

ELEMENTARY METHODS IN THE THEORY OF PRIMES

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Introduction. To facilitate the discussion of elementary formulas in the first section, we introduce the ring of formal Dirichlet series on the integral ideals of an algebraic number field. We show how formal Dirichlet series can be employed systematically to give proofs of some well-known results in elementary number theory. Then we give a new proof of Selberg's fundamental formula for algebraic number fields and a very simple proof of a somewhat stronger result first proved by Amitsur using his symbolic calculus. Next we consider formal Dirichlet series involving characters of ideal classes modulo an integral ideal \mathfrak{m} , and obtain several new asymptotic formulas involving these characters. Using these formulas we give a new proof of Selberg's formula for ideals in ideal classes and derive a new result standing in the same relation to Selberg's formula for ideal classes as Amitsur's result is to Selberg's formula for number fields.

In the second section, motivated by some results in the first section, we are led to consider certain Tauberian theorems in order to get a new proof of the equidistribution of prime ideals in ideal classes. Certainly these methods yield neither the most elegant nor the most elementary proof of this result but the methods may be of some interest in themselves. We are led to consider Tauberian theorems involving complex-valued functions where previously only real-valued functions had been considered and we show our results are, in some sense, best possible. We also consider the situation where we have several functions satisfying simultaneous and interdependent Tauberian conditions.

I. Elementary formulas.

DEFINITION 1.1. Let k be an algebraic number field with ring of integers R , then the ring D of formal Dirichlet series on k is the set of complex-valued functions defined on the set I of integral ideals of k with the following operations:

- (a) For $\alpha, \beta \in D$ and $\mathfrak{a} \in I$; $(\alpha + \beta)(\mathfrak{a}) = \alpha(\mathfrak{a}) + \beta(\mathfrak{a})$.
- (b) For $\alpha, \beta \in D$ and $\mathfrak{a} \in I$; $(\alpha\beta)(\mathfrak{a}) = \sum_{\mathfrak{b} \supseteq \mathfrak{a}} \alpha(\mathfrak{a})\beta(\mathfrak{a}\mathfrak{b}^{-1})$.

It is easily seen that with these operations D becomes a commutative ring with identity e , where $e(R) = 1$ and $e(\mathfrak{a}) = 0$ for $\mathfrak{a} \neq R$. Furthermore $\alpha \in D$ has an inverse if and only if $\alpha(R) \neq 0$.

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DEFINITION 1.2.

(a) We say $\alpha \in D$ is strongly multiplicative if $\alpha(R) = 1$ and $\alpha(ab) = \alpha(a)\alpha(b)$ for all $a, b \in I$.

(b) We define $\zeta \in D$ by $\zeta(a) = 1$ for all $a \in I$.

(c) We define $\mu(a) = (-1)^r$ if $a = p_1 \cdots p_r$, where p_1, \dots, p_r are distinct prime ideals, and $\mu(a) = 0$ otherwise.

(d) If $\alpha \in D$ we define $\alpha' \in D$ by $\alpha'(a) = -\alpha(a) \log Na$ for all $a \in I$, where Na denotes the norm of a .

(e) For $\alpha \in D, x \geq 0$, let $C(\alpha, x) = \sum_{Na \leq x} \alpha(a)$.

(f) We define $\Lambda(a) = \log Na$ if $a = p^n$ for some prime ideal p and $\Lambda(a) = 0$ otherwise.

Using the above definitions and unique factorization for ideals in algebraic number fields, it is straightforward to verify the following:

$$(1.1) \quad \log Na = \sum_{b \supseteq a} \Lambda(b).$$

$$(1.2) \quad \text{For } \alpha, \beta \in D, (\alpha + \beta)' = \alpha' + \beta' \text{ and } (\alpha\beta)' = \alpha'\beta + \alpha\beta'.$$

$$(1.3) \quad \text{If } \alpha \in D \text{ and } \alpha(R) \neq 0, \text{ then } (\beta\alpha^{-1})' = (\alpha\beta' - \beta\alpha')(\alpha^{-1})^2.$$

