ON THE CENTRAL DECOMPOSITION FOR POSITIVE FUNCTIONALS ON C*-ALGEBRAS

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1. Introduction. Let $A$ be a $C^*$-algebra with unit 1, $A^*$ the dual Banach space of $A$, $S$ the set of all states $\phi$ on $A$ (namely, $\phi(a^*a) \geq 0$ for $a \in A$ and $\phi(1) = 1$), $\Omega$ the set of all extreme points of $S$.

Then $S$ is a $\sigma(A^*,A)$-compact space. Let $C(S)$ be the Banach algebra of all continuous complex-valued functions on the compact space $S$ with the usual supremum norm, then the $A$ may be topologically embedded into $C(S)$; moreover, the self-adjoint portion $A^0$ of $A$ may be order-isomorphically embedded into the real Banach space $C_r(S)$ of all continuous real-valued functions on $S$.

Any positive linear functional $\psi$ on $A$ extends to a positive linear functional on $C(S)$ and thus, by the Riesz representation theorem, there will be some positive Radon measure $\mu$ on $S$ so that

\[ \psi(a) = \int \phi(a) d\mu(\phi) \quad \text{for} \quad a \in A. \]

Let $\mathcal{M}(\psi)$ be the set of all positive Radon measures on $S$ satisfying the equality $(\ast)$. In general, $\mathcal{M}(\psi)$ consists of many different measures. An important family of measures belonging to $\mathcal{M}(\psi)$ is the one consisting of measures concentrated on $\Omega$. M. Tomita [11], [18] showed that if $A$ is separable, $\mathcal{M}(\psi)$ contains a measure $\mu$ such that $\mu(\Omega) = \mu(S)$ and gave a refinement of the Mautner-Godement-Segal theorem [10], [7], [16] that an arbitrary separable $*$-representation of $A$ can be expressed as a direct integral of irreducible $*$-representations—that is, he showed that the Mautner-Godement-Segal decomposition can be realized by a Radon measure on the compact space $S$. Moreover, at the present time, we can obtain this result of Tomita and, furthermore, show that $\Omega$ is a $G_\delta$-set, if $A$ is separable, from the general theory of Choquet [2], Bishop and de Leeuw [1].

Also, L. Loomis [8] gave further developments along this line.

If $A$ is commutative, the $\mu$ such that $\mu \in \mathcal{M}(\psi)$ and $\mu(\Omega) = \mu(S)$ is unique; this implies the classical theorem that the problem of determining all cyclic unitary representations of a locally compact abelian group $G$ can be reduced to the prob-

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Problem of determining all equivalence classes of bounded positive Radon measures on the dual group $\hat{G}$.

If $A$ is not commutative, such $\mu$ is, in general, not unique even for a finite-dimensional $A$. Therefore, in spite of their importance, such measures are not suitable for the decomposition theory of representations.

On the other hand, von Neumann's reduction theory (cf. [10], [12]) insures that every separable $\ast$-representation of $A$ may be expressed as a direct integral of factor $\ast$-representations (called the central decomposition) and such a decomposition is essentially unique.

From this fact, for a separable $A$, we can guess the existence and the unicity of a distinguished Radon measure belonging to $\mathcal{M}(\psi)$ which realizes the central decomposition of the $\ast$-representation of $A$ constructed via $\psi$ on the compact space $\mathcal{S}$.

The main purpose of this paper is to show that this guess is true. We say that $\phi$ is primary if the $\ast$-representation of $A$ constructed via $\phi$ is a factor representation. Let $K$ be the set of all primary states on $A$. Then we shall show: if $A$ is separable, $K$ is measurable for all Radon measures $\nu$ on the compact space $\mathcal{S}$ (Theorem 1) and, moreover, there is one and only one measure $\mu$ belonging to $\mathcal{M}(\psi)$ such that $\mu(K) = \mu(\mathcal{S})$ and the expression $\psi(a) = \int_\mathcal{S} \phi(a) d\mu(\phi)$ realizes the central decomposition of the $\ast$-representation of $A$ constructed via $\psi$ (Theorems 2, 3 and 4).

We call such a unique $\mu$ the central Radon measure of $\psi$.

2. Preliminaries. Let $\mathfrak{A}$ be a $C^*$-algebra. The term "$\ast$-representation" of $\mathfrak{A}$ shall mean a homomorphism $a \rightarrow \pi_a$ of $\mathfrak{A}$ into the algebra of all bounded operators on some Hilbert space $\mathcal{H}$ such that $(\pi_a)^* = \pi_{a^*}$ for $a \in \mathfrak{A}$, and denote it by $\pi(\mathcal{H})$. Two $\ast$-representations $\pi^1(\mathcal{H}_1)$, $\pi^2(\mathcal{H}_2)$ are said to be equivalent if there exists a unitary mapping $U$ of $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $U\pi_a^1 U^{-1} = \pi_a^2$ for $a \in \mathfrak{A}$.

Let $\mathfrak{A}^{**}$ be the second dual Banach space of $\mathfrak{A}$; then, according to a result of Sherman [17], $\mathfrak{A}^{**}$ may be regarded as a $W^*$-algebra, whose associated space is the dual Banach space $\mathfrak{A}^*$ of $\mathfrak{A}$, and $\mathfrak{A}$ may be regarded as a $\sigma$-dense $C^*$-subalgebra (the $\sigma$-topology on $\mathfrak{A}^{**}$ shall mean the weak-$\ast$-topology on $\mathfrak{A}^{**}$) of $\mathfrak{A}^{**}$, when $\mathfrak{A}$ is canonically embedded into $\mathfrak{A}^{**}$.

Definition 1. Let $G$ be a locally compact group, and $A(G)$ the group $C^*$-algebra of $G$ (cf. 6); then the $W^*$-algebra $A(G)^{**}$ is called the group $W^*$-algebra of $G$ and denoted by $W(G)$.

Definition 2. Let $M$ be a $W^*$-algebra; then a $W^*$-representation $\pi(\mathcal{H})$ of $M$ is a continuous $\ast$-representation of $M$ with the $\sigma$-topology (the $\sigma$-topology on $M$ shall mean the weak $\ast$-topology on $M$ (cf. [14])) into the algebra $B(\mathcal{H})$, with the weak operator topology, of all bounded operators on a Hilbert space $\mathcal{H}$ such that $\pi_1$ is the identity operator on $\mathcal{H}$ for the identity 1 of $M$.

