^{i\Phi(z_0)}\), and we proved \(n = 1\) (for \(r < r_0\)) by showing \(G(z)\) is starlike; hence \(z \partial F/\partial z \geq 0\) for only one value of \(z\) on \(|z| = 1\). But under a conformal mapping \(z = e^{i\phi}(w - \alpha)/(1 - \bar{\alpha}w)\) of \(|w| < 1\) onto \(|z| < 1\),
\[ z_v \frac{\partial F}{\partial z_v} = (1 - |\alpha|^2)^{-1} |w_v - \alpha|^2 w_v \frac{\partial F}{\partial w_v}, \quad |w_v| = 1. \]

Hence \( w_v \frac{\partial F}{\partial w_v} \geq 0 \) can happen for only one value of \( w_v, |w_v| = 1 \).

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University of Michigan,
Ann Arbor, Michigan