$$(1.4) \quad \text{For all } a \in I, \zeta^{-1}(a) = \mu(a).$$

$$(1.5) \quad \text{If } \alpha \in D \text{ is strongly multiplicative then } \alpha^{-1}(a) = \mu(a)\alpha(a) \text{ for all } a \in I.$$

$$(1.6) \quad \text{If } \alpha \in D \text{ is strongly multiplicative then } -\alpha'\alpha^{-1}(a) = \Lambda(a)\alpha(a) \text{ for all } a \in I.$$

$$(1.7) \quad \text{For } \alpha, \beta \in D \text{ we have}$$

$$C(\alpha\beta, x) = \sum_{Na \leq x} \alpha(a) C\left(\beta, \frac{x}{Na}\right) = \sum_{Na \leq x} \beta(a) C\left(\alpha, \frac{x}{Na}\right).$$

Next we prove an identity from which we will be able to deduce several asymptotic formulas.

THEOREM 1.1. *Let $\alpha, \phi \in D$ with α strongly multiplicative; then*

$$(1.8) \quad \begin{aligned} C(\phi, x) \log x + \sum_{Na \leq x} \alpha(a) \Lambda(a) C\left(\phi, \frac{x}{Na}\right) \\ = \sum_{Na \leq x} \mu(a) \alpha(a) \log \frac{x}{Na} C\left(\phi\alpha, \frac{x}{Na}\right). \end{aligned}$$

Proof. Consider the expression

$$\begin{aligned} \sum_{Na \leq x} \mu(a) \alpha(a) \log \frac{x}{Na} C\left(\phi\alpha, \frac{x}{Na}\right) \\ = \log x \sum_{Na \leq x} \mu(a) \alpha(a) C\left(\phi\alpha, \frac{x}{Na}\right) \\ + \sum_{Na \leq x} \left\{ -\mu(a) \alpha(a) \log Na C\left(\phi\alpha, \frac{x}{Na}\right) \right\}. \end{aligned}$$

By (1.5) and (1.7) the first term is just

$$\begin{aligned} \log x \sum_{N\alpha \leq x} \alpha^{-1}(\alpha) C\left(\phi\alpha, \frac{x}{N\alpha}\right) \\ = \log x C(\alpha^{-1}\phi\alpha, x) = \log x C(\phi, x). \end{aligned}$$

Next we note that $-\mu(\alpha)\alpha(\alpha)\log N\alpha = (\alpha^{-1})'(\alpha)$ and hence by (1.7), the second term is just

$$\begin{aligned} \sum_{N\alpha \leq x} (\alpha^{-1})'(\alpha) C\left(\phi\alpha, \frac{x}{N\alpha}\right) \\ = C((-\alpha^{-1})'\phi\alpha, x) = C\left(-\left|\frac{\alpha'}{\alpha^2}\right|\phi\alpha, x\right) = C\left(-\left|\frac{\alpha'}{\alpha}\right|\phi, x\right) \\ = \sum_{N\alpha \leq x} \alpha(\alpha)\Lambda(\alpha) C\left(\phi, \frac{x}{N\alpha}\right) \end{aligned}$$

by (1.3), (1.6), and (1.7). This proves the theorem.

Before we proceed we need a few preliminary estimates. First of all:

(1.9) There exists $\lambda > 0$ such that $C(\zeta, x) = \sum_{N\alpha \leq x} 1 = \lambda x + O(x^{1-1/n})$ where n is the degree of the field k over the rationals, this result can be found in [2].

Suppose $\alpha \in D$, $\alpha(\alpha) \geq 0$ for all $\alpha \in I$ and there exists $\xi > 0$ such that $C(\alpha, x) = \xi x + O(x^{1-1/n})$; then:

$$(1.10) \quad \sum_{y < N\alpha \leq x} \frac{1}{N\alpha^{1-\delta}} = O(x^\delta - y^\delta), \quad 0 < \delta < 1, \quad x \rightarrow \infty.$$