We shall denote by $\pi[M]$ the image of $M$ under $\pi$. If $\pi(\mathcal{H})$ is a $W^*$-representation
of a $W^*$-algebra $M$, $\pi[M]$ is a weakly closed $\ast$-subalgebra (called a concrete $W^*$-algebra) of $B(\mathcal{H})$ (cf. 14).

Now let $\pi(\mathcal{A})$ be a $\ast$-representation of a $C^*$-algebra $\mathcal{A}$ such that $\{\xi | \pi_a^*\xi = 0 \text{ for } \xi \in \mathcal{H} \text{ and all } a \in \mathcal{A}\} = (0)$ (called a nowhere trivial $\ast$-representation), then $\pi(\mathcal{A})$ can be uniquely extended to a $W^*$-representation of $\mathcal{A}^{**}$ (cf. [14, Theorem in Appendix]). We shall denote also by $\pi(\mathcal{A})$ this $W^*$-representation of $\mathcal{A}^{**}$.

Conversely, let $\tilde{\pi}(\mathcal{A})$ be a $W^*$-representation of $\mathcal{A}^{**}$, then its restriction on $\mathcal{A}$ is a nowhere trivial $\ast$-representation $\pi(\mathcal{A})$ of $\mathcal{A}$. Moreover, the $W^*$-representation of $\mathcal{A}^{**}$ obtained from the $\ast$-representation $\pi(\mathcal{A})$ of $\mathcal{A}$ coincides clearly with the $\tilde{\pi}(\mathcal{A})$.

The $W^*$-representation of $\mathcal{A}^{**}$ is irreducible if and only if the corresponding $\ast$-representation of $\mathcal{A}$ is irreducible; two $W^*$-representations of $\mathcal{A}^{**}$ are equivalent if and only if the corresponding two $\ast$-representations of $\mathcal{A}$ are equivalent. Therefore the theory of $\ast$-representations of $\mathcal{A}$ can be reduced to the $W^*$-representation theory of $\mathcal{A}^{**}$, so that the unitary representation theory of a locally compact group $G$ can be reduced to the $W^*$-representation theory of the group $W^*$-algebra $W(G)$.

Let $M$ be a $W^*$-algebra, and $\pi(\mathcal{A})$ a $W^*$-representation of $M$; then the kernel $\mathcal{Z} = \{a | \pi_a = 0, a \in M\}$ is a $\sigma$-closed ideal of $M$; hence there is a unique central projection $\pi$ such that $\mathcal{Z} = M_\pi$ (cf. [14]). Put $1 - \pi = s(\pi)$; $s(\pi)$ is called the support of $\pi$. The support of $\pi$ is a nonzero central projection and the restriction of $\pi$ on $M(\pi)$ is one-to-one.

**Definition 3.** Let $\pi_1(\mathcal{A}_1), \pi_2(\mathcal{A}_2)$ be two $W^*$-representations of $M$. If $s(\pi_1) = s(\pi_2)$, we say that $\pi_1(\mathcal{A}_1)$ is quasi-equivalent to $\pi_2(\mathcal{A}_2)$.

Clearly, the quasi-equivalence is a usual equivalence relation, so that by this relation we can classify $W^*$-representations of $M$ into quasi-equivalence classes.

Let $\mathcal{D}(M)$ be the family of all quasi-equivalence classes of the $W^*$-representations of $M$; then, for each element $\rho \in \mathcal{D}(M)$, there corresponds a unique nonzero central projection $c(\rho)$ of $M$ such that $c(\rho) = s(\pi)$ for every $\pi(\mathcal{A}) \in \rho$.

Conversely, let $\pi$ be a central nonzero projection; then $M_\pi$ is a $W^*$-algebra, so that it has a faithful $W^*$-representation $\pi(\mathcal{A})$ (cf. [13]).

Then the mapping $a \mapsto \pi_{ax}$ of $M$ into $B(\mathcal{A})$ is a $W^*$-representation of $M$; hence there is unique element $\rho$ of $\mathcal{D}(M)$ such that $c(\rho) = \pi$.

Therefore, there is a one-to-one correspondence between all elements of $\mathcal{D}(M)$ and all nonzero central projections of $M$.

For convenience, we shall add an imaginary 0-element to $\mathcal{D}(M)$ and we shall denote by $\mathcal{D}'(M)$ the set $\mathcal{D}(M) \cup \{0\}$; moreover, we shall make the 0-element of $M$ correspond to this 0-element of $\mathcal{D}'(M)$. Then there is a one-to-one correspondence between $\mathcal{D}'(M)$ and the set $Z_\rho$ of all central projections of $M$. Since $Z_\rho$ is a complete Boolean algebra, we can canonically introduce its Boolean structure into $\mathcal{D}'(M)$.

Also, we can regard every element $\rho$ of $\mathcal{D}(M)$ as a $\sigma$-continuous $\ast$-homomorphism of $M$ onto the $W^*$-algebra $M_{c(\rho)}$. Therefore, within quasi-equivalence,
the $W^*$-representation theory of $M$ can be completely reduced to the structure theory of the $W^*$-algebra $M$.

So we shall freely use various definitions and theorems concerning the structure of $W^*$-algebras. For them, the reader should be referred to [3] and [14].

Let $\pi(\mathcal{H})$ be a nowhere trivial $*$-representation of a $C^*$-algebra $\mathcal{H}$. Then the support $s(\pi)$ of $\pi(\mathcal{H})$ shall mean the support of the corresponding $W^*$-representation $\pi(\mathcal{H})$ of the $W^*$-algebra $\mathcal{H}^\ast$.

Let $\pi^1(\mathcal{H}_1), \pi^2(\mathcal{H}_2)$ be two nowhere trivial $*$-representations of $\mathcal{H}$. If the corresponding two $W^*$-representations of $\mathcal{H}^\ast$ are quasi-equivalent, we say that $\pi^1(\mathcal{H}_1)$ and $\pi^2(\mathcal{H}_2)$ are quasi-equivalent.

3. The decomposition theory. Let $A$ be a $C^*$-algebra with unit $1$. In this section we shall always assume that $A$ is uniformly separable. Let $A^\ast$ be the dual Banach space of $A$ and $\mathcal{G}$ the set of all states on $A$; then $\mathcal{G}$ is $\sigma(A^\ast, A)$-compact; let \{a$_n$\} be a sequence of nonzero elements which is uniformly dense in the selfadjoint portion $A^\ast$ of $A$.