(1.11) There exists β such that

$$\sum_{N\alpha \leq x} \frac{\alpha(\alpha)}{N\alpha} = \xi\{\log x + \beta\} + O(x^{-1/n}), \quad x \rightarrow \infty.$$

$$(1.12) \quad \sum_{N\alpha \leq x} \alpha(\alpha)\log N\alpha = \xi\{x \log x - x\} + O(x^{1-1/n}\log x), \quad x \rightarrow \infty.$$

$$(1.13) \quad \sum_{N\alpha \leq x} \alpha^2(\alpha) = C(\alpha^2, x) = \xi^2[x \log x + (2\beta - 1)x] + O(x^{1-1/2n}), \quad x \rightarrow \infty.$$

$$(1.14) \quad \sum_{N\alpha \leq x} \alpha(\alpha) \frac{\log \frac{x}{N\alpha}}{N\alpha^{1-\delta}} = O(x^\delta), \quad 0 < \delta < 1, \quad x \rightarrow \infty.$$

$$(1.15) \quad \sum_{N\alpha \leq x} \alpha(\alpha) \frac{\log N\alpha}{N\alpha} = \xi \frac{\log^2 x}{2} + O(1), \quad x \rightarrow \infty.$$

The proofs of (1.10)–(1.15) are based on summation by parts and use of $C(\alpha, x) = \xi x + O(x^{1-1/n})$.

Four well-known results of elementary number theory will now be derived by use of the estimates and identities listed above.

THEOREM 1.2. $\sum_{N\alpha \leq x} (\mu(\alpha)/N\alpha) = O(1)$.

Proof.

$$\begin{aligned} x \sum_{N\mathfrak{a} \leq x} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} &= \sum_{N\mathfrak{a} \leq x} \mu(\mathfrak{a}) \frac{x}{N\mathfrak{a}} \\ &= \sum_{N\mathfrak{a} \leq x} \mu(\mathfrak{a}) \frac{1}{\lambda} \left\{ C\left(\zeta, \frac{x}{N\mathfrak{a}}\right) + O\left(\left(\frac{x}{N\mathfrak{a}}\right)^{1-1/n}\right) \right\} \\ &= \frac{1}{\lambda} + O(x^{1-1/n}) \sum_{N\mathfrak{a} \leq x} \frac{1}{N\mathfrak{a}^{1-1/n}} = O(x). \end{aligned}$$

THEOREM 1.3. $\sum_{N\mathfrak{a} \leq x} (\mu(\mathfrak{a})/N\mathfrak{a}) \log(x/N\mathfrak{a}) = O(1)$.

Proof.

$$\begin{aligned} \sum_{N\mathfrak{a} \leq x} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log \frac{x}{N\mathfrak{a}} &= \sum_{N\mathfrak{a} \leq x} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \left\{ \frac{1}{\lambda} \sum_{N\mathfrak{b} \leq x/N\mathfrak{a}} \frac{1}{N\mathfrak{b}} - \beta + O\left(\left(\frac{x}{N\mathfrak{a}}\right)^{-1/n}\right) \right\} \\ &= \frac{1}{\lambda} C(\alpha^{-1}, x) - \beta \sum_{N\mathfrak{a} \leq x} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \\ &\quad + O(x^{1-n}) \sum_{N\mathfrak{a} \leq x} \frac{1}{N\mathfrak{a}^{1-1/n}} \quad \text{where } \alpha(\mathfrak{a}) = \frac{1}{N\mathfrak{a}}. \end{aligned}$$

The result then follows from Theorem 1.2 and (1.10).

THEOREM 1.4. $\psi(x) = \sum_{N\mathfrak{a} \leq x} \Lambda(\mathfrak{a}) = O(x)$.