For $\phi, \psi \in \mathcal{G}$, define

$$d(\phi, \psi) = \sum_{i=1}^\infty \frac{|(\phi - \psi)(a_n)|}{2^n \|a_n\|};$$

then $d$ is a metric on $\mathcal{G}$ which is equivalent to $\sigma(\mathcal{G}, A)$; therefore, $\mathcal{G}$ is considered a compact metric space with respect to $\sigma(\mathcal{G}, A)$; hence, the compact space $\mathcal{G}$ satisfies the second countability axiom, because a compact metric space is always separable. In this section we shall always consider the $\mathcal{G}$ as the compact space with the topology $\sigma(\mathcal{G}, A)$.

Let $\phi \in \mathcal{G}$, and let $V_\phi$ be the invariant closed subspace under $R_\phi, L_\phi(a \in A)$ generated by $\phi$; then there is a unique central projection $z_\phi$ such that $L_{z_\phi}A^\ast = V_\phi$ (cf. [14], [19]); clearly $z_\phi$ is the least central projection of $A^\ast$ containing the support $s(\phi)$ of $\phi$ on $A^\ast$ (cf. [14]) and, moreover, $s(\pi^\phi) = z_\phi$, where $\pi^\phi(\mathcal{H}_\phi)$ is the $*$-representation of $A$ constructed via $\phi$.

Now we shall show

**Theorem 1.** Let $K$ be a subset of $\mathcal{G}$ consisting of all primary states $\phi$ (namely, the $*$-representation $\pi^\phi(\mathcal{H}_\phi)$ of $A$ is a factor representation); then $K$ is $\nu$-measurable for all Radon measures $\nu$ on the compact space $\mathcal{G}$.

**Proof.** It is clear that the primarity of $\phi$ is equivalent to the fact that the support $s(\pi^\phi)$ of $\pi^\phi(\mathcal{H}_\phi)$ is a minimal central projection of $A^\ast$.

Now put $\mathcal{G} = \{\phi \mid \phi \geq 0, \phi(1) \leq 1, \phi \in A^\ast\}$; then $\mathcal{G}$ is $\sigma(A^\ast, A)$-compact; let \{a$_n$\} be a sequence of elements which is uniformly dense in $A$. Consider the product space $\mathcal{G} \times \mathcal{G}$ and put

$$G_{m,n} = \{ (\phi, \psi) \mid \|R_{a_m}L_{a_m}^\ast \phi - R_{a_n}L_{a_n}^\ast \psi\| = \|R_{a_m}L_{a_m}^\ast \phi + R_{a_n}L_{a_n}^\ast \psi\|, (\phi, \psi) \in \mathcal{G} \times \mathcal{G}\}.$$
Since
\[
\|R_{an}L_{an}\phi \pm R_{an}L_{an}\psi\| = \sup_{\|\alpha\| \leq 1} |\phi(a_n^*aa_n) \pm \psi(a_n^*aa_n)|
\]
and the function \((\phi, \psi) \mapsto \phi(a_n^*aa_n) \pm \psi(a_n^*aa_n)\) on the compact space \(\mathcal{F} \times \mathcal{F}\) is continuous, \(\|R_{an}L_{an} \pm R_{an}L_{an}\psi\|\) is lower semi-continuous; hence, \(G_{m,n}\) is a Borel set in the compact space \(\mathcal{F} \times \mathcal{F}\); put \(F = \{(\phi, \psi) \mid \phi(1) + \psi(1) = 1, (\phi, \psi) \in \mathcal{F} \times \mathcal{F}\}\); then \(F\) is a closed set in \(\mathcal{F} \times \mathcal{F}\); and put \(\mathcal{F}_0 = \{\phi \mid \phi > 0, \phi(1) \leq 1\}\); then \(\mathcal{F}_0 \times \mathcal{F}_0\) is an open set in \(\mathcal{F} \times \mathcal{F}\).

Now put \(U = (\mathcal{F}_0 \times \mathcal{F}_0) \cap (F \cap \bigcap_{m,n=1}^{\infty} G_{m,n})\); then, clearly, \(U\) is a Borel set in \(\mathcal{F} \times \mathcal{F}\).

Consider the mapping \((\phi, \psi) \mapsto \phi + \psi\) of \(F\) onto \(\mathcal{E}\); it is continuous, so that \(\Phi(U)\) is analytic and so \(\nu\)-measurable for all Radon measures \(\nu\) on the compact space \(\mathcal{E}\) (cf. [3, Appendix (V)]).

If \(z \in \Phi(U)\), then there is an element \((\phi, \psi)\) in \(U\) such that \(z = \phi + \psi\); \((\phi, \psi) \in U\) implies \(s(R_{an}L_{an}^*\phi) \cdot s(R_{an}L_{an}^*\psi) = 0\) for all \(m, n\) (cf. [14]) and so the closed invariant subspaces generated by \(\phi\) and \(\psi\), respectively, are mutually disjoint; hence \(s(\pi^z) \cdot s(\pi^y) = 0\), so that \(s(\pi^z) = s(\pi^x) + s(\pi^y)\) is not minimal. Conversely, suppose \(z \notin K\), then there is a nonzero central projection \(z\) of \(A^{**}\) such that \(z < s(\pi^z)\), then \(\xi(a) = \xi(za) + \xi(s(\pi^z) - za)\) for \(a \in A\); clearly \((L_{za}, L_{s(\pi^z) - za}) \in U\), so that \(\xi \in \Phi(U)\); Hence \(K = \mathcal{E} - \Phi(U)\). This completes the proof.

**PROBLEM 1.** Can we conclude that \(K\) is a Borel subset in the compact space \(\mathcal{E}\)?

Now let \(\mu\) be a positive Radon measure on the compact space \(\mathcal{E}\). Put \(\psi(a) = \int_{\mathcal{E}} \phi(a) d\mu(\phi)\) for \(a \in A\); then \(\psi\) is a positive linear functional on \(A\). We shall always consider the \(\psi\) as a \((A^{**}, A^*)\)-continuous positive linear functional on the \(W^*\)-algebra \(A^{**}\), because positive linear functionals on \(A\) extend canonically to \((A^{**}, A^*)\)-continuous positive linear functionals on \(A^{**}\). Then,

**DEFINITION 4.** We say \(\mu\) is a central Radon measure if it satisfies the following condition:

\((**\)\) There is a \(\sigma\)-continuous homomorphism \(\Phi\) of the center \(Z\) of the \(W^*\)-algebra \(A^{**}\) onto the \(W^*\)-algebra \(L^\infty(\mathcal{E}, \mu)\) of all essentially bounded \(\mu\)-measurable functions on \(\mathcal{E}\) such that

\[
\psi(za) = \int_{\mathcal{E}} \Phi(z)(\phi) \phi(a) d\mu(\phi)
\]

for \(z \in Z\) and \(a \in A\).