Proof.

$$\begin{aligned} \sum_{N\mathfrak{a} \leq x} \Lambda(\mathfrak{a}) &= C(-\zeta'/\zeta, x) \\ &= \sum_{N\mathfrak{a} \leq x} \mu(\mathfrak{a}) \sum_{N\mathfrak{b} \leq x/N\mathfrak{a}} \log N\mathfrak{b} \\ &= \sum_{N\mathfrak{a} \leq x} \mu(\mathfrak{a}) \left[\lambda \frac{x}{N\mathfrak{a}} \log \frac{x}{N\mathfrak{a}} - \frac{\lambda x}{N\mathfrak{a}} + O\left(\left(\frac{x}{N\mathfrak{a}}\right)^{1-1/n} \log \frac{x}{N\mathfrak{a}}\right) \right] \\ &= \lambda x \left[\sum_{N\mathfrak{a} \leq x} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log \frac{x}{N\mathfrak{a}} - \sum_{N\mathfrak{a} \leq x} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \right] \\ &\quad + O(x^{1-1/n}) \sum_{N\mathfrak{a} \leq x} \frac{\log \frac{1}{N\mathfrak{a}}}{N\mathfrak{a}^{1-1/n}} = O(x) \end{aligned}$$

by Theorems 1.2 and 1.3 and by (1.14).

THEOREM 1.5. $\sum_{N\mathfrak{a} \leq x} (\Lambda(\mathfrak{a})/N\mathfrak{a}) = \log x + O(1)$.

Proof.

$$\begin{aligned} x \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} &= \sum_{N\mathfrak{a} \leq x} \Lambda(\mathfrak{a}) \left\{ \frac{1}{\lambda} C\left(\zeta, \frac{x}{N\mathfrak{a}}\right) + O\left(\left(\frac{x}{N\mathfrak{a}}\right)^{1-1/n}\right) \right\} \\ &= \frac{1}{\lambda} C\left(-\frac{\zeta'}{\zeta}, \zeta, x\right) + O(x^{1-1/n}) \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}^{1-1/n}}. \end{aligned}$$

Summing by parts and using the fact that $\sum_{N\mathfrak{a} \leq x} \Lambda(\mathfrak{a}) = O(x)$ we see that the second term is $O(x)$. The first term is just

$$\frac{1}{\lambda} C(-\zeta', x) = \frac{1}{\lambda} \sum_{N\mathfrak{a} \leq x} \log N\mathfrak{a} = x \log x + O(x)$$

by (1.12). This proves the result.

We return to (1.8). This identity says that if we want information about $C(\phi, x)$ for some $\phi \in D$, an estimate for $C(\phi\alpha, x)$ for some strongly multiplicative $\alpha \in D$ might help. If we let $\phi = -\zeta'/\zeta - (1/\lambda)\zeta + 2\beta e$ then $C(\phi, x) = \sum_{N\mathfrak{a} \leq x} \Lambda(\mathfrak{a}) - x + o(x)$ and hence the proposition $C(\phi, x) = o(x)$ is the prime ideal theorem. If we let $\alpha = \zeta$, then

$$C(\phi\alpha, x) = C\left(-\zeta' - \frac{1}{\lambda}\zeta^2 + 2\beta\zeta, x\right) = O(x^{1-1/2n})$$

by (1.12), (1.13), and (1.9) and hence

$$\sum_{N\mathfrak{a} \leq x} \mu(\mathfrak{a})\alpha(\mathfrak{a}) \log \frac{x}{N\mathfrak{a}} C\left(\phi\alpha, \frac{x}{N\mathfrak{a}}\right) = \sum_{N\mathfrak{a} \leq x} O\left(\left(\frac{x}{N\mathfrak{a}}\right)^{1-\varepsilon}\right)$$

for sufficiently small $\varepsilon > 0$ and this is just $O(x)$ by (1.10). Therefore by Theorem 1.1

$$C(\phi, x) \log x + \sum_{N\mathfrak{a} \leq x} \Lambda(\mathfrak{a}) C\left(\phi, \frac{x}{N\mathfrak{a}}\right) = O(x)$$

or,

$$\begin{aligned} \psi(x) \log x + \sum_{N\mathfrak{a} \leq x} \Lambda(\mathfrak{a}) \psi\left(\frac{x}{N\mathfrak{a}}\right) &= \log x C\left(\frac{1}{\lambda}\zeta, x\right) + \sum_{N\mathfrak{a} \leq x} \Lambda(\mathfrak{a}) C\left(\frac{1}{\lambda}\zeta, \frac{x}{N\mathfrak{a}}\right) + O(x) \\ &= x \log x + \frac{1}{\lambda} C(-\zeta', x) + O(x) = 2x \log x + O(x). \end{aligned}$$

This is just Selberg's fundamental formula for algebraic number fields.