Now let \(\mu\) be a positive central Radon measure on the compact space \(\mathcal{E}\) and put \(\psi(a) = \int_{\mathcal{E}} \phi(a) d\mu(\phi)\) for \(a \in A\).

Consider the \(*\)-representation \(\pi^\psi(\mathcal{E}_\phi)\) of \(A\) and the corresponding \(W^*\)-representation \(\pi^\psi(\mathcal{E}_\phi)\) of \(A^{**}\). We shall define a linear mapping of \(\mathcal{E}_\phi\) into \(\mathcal{E}_\phi\)-valued functions on \(\mathcal{E}\) in the following way.

Let \(\xi = \sum_{i=1}^n c_i \pi_{a_i}^\psi 1_\psi \in \mathcal{E}_\phi\), where \(c_i \in \mathbb{Z}\), \(a_i \in A\) and \(1_\psi\) is the image of \(1\) in \(\mathcal{E}_\phi\).
Then

\[ \| \xi \|^2 = \left\langle \sum_{i=1}^{n} \pi_{c_i}^\psi \pi_{a_i}^\psi 1_\phi, \sum_{i=1}^{n} \pi_{c_i}^\psi \pi_{a_i}^\psi 1_\phi \right\rangle \]

\[ = \sum_{i,j=1}^{n} \langle \pi_{c_i a_i}^\psi 1_\phi, \pi_{c_j a_j}^\psi 1_\phi \rangle \]

\[ = \sum_{i,j=1}^{n} \psi(a_j^* c_j c_i a_i) \]

\[ = \sum_{i,j=1}^{n} \psi(c_j^* c_i a_j a_i) \]

\[ = \sum_{i,j=1}^{n} \int_{\mathbb{G}} \Phi(c_j)(\phi) \Phi(c_i)(\phi) (a_j^* a_i) \, d\mu(\phi). \]

On the other hand, for each \( \phi \in \mathbb{G} \), we consider the \( * \)-representation \( \pi^\psi(\mathcal{H}_\phi) \) of \( A \). For \( a \in A \), the image of \( a \) in \( \mathcal{H}_\phi \) is denoted by \( a^\phi \). Then the function \( \phi \rightarrow a^\phi \) on \( \mathbb{G} \) is a \( \mathcal{H}_\phi \)-valued function. We shall denote this function by \( \bar{a} \). The function \( \phi \rightarrow \sum_{i=1}^{n} \Phi(c_i)(\phi) a_i,^\phi \) is a \( \mathcal{H}_\phi \)-valued function and we have

\[ \| \xi \|^2 = \int_{\mathbb{G}} \left\| \sum_{i=1}^{n} \Phi(c_i)(\phi) a_i,^\phi \right\|^2 \, d\mu(\phi). \]

Hence the mapping \( \xi \rightarrow \sum_{i=1}^{n} \Phi(c_i) a_i,^\phi \) extends uniquely a unitary mapping \( U \) of \( \mathcal{H}_\phi \) onto the direct integral \( \int_{\mathbb{G}} \mathcal{H}_\phi \, d\mu(\phi) \) of the spaces \( \mathcal{H}_\phi \) with respect to the measure \( \mu \) (cf. [3]), because elements of the form \( \xi \) are dense in \( \mathcal{H}_\phi \).

Hence, under this unitary mapping, we can write \( \mathcal{H}_\psi = \int_{\mathbb{G}} \mathcal{H}_\phi \, d\mu(\phi) \).

Moreover, for \( c \in Z \),

\[ (U \pi_c^\psi \xi)(\phi) = U \pi_c^\psi \sum_{i=1}^{n} \pi_{c_i}^\psi \pi_{a_i}^\psi 1_\phi = U \sum_{i=1}^{n} \pi_{c_i}^\psi \pi_{a_i}^\psi 1_\phi \]

\[ = \sum_{i=1}^{n} \Phi(c c_i)(\phi) a_i,^\phi = \Phi(c)(\phi) \sum_{i=1}^{n} \Phi(c_i)(\phi) a_i,^\phi \]

\[ = \Phi(c)(\phi)(U \xi)(\phi); \]

hence \( U \pi^\psi_c U^* = \Phi(c) \).

Since \( \{ \pi_c^\psi \mid c \in Z \} \) = the center of the concrete \( W^* \)-algebra \( \pi^\psi[A^{**}] \) and \( \Phi(Z) = L^\infty(\mathbb{G}, \mu) \), under the realization \( \mathcal{H}_\psi = \int_{\mathbb{G}} \mathcal{H}_\phi \, d\mu(\phi) \), the center of \( \pi^\psi[A^{**}] \) is the set of all diagonalizable operators.

Hence we have the central decomposition (cf. [12])
\[ \pi^\psi[A^{**}] = \int \{ \pi^\psi[A^{**}] \}(\phi) \, d\mu(\phi), \]

and \( \{ \pi^\psi[A^{**}] \}(\phi) \) is a factor for \( \mu \)-almost all \( \phi \in \mathcal{S} \). \( \{ \pi^\psi[A^{**}] \}(\phi) = \pi^\psi[A^{**}] \) for \( \mu \)-almost all \( \phi \in \mathcal{S} \), because \( \pi^\psi[A] \) (respectively, \( \pi^\psi[A] \)) is weakly dense in \( \pi^\psi[A^{**}] \) (respectively, \( \pi^\psi[A^{**}] \)). Therefore we have the following theorem.

**Theorem 2.** Let \( \mu \) be a positive central Radon measure on the compact space \( \mathcal{S} \) and put \( \psi(a) = \int \phi(a) \, d\mu(\phi) \) for \( a \in A \). Then the \( \ast \)-representation \( \pi^\psi(\mathcal{S}_\psi) \) of \( A \) is expressed as a direct integral of the \( \ast \)-representations \( \pi^\psi(\mathcal{S}_\psi) \) of \( A \) with respect to the measure \( \mu \), on the compact space \( \mathcal{S} \). Moreover, this integral is the central decomposition, and so it satisfies the following properties:

1. \( \pi^\psi(\mathcal{S}_\psi) \) is a factor representation for \( \mu \)-almost all \( \phi \in \mathcal{S} \), i.e., \( \mu(K) = \mu(\mathcal{S}) \),
2. there is a Borel subset \( \Delta \mu \) of the compact space \( \mathcal{S} \) such that \( \Delta \mu \subset K \),
3. \( s(\pi^{\psi_1}) \cdot s(\pi^{\psi_2}) = 0 \) for two arbitrary different \( \phi_1, \phi_2 \in \Delta \mu \) and \( \mu(\Delta \mu) = \mu(\mathcal{S}) \).

For the proof of property (ii) the reader is referred to [5, proof of Proposition 3].

**Theorem 3.** Suppose that \( \mu_1, \mu_2 \) are two positive central Radon measures on the compact space \( \mathcal{S} \) such that \( \psi(a) = \int \phi(a) \, d\mu_1(\phi) = \int \phi(a) \, d\mu_2(\phi) \) for all \( a \in A \); then \( \mu_1 = \mu_2 \).

**Proof.** Let \( \Phi_1 \) (respectively, \( \Phi_2 \)) be a \( \sigma \)-continuous homomorphism of \( Z \) onto \( L^\infty(\mathcal{S}, \mu_1) \) (respectively, \( L^\infty(\mathcal{S}, \mu_2) \)) such that \( \psi(za) = \int \Phi_i(z)(\phi) \phi(a) \, d\mu_i(\phi) \) for \( z \in Z \) and \( a \in A \) \( (i = 1, 2). \) From the discussions in the proof of Theorem 2, clearly the kernel of \( \Phi_1 \) (respectively, \( \Phi_2 \)) is \( (1 - s(\pi^{\psi_1}))(\mathcal{S}) \) (respectively, \( (1 - s(\pi^{\psi_2}))(\mathcal{S}) \)).

For \( a \) \( (\geq 0) \) \( a \in A \), the continuous function \( \phi \rightarrow \phi(a) \) on \( \mathcal{S} \) belongs to \( L^\infty(\mathcal{S}, \mu_1) \) and \( L^\infty(\mathcal{S}, \mu_2), \) respectively, so that there are positive elements \( c_1^a, c_2^a \) in \( \mathcal{Z} \) such that \( \phi(a) = \Phi_1(c_1^a)(\phi) \mu_1 - \text{a.e.} \) and \( \phi(a) = \Phi_2(c_2^a)(\phi) \mu_2 - \text{a.e.} \).

Suppose that \( c_1^a \neq c_2^a \), then there is a projection \( z \) of \( Z \) such that \( c_1^a z < c_2^a z \) or \( c_1^a z > c_2^a z \); assume that \( c_1^a z < c_2^a z \), then

\[
\psi(c_1^a z) = \int \Phi_1(c_1^a)(\phi) \Phi_1(z)(\phi) \, d\mu_1(\phi)
= \int \Phi_1(z)(\phi) \phi(a) \, d\mu_1(\phi)
= \psi(az)
\]

and, analogously,

\[
\psi(c_2^a z) = \psi(az);
\]

hence \( c_2^a z - c_1^a z = 0 \), a contradiction; hence we have \( c_1^a = c_2^a = c^a \). Therefore,
\[ \int \phi(a_1) \phi(a_2) \cdots \phi(a_n) \, d\mu_1(\phi) \]

\[ = \int \Phi_1(c^{a_1}) (\phi) \Phi_1(c^{a_2}) (\phi) \cdots \Phi(c^{a_n}) (\phi) \, d\mu_1(\phi) \]

\[ = \psi(c^{a_1} c^{a_2} \cdots c^{a_n}) \]

\[ = \int \Phi_2(c^{a_1} c^{a_2} \cdots c^{a_n}) (\phi) \, d\mu_2(\phi) \]

\[ = \int \Phi_2(c^{a_1}) (\phi) \Phi_2(c^{a_2}) (\phi) \cdots \Phi_2(c^{a_n}) (\phi) \, d\mu_2(\phi) \]

\[ = \int \phi(a_1) \phi(a_2) \cdots \phi(a_n) \, d\mu_2(\phi) \]

for \( a_1, a_2, \ldots, a_n \in A \).

Therefore, by the Stone-Weierstrass Theorem,

\[ \int f(\phi) \, d\mu_1(\phi) = \int f(\phi) \, d\mu_2(\phi) \]

for \( f \in c(\mathfrak{S}) \), where \( c(\mathfrak{S}) \) is the Banach algebra of all continuous functions on the compact space \( \mathfrak{S} \); hence \( \mu_1 = \mu_2 \).

Moreover, the *-algebra generated by \( \{ e^a \mid a \in A \} \) is \( \sigma \)-dense in \( Zs(\pi^W) \), because \( c(\mathfrak{S}) \) is \( \sigma \)-dense in \( L^\infty(\mathfrak{S}, \mu_i) \) \( (i = 1, 2) \) and \( \Phi_i \) is isomorphic on \( Zs(\pi^W) \); hence \( \Phi_1 = \Phi_2 \). This completes the proof.

**Definition 5.** From the proof of Theorem 3, the corresponding homomorphism \( \Phi \) to a positive central Radon measure \( \mu \) on the compact space \( \mathfrak{S} \) is unique. This unique \( \Phi \) is called the homomorphism of \( \mu \) and denoted by \( \Phi(\mu) \).

By Theorem 3, the mapping \( \mu \to \psi = \int \phi \, d\mu \) of the set of all positive central Radon measures on the compact space \( \mathfrak{S} \) into the set of all positive linear functionals on \( A \) is one-to-one.

Now we shall show this mapping to be **onto**.

**Theorem 4.** Let \( \psi \) be a positive linear functional on \( A \); then there is a positive central Radon measure \( \mu \) on the compact space \( \mathfrak{S} \) such that \( \psi(a) = \int \phi(a) \, d\mu(\phi) \) for \( a \in A \).

To prove Theorem 4, we shall provide some considerations.

For \( \psi(\geq 0) \in A^* \), we shall consider \( \psi \) as a \( \sigma(A^{**}, A^*) \)-continuous positive linear functional on \( A^{**} \) canonically.

The \( W^* \)-algebra \( A^{**}(\pi^W) \) has the separable associated space; therefore, there is a homogeneous type \( I_{\mathfrak{R}_0} \) \( W^* \)-algebra \( N \) such that \( A^{**}(\pi^W) \subset N \) and the center \( Zs(\pi^W) \) of \( A^{**}(\pi^W) \) = the center of \( N \).
We shall represent the $N$ as follows: $N = L^\infty(B, Q, \omega)$, $N_* = L^1(B_*, Q, \omega)$, where $B$ is a type $I_{K_0}$-factor, $N_*$ (respectively, $B_*$) is the associated space of $N$ (respectively, $B$), $Q$ is a compact space satisfying the second countability axiom and $\omega$ is a positive Radon measure on $Q$ with $\omega(Q) = 1$, and $L^\infty(B, Q, \omega)$ is the $W^*$-algebra of all essentially bounded weakly $*-$measurable $B$-valued functions on the $Q$ and $L^1(B_*, Q, \omega)$ is the Banach space of all strongly $\omega$-integrable $B_*$-valued functions on the $Q$ (cf. [14], [15]).