However this technique will yield a stronger result. If we let $\alpha \in D$ be defined by $\alpha(\mathfrak{a}) = 1/N\mathfrak{a}$ and $\phi = -\alpha'/\alpha - (1/\lambda)\alpha + 2\beta e$, then

$$C(\phi, x) = \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} - \log x + \beta + o(1).$$

Since it is known that $\sum_{N\mathfrak{a} \leq x} (\Lambda(\mathfrak{a})/N\mathfrak{a}) = \log x - \beta + o(1)$ is an elementary equivalent of the prime ideal theorem, so is the condition $C(\phi, x) = o(1)$. Using the estimates already derived it is not difficult to show that

$$C(\phi\alpha, x) = C\left(-\alpha' - \frac{1}{\lambda}\alpha^2 + 2\beta\alpha, x\right) = A + O(x^{-1/2n})$$

for some constant A . Therefore

$$\begin{aligned} & \sum_{N\mathfrak{a} \leq x} \mu(\mathfrak{a}) \alpha(\mathfrak{a}) \log \frac{x}{N\mathfrak{a}} C\left(\phi\alpha, \frac{x}{N\mathfrak{a}}\right) \\ &= A \sum_{N\mathfrak{a} \leq x} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log \frac{x}{N\mathfrak{a}} + O(x^{-1/2n}) \sum_{N\mathfrak{a} \leq x} \frac{\log(x/N\mathfrak{a})}{N\mathfrak{a}^{1-1/2n}} = O(1) \end{aligned}$$

by Theorem 1.3 and (1.14).

Hence we have

$$\log x C(\phi, x) + \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} C\left(\phi, \frac{x}{N\mathfrak{a}}\right) = O(1).$$

Using the definition of ϕ we get

$$\begin{aligned} & \log x \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} + \sum_{N\mathfrak{a}\mathfrak{b} \leq x} \frac{\Lambda(\mathfrak{a})\Lambda(\mathfrak{b})}{N\mathfrak{a}\mathfrak{b}} \\ &= \frac{1}{\lambda} \log x C(\alpha, x) - 2\beta \log x + \frac{1}{\lambda} \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} C\left(\alpha, \frac{x}{N\mathfrak{a}}\right) \\ &\quad - 2\beta \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} + O(1). \end{aligned}$$

We note that the third term is just $\lambda^{-1} C(-(\alpha'/\alpha) \cdot \alpha, x) = \lambda^{-1} \sum_{N\mathfrak{a} \leq x} (\log N\mathfrak{a})/N\mathfrak{a}$. Hence every term can be estimated up to a constant and the end result is:

$$(1.16) \quad \log x \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} + \sum_{N\mathfrak{a}\mathfrak{b} \leq x} \frac{\Lambda(\mathfrak{a})\Lambda(\mathfrak{b})}{N\mathfrak{a}\mathfrak{b}} = \frac{3}{2} \log^2 x - 3\beta \log x + O(1).$$

A simple elementary argument shows that

$$\begin{aligned} \log x \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a})}{N(\mathfrak{a})} - \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a}) \log N\mathfrak{a}}{N\mathfrak{a}} &= \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} \log \frac{x}{N\mathfrak{a}} \\ &= \frac{\log^2 x}{2} + \beta \log x + O(1) \end{aligned}$$

and hence it follows from (1.16) that

$$(1.17) \quad \sum_{N\mathfrak{a} \leq x} \frac{\Lambda(\mathfrak{a}) \log N\mathfrak{a}}{N\mathfrak{a}} + \sum_{N\mathfrak{a}\mathfrak{b} \leq x} \frac{\Lambda(\mathfrak{a})\Lambda(\mathfrak{b})}{N\mathfrak{a}\mathfrak{b}} = \log^2 x - 2\beta \log x + O(1).$$

This result was first proved (by ‘‘elementary’’ methods) by Amitsur [1]. His proof is somewhat more complicated as it involves the use of his ‘‘symbolic calculus.’’ It is easy to show that (1.17) implies Selberg’s fundamental formula (Amitsur [1]).