Let $\tilde{\psi}$ be the restriction of $\psi$ on the $W^*$-algebra $A**s(\pi^*)$, then it can be extended to a $\sigma$-continuous linear functional $\tilde{\psi}$ on $N$ (cf. [14]).

Under the above representation, we have

$$\psi = \int_Q \tilde{\psi}_t \, d\omega(t),$$

where the function $\tilde{\psi}_t$ on $Q$ belongs to $L^1(B_*, Q, \omega)$ and $\tilde{\psi}_t$ is $\mu$-almost everywhere positive (cf. [14]).

Since $||\tilde{\psi}_t|| = \int_Q ||\tilde{\psi}_t|| \, d\omega(t)$, $\tilde{\psi}_t(1)$ is $\omega$-integrable; moreover, let $F = \{ t \mid ||\tilde{\psi}_t|| = 0, t \in Q \}$, then the function $\chi_F(t) \cdot 1 = z \in Zs(\pi^*)$, where $\chi_F$ is the characteristic function of $F$.

Since $\int_Q \tilde{\psi}_t(\chi_F(t) \cdot 1) \, d\omega(t) = \tilde{\psi}(1) = 0$, $\pi^*_z = 0$ and so $z = 0$; therefore $\tilde{\psi}(1) > 0$ $\omega$-a.e.

For $a \in As(\pi^*) \subset A**s(\pi^*)$, we write $a = \int_Q a(t) \, d\omega(t)$ in $L^\infty(B, Q, \omega)$; then the function $\tilde{\psi}_t(a(t))/\tilde{\psi}_t(1)$ on $Q$ is bounded $\omega$-measurable.

Let $C$ be a commutative $C^*$-subalgebra of the $W^*$-algebra $L^\infty(Q, \omega)$ generated by the family $\{ \tilde{\psi}_t(a(t))/\tilde{\psi}_t(1) \mid a \in As(\pi^*) \}$; then we shall show

**Lemma 1.** $C$ is $\sigma$-dense in the $W^*$-algebra $L^\infty(Q, \omega)$.

**Proof.** Suppose that there is an $\omega$-integrable complex-valued function $g$ on $Q$ such that $\int_Q (\tilde{\psi}_t(a(t))/\tilde{\psi}_t(1)) \, g(t) \, d\omega(t) = 0$ for all $a \in As(\pi^*)$. Since the function $\tilde{\psi}_t \in L^1(B_*, Q, \omega)$, the function $(g(t)/\tilde{\psi}_t(1)) \tilde{\psi}_t$ is strongly $\omega$-measurable and

$$\int_Q \left\| \frac{g(t)}{\tilde{\psi}_t(1)} \tilde{\psi}_t \right\| \, d\omega(t) = \int_Q |g(t)| \, d\omega(t) < +\infty;$$

hence

$$\frac{g(t)}{\tilde{\psi}_t(1)} \tilde{\psi}_t \in L^1(B_*, Q, \omega)$$

and so there is an element $\check{f}$ of $N_*$ such that

$$\check{f}(x) = \int_Q \frac{g(t)}{\tilde{\psi}_t(1)} \tilde{\psi}_t(x(t)) \, d\omega(t)$$

for

$$x = \int_Q x(t) \, d\omega(t) \in L^\infty(B, Q, \omega) = N.$$
Since $\int (As(\pi^\psi)) = 0$, $\int (A^* s(\pi^\psi)) = 0$, so that $\int (Zs(\pi^\psi)) = 0$; therefore

$$\int_Q \frac{g(t)}{\psi_f(t)} \psi_f(h(t) \cdot 1) d\omega(t) = \int_Q h(t) g(t) d\omega(t) = 0$$

for all $h \in L^\infty(Q, \omega)$; hence $g(t) = 0$ $\omega$-a.e. This completes the proof.

Let $Q_1$ be the spectrum space of $C$, then $\omega(g) = \int_Q g(t) d\omega(t)$ for $g \in C$ defines a positive Radon measure $\omega_1$ on $Q_1$ such that $\int_Q g(t) d\omega(t) = \int_{Q_1} g(v) d\omega_1(v)$ for $g \in C$.

Since $C$ is $\sigma$-dense in $L^\infty(Q, \omega)$, the above equality implies that the isomorphism $g(t) \rightarrow g(v)$ of $C$ in $L^\infty(Q, \omega)$ onto $C$ in $L^\infty(Q_1, \omega_1)$ can be uniquely extended to an isomorphism $\Psi$ of $L^\infty(Q, \omega)$ onto $L^\infty(Q_1, \omega_1)$ (cf. [14]).

Since

$$\left| \frac{\tilde{\psi}_f(a(t) - b(t))}{\tilde{\psi}_f(1)} \right| \leq \frac{\tilde{\psi}_f(1)}{\tilde{\psi}_f(1)} \| a - b \| \omega$-a.e.

and the $C\ast$-algebra $As(\pi^\psi)$ is uniformly separable, $C$ is uniformly separable; hence $Q_1$ satisfies the second countability axiom (cf. [3, Appendix (I)]).

Therefore, by the theorem of von Neumann (cf. [3, Appendix (IV)]), there is a $\omega$-null set $P$ in $Q$, an $\omega_1$-null set $P_1$ in $Q_1$ and a one-to-one measurable mapping $\eta$ of $Q - P$ onto $Q_1 - P_1$ such that $\Psi(f)(\eta(t)) = f(t)$ for $f \in L^\infty(Q, \omega)$ and $t \in Q - P$; moreover, in our case, clearly the $\eta$ is measure-preserving.

Therefore, using $\eta$, we can translate the structures $L^\infty(B, Q, \omega)$ and $L^1(B_*, Q, \omega)$ on the measure space $(Q, \omega)$ to the structures $L^\infty(B, Q_1, \omega_1)$ and $L^1(B_*, Q_1, \omega_1)$ on the measure space $(Q_1, \omega_1)$.