Before we proceed we need a few more definitions.

DEFINITION 1.3. Let \mathfrak{m} be an integral ideal of k , then $A^{(\mathfrak{m})}$ is the group of fractional ideals of k prime to \mathfrak{m} and $H^{(\mathfrak{m})}$ is the group of principal ideals (α) where

(a) α is totally positive,

(b) $v_p(\alpha - 1) \geq v_p(\mathfrak{m})$ for all $p \geq \mathfrak{m}$, i.e., for all prime ideals p dividing \mathfrak{m} the exponent of p in the unique factorization of $(\alpha - 1)$ is greater than the exponent of p in the unique factorization of \mathfrak{m} .

We assume the following facts found in Landau [1].

(1.18) $A^{(\mathfrak{m})}/H^{(\mathfrak{m})}$ is a finite group of order h .

(1.19) If $g \in G_{\mathfrak{m}} = A^{(\mathfrak{m})}/H^{(\mathfrak{m})}$ then $\sum_{N\mathfrak{a} \leq x, \mathfrak{a} \in g} 1 = \tau x + O(x^{1-1/n})$ where $\tau > 0$ is independent of g and n is the degree of k over the rationals. If χ is a character of $G_{\mathfrak{m}}$ then for any $\mathfrak{a} \in I$ we define $\chi(\mathfrak{a}) = \chi(\mathfrak{a}H^{(\mathfrak{m})})$ if $\mathfrak{a} \in A^{(\mathfrak{m})}$, and $\chi(\mathfrak{a}) = 0$ otherwise. Then χ defines a strongly multiplicative formal Dirichlet series. From (1.19) and elementary properties of characters of finite abelian groups one obtains

(1.20) For any nonprincipal character χ of $G_{\mathfrak{m}}$ (i.e., $\chi \neq 1$) we have

$$\sum_{N\mathfrak{a} \leq x} \chi(\mathfrak{a}) = O(x^{1-1/n}).$$

(1.21) If ε is the principal character of $G_{\mathfrak{m}}$ (i.e., $\varepsilon \equiv 1$) then

$$\sum_{N\mathfrak{a} \leq x} \varepsilon(\mathfrak{a}) = \sum_{N\mathfrak{a} \leq x, \mathfrak{a} \in A^{(\mathfrak{m})}} 1 = \omega x + O(x^{1-1/n}),$$

where $\omega = h\tau > 0$.

If ε is the principal character of $G_{\mathfrak{m}}$ then

$$(1.22) \quad \log x \sum_{N\mathfrak{a} \leq x} \varepsilon(\mathfrak{a})\Lambda(\mathfrak{a}) + \sum_{N\mathfrak{a}\mathfrak{b} \leq x} \varepsilon(\mathfrak{a}\mathfrak{b})\Lambda(\mathfrak{a})\Lambda(\mathfrak{b}) = 2x \log x + O(x).$$

This follows when one notices that the difference between the left-hand side of (1.22) and the left-hand side of Selberg's fundamental formula is $O(x)$.

If χ is a nonprincipal character of $G_{\mathfrak{m}}$ then we apply Theorem 1.1 with $\phi = -\chi'/\chi$ and $\alpha = \chi$. Now

$$C(\phi\alpha, x) = C(-\chi', x) = \sum_{N\mathfrak{a} \leq x} \chi(\mathfrak{a}) \log N\mathfrak{a} = O(x^{1-\varepsilon})$$

for $0 < \varepsilon < 1/n$, as can be seen by summation by parts and application of (1.20). Therefore,

$$\sum_{N\mathfrak{a} \leq x} \mu(\mathfrak{a})\alpha(\mathfrak{a}) \log \frac{x}{N\mathfrak{a}} C\left(\phi\alpha, \frac{x}{N\mathfrak{a}}\right) = \sum_{N\mathfrak{a} \leq x} O\left(\left(\frac{x}{N\mathfrak{a}}\right)^{1-\varepsilon/2}\right) = O(x)$$