Then $\tilde{\psi}_v = \int_Q \tilde{\psi}_f d\omega(t) = \int_{Q_1} \tilde{\psi}_v d\omega_1(v)$ and $a = \int_Q a(t) d\omega(t) = \int_{Q_1} a(v) d\omega_1(v)$ for $a \in As(\pi^\psi)$, where $v = \eta(t)$.

$$\frac{\tilde{\psi}_v(a(v))}{\tilde{\psi}_v(1)} = \frac{\tilde{\psi}_v(a(t))}{\tilde{\psi}_v(1)} = \Psi(f)(v), \text{ where } f(t) = \frac{\tilde{\psi}_v(a(t))}{\tilde{\psi}_v(1)};$$

since $f \in C$, there is a continuous function $\xi_a(= \Psi(f))$ on $Q_1$ such that

$$\frac{\tilde{\psi}_v(a(v))}{\tilde{\psi}_v(1)} = \xi_a(v) \omega_1$-a.e.;$$

moreover, such a continuous function is unique—in fact, $\xi_a(v) = h(v) \omega_1$-a.e. for a continuous $h$ on $Q_1$ implies $\Psi^{-1}(h) = \Psi^{-1}(\xi_a)$, so that $\xi_a = h$ in $C$; hence $\xi_a(v) = h(v)$ for all $v \in Q_1$.

$$\frac{\tilde{\psi}_v((a \ast a)(v))}{\tilde{\psi}_v(1)} \geq 0 \omega_1$-a.e.$$

implies $\xi_{a \ast b}(v) \geq 0$ for all $v \in Q_1$, and, analogously, we have $\xi_{a + b}(v) = \xi_a(v) + \xi_b(v)$,
\( \xi_a(v) = \lambda \xi_a(v) \) and \( \xi_{a(\pi^g)}(v) = 1 \) for all \( v \in Q_1 \), where \( a, b \in As(\pi^g) \), \( \lambda \) a complex number.

Now put \( \phi_v(x) = \xi_{x(x^v)}(v) \) for \( x \in A \); then \( \phi_v \) is a state on \( A \) for all \( v \in Q_1 \); moreover, suppose \( v_1 \neq v_2 \); then there is a function \( \xi_a \) such that \( \xi_a(v_1) \neq \xi_a(v_2) \) \((a \in As(\pi^g))\), because the family \( \{\xi_a \mid a \in As(\pi^g)\} \) of continuous functions on \( Q_1 \) generates \( C \); hence, for some \( x \in A \), we have \( \phi_v(x) \neq \phi_v(x) \), and so \( \phi_{v_1} \neq \phi_{v_2} \).

**Lemma 2.** The mapping \( v \mapsto \phi_v \) of the compact space \( Q_1 \) into the compact space \( \mathcal{S} \) is homeomorphic.

**Proof.** Since the above consideration shows that \( p \) is one-to-one, it is enough to show the continuity of \( p \).

Suppose that \( v_x \to v \) in \( Q_1 \); then \( \xi_a(v_x) \to \xi_a(v) \) for all \( a \in As(\pi^g) \), so that \( \phi_{v_x}(x) \to \phi_v(x) \) for \( x \in A \). This completes the proof.

By this lemma, the Radon measure \( \omega_1 \) on \( Q_1 \) may be canonically considered a Radon measure \( \omega_1 \) on the compact space \( \mathcal{S} \) with the support \( p(Q_1) \).

Now we shall show

**Proof of Theorem 4.** Put \( d\mu(\phi) = \check{\psi}_\phi(1)d\omega_1(\phi) \), where \( \omega_1(\phi) \) is the Radon measure on \( \mathcal{S} \) defined in the preceding discussion, and \( \check{\psi}_\phi(1) = \check{\psi}_\psi(1) \) for \( \phi = p(v) \) \((v \in Q_1)\) and \( \check{\psi}_\phi(1) = 0 \) for \( \phi \notin p(Q_1) \).

Since \( \mu \) is equivalent to \( \omega_1 \), \( L^\infty(B,\mathcal{S},\mu) = L^\infty(B,\mathcal{S},\omega_1) \). Put \( \check{\psi} = \int_\mathcal{S} \check{\tau}_\phi d\mu(\phi) \) in \( L^1(B_\phi,\mathcal{S},\mu) \), then

\[
\check{\psi} = \int_\mathcal{S} \check{\tau}_\phi d\mu(\phi) = \int_\mathcal{S} \check{\psi}_\phi \cdot \frac{1}{\check{\psi}_\phi(1)} \check{\psi}_\phi(1)d\omega_1(\phi);
\]

hence \( \check{\tau}_\phi = \check{\psi}_\phi \check{\psi}_\phi(1) \) for \( \mu \)-almost all \( \phi \in \mathcal{S} \).

Therefore, for \( a \in A \),

\[
\psi(a) = \check{\psi}(as(\pi^g)) = \int \check{\psi}_\phi(as(\pi^g))(\phi) \frac{d\mu(\phi)}{\check{\psi}_\phi(1)} = \int_\mathcal{S} \xi_{as(\pi^g)}(\phi)d\mu(\phi) = \int_\mathcal{S} \phi(a)d\mu(\phi).
\]

Moreover, for \( z \in Z \), \( zs(\pi^g) \) belongs to the center of \( N \); here there is a unique essentially bounded \( \mu \)-measurable function \( f \) on \( \mathcal{S} \) such that \( (zs(\pi^g))(\phi) = f(\phi) \cdot 1 \) \( \mu \)-a.e. Put \( \Phi(z) = \text{the function } f(\phi) \); then \( \Phi \) is a \( \sigma \)-continuous homomorphism of \( Z \) onto the \( W^* \)-algebra \( L^\infty(\mathcal{S},\mu) \).

For \( z \in Z \) and \( a \in A \),
\[ \psi(z \alpha) = \tilde{\psi}(zas(\pi^*)) = \int_{\mathbb{S}} \tilde{\xi}(zas(\pi^*)) \, d\mu(\phi) \]
\[ = \int_{\mathbb{S}} \tilde{\xi}(\Phi(z) \, \phi) \, d\mu(\phi) \]
\[ = \int_{\mathbb{S}} \Phi(z) \, \tilde{\xi}(zas(\pi^*)) \, d\mu(\phi) \]
\[ = \int_{\mathbb{S}} \Phi(z) \, \phi(a) \, d\mu(\phi). \]

Hence, \( \mu \) is a positive central Radon measure on the compact space \( \mathbb{S} \). This completes the proof.

Next, in order to make our theorem applicable for the unitary representation theory of locally compact groups, we shall consider a separable \( C^* \)-algebra \( \mathcal{A} \) without unit.

Let \( \mathbb{S}_0 \) be the set of all positive linear functionals \( \phi \) on \( \mathcal{A} \) such that \( \phi \geq 0 \) and \( \| \phi \| \leq 1 \), and \( \mathbb{S}_1 \) the set of all positive linear functionals \( \phi \) on \( \mathcal{A} \) such that \( \| \phi \| = 1 \); then \( \mathbb{S}_0 \) is a \( (\mathbb{S}_0, \mathcal{A}) \)-compact space.

The function \( \phi \rightarrow \| \phi \| \) on the compact space \( \mathbb{S}_0 \) is lower semi-continuous, so that \( \mathbb{S}_1 \) is a Borel set in the compact space \( \mathbb{S}_0 \). Let \( \mathcal{A} \) be the \( C^* \)-algebra adjoining the unit \( 1 \) to \( \mathcal{A} \) and \( \phi_0 \) be a state on \( \mathcal{A} \) such that \( \phi_0(\mathcal{A}) = 0 \).

For \( \phi \in \mathbb{S}_0 \), we can uniquely extend \( \phi \) to a positive linear functional with the same norm on \( \mathcal{A} \), and we shall denote it by the same notation \( \phi \).

Define a mapping \( \alpha \) of \( \mathbb{S}_0 \) onto \( \mathbb{S} \) as follows:
\[ \alpha(\phi) = \phi + (1 - \phi(1))\phi_0. \]

Then \( \alpha \) is a homeomorphic mapping of the compact space \( \mathbb{S}_0 \) onto the compact space \( \mathbb{S} \).

Now let \( \psi \) be a bounded positive linear functional on \( \mathcal{A} \), and we shall extend uniquely \( \psi \) on \( \mathcal{A} \) with the same norm. Then there is a unique positive central Radon measure \( \mu \) on \( \mathbb{S} \) such that
\[ \psi(a) = \int_{\mathbb{S}} \phi(a) \, d\mu(\phi) \quad \text{for } a \in \mathcal{A}. \]

Let \( \pi^*(\mathcal{S}_\phi) \) be the \*-representation of \( \mathcal{A} \) constructed via \( \psi \); then there is a sequence of positive elements \( (h_n) \) in \( \mathcal{A} \) such that \( \| h_n \| \leq 1 \), \( \{ \pi^*_{h_n} \} \) converges strongly to \( 1_{\mathcal{S}_\phi} \), where \( 1_{\mathcal{S}_\phi} \) is the identity operator on \( \mathcal{S}_\phi \). Therefore,
\[ \lim_n \psi(1 - h_n) = \int_{\mathbb{S}} \phi(1 - h_n) \, d\mu(\phi) = 0. \]

Since \( \phi(1 - h_n) \geq 0 \), there is a subsequence of \( (n_j) \) of \( (n) \) such that \( \lim_j \phi(1 - h_{n_j}) = 0 \).
\( \mu \)-a.e.; hence \( \sup_{\|x\| \leq 1; x \in \mathcal{A}} |\phi(x)| = |\phi(1)| = 1 \) for \( \mu \)-a.e., \( \phi \in \mathcal{S} \). Therefore \( \mu \) is concentrated on the Borel set \( \alpha(\mathcal{S}_1) \).

Hence we can easily prove the following theorem.

**Theorem 5.** Let \( \mathcal{A} \) be a \( \mathcal{C}^* \)-algebra, \( \mathcal{S}_0 \) the set of all positive linear functionals \( \phi \) on \( \mathcal{A} \) such that \( \phi \geq 0 \) and \( \|\phi\| \leq 1 \), \( \mathcal{S}_1 \) the set of all positive linear functionals \( \phi \) on \( \mathcal{A} \) such that \( \|\phi\| = 1 \), \( K_0 \) the set of all primary positive linear functionals \( \phi \) on \( \mathcal{A} \) such that \( \|\phi\| \leq 1 \); then \( \mathcal{S}_1 \) is a Borel subset in the \( \sigma(\mathcal{S}_0, \mathcal{A}) \)-compact space \( \mathcal{S}_0 \) and \( K_0 \) is measurable for all Radon measures \( \nu \) on the compact space \( \mathcal{S}_0 \).

Moreover, let \( \psi \) be a bounded positive linear functional on \( \mathcal{A} \); then there is one and only one positive Radon measure \( \mu \) on the compact space \( \mathcal{S}_0 \) satisfying the following conditions:

(i) \( \psi(a) = \int_{\mathcal{S}_0} \phi(a) d\mu(\phi) \) for \( a \in \mathcal{A} \).

(ii) There is a unique \( \sigma \)-continuous homomorphism \( \Phi_n \) of the center \( Z \) of the \( \mathcal{W}^* \)-algebra \( \mathcal{A}^{**} \) onto the \( \mathcal{W}^* \)-algebra \( \mathcal{L}(\mathcal{S}_0, \mu) \) such that

\[
\psi(za) = \int_{\mathcal{S}_0} \Phi_n(z)(\phi) \phi(a) d\mu(\phi) \quad \text{for} \quad z \in Z \quad \text{and} \quad a \in \mathcal{A},
\]

where "the \( \psi \) on \( \mathcal{A}^{**} \)" is the unique canonical extension of "the \( \psi \) on \( \mathcal{A} \)."

(iii) There is a Borel subset \( U_\psi \) in the compact space \( \mathcal{S}_0 \) such that \( U_\psi \subseteq K_0 \cap \mathcal{S}_1 \), \( \mu(U_\psi) = \mu(\mathcal{S}_0) \) and \( s(\pi^{\phi_1}) \cdot s(\pi^{\phi_2}) = 0 \) for two different \( \phi_1, \phi_2 \in U_\psi \).

We shall call this unique Radon measure \( \mu \) the central Radon measure of \( \psi \).

Finally we shall state some problems.

The problem of characterizing intrinsically central Radon measures is very interesting — in particular, it is important in the unitary representation theory of locally compact groups.

Therefore we shall put

**Problem 2.** Is there an intrinsical characterization of central Radon measures?

In particular, can we have a purely measure-theoretic characterization of those measures?

Suppose \( \psi \) is a positive linear functional on the \( \mathcal{C}^* \)-algebra \( \mathcal{A} \), \( \mathcal{M}(\psi) \) the set of all positive Radon measures on the compact space \( \mathcal{S} \) such that \( \psi(a) = \int_{\mathcal{S}} \phi(a) d\mu(\phi) \) for \( a \in \mathcal{A} \).

Then, \( \mathcal{M}(\psi) \) contains one and only one central Radon measure \( \mu_0 \).

**Problem 3.** Is there a geometrical characterization of \( \mu_0 \) in the \( \mathcal{M}(\psi) \)?

**Problem 4.** Without use of the reduction theory of von Neumann, can we show the existence of a central Radon measure in the \( \mathcal{M}(\psi) \)?

*Added in proof.* Problem 1 is solved by J. Feldman and E. Effros.

**References**


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