as we have seen. It follows that

$$(1.23) \quad \log x \sum_{N\mathfrak{a} \leq x} \chi(\mathfrak{a})\Lambda(\mathfrak{a}) + \sum_{N\mathfrak{a}\mathfrak{b} \leq x} \chi(\mathfrak{a}\mathfrak{b})\Lambda(\mathfrak{a})\Lambda(\mathfrak{b}) = \eta_{\chi}(x),$$

where $\eta_\chi(x) = O(x)$ for $\chi \neq \varepsilon$ and $\eta_\chi(x) = 2x \log x + O(x)$ for $\chi = \varepsilon$. Rearranging, we have

$$\sum_{g \in G_m} \chi(g) \left\{ \log x \sum_{N\mathfrak{a} \leq x, \mathfrak{a} \in g} \Lambda(\mathfrak{a}) + \sum_{N\mathfrak{ab} \leq x, \mathfrak{ab} \in g} \Lambda(\mathfrak{a})\Lambda(\mathfrak{b}) \right\} = \eta_\chi(x).$$

Solving these equations we get, for each $g \in G$

$$(1.23a) \quad \log x \sum_{N\mathfrak{a} \leq x, \mathfrak{a} \in g} \Lambda(\mathfrak{a}) + \sum_{N\mathfrak{ab} \leq x, \mathfrak{ab} \in g} \Lambda(\mathfrak{a})\Lambda(\mathfrak{b}) = \frac{1}{h} 2x \log x + O(x).$$

This is Selberg's fundamental formula for ideals in ideal classes modulo m .

Now we show how (1.17) can be generalized to ideal classes. From (1.11) there exists c_0 such that

$$\sum_{N\mathfrak{a} \leq x} \frac{\varepsilon(\mathfrak{a})}{N\mathfrak{a}} = \omega[\log x + c_0] + O(x^{-1/n}).$$

Furthermore we can show that

$$(1.24) \quad \sum_{N\mathfrak{a} \leq x} \frac{\varepsilon(\mathfrak{a})\mu(\mathfrak{a})}{N\mathfrak{a}} = O(1),$$

$$(1.25) \quad \sum_{N\mathfrak{a} \leq x} \frac{\varepsilon(\mathfrak{a})\mu(\mathfrak{a})}{N\mathfrak{a}} \log \frac{x}{N\mathfrak{a}} = O(1),$$

$$(1.26) \quad \sum_{N\mathfrak{a} \leq x} \frac{\varepsilon(\mathfrak{a})\Lambda(\mathfrak{a})}{N\mathfrak{a}} = \log x + O(1).$$

The proofs of (1.24)–(1.26) are almost identical with the proofs of (1.2), (1.3), and (1.5). The important thing is that ε defines a strongly multiplicative formal Dirichlet series such that $C(\varepsilon, x) = \omega x + O(x^{1-1/n})$ for some $\omega > 0$. With these results we can prove

$$(1.27) \quad \begin{aligned} \log x \sum_{N\mathfrak{a} \leq x} \frac{\varepsilon(\mathfrak{a})\Lambda(\mathfrak{a})}{N\mathfrak{a}} + \sum_{N\mathfrak{ab} \leq x} \frac{\varepsilon(\mathfrak{ab})\Lambda(\mathfrak{a})\Lambda(\mathfrak{b})}{N\mathfrak{ab}} \\ = \frac{3}{2} \log^2 x - 3c_0 \log x + O(1) \end{aligned}$$

in exactly the same way that (1.16) was proved. Furthermore from (1.27) we can deduce

$$(1.28) \quad \begin{aligned} \sum_{N\mathfrak{a} \leq x} \frac{\varepsilon(\mathfrak{a})\Lambda(\mathfrak{a})\log N\mathfrak{a}}{N\mathfrak{a}} + \sum_{N\mathfrak{ab} \leq x} \frac{\varepsilon(\mathfrak{ab})\Lambda(\mathfrak{a})\Lambda(\mathfrak{b})}{N\mathfrak{ab}} \\ = \log^2 x - 2c_0 \log x + O(1) \end{aligned}$$

in the same way that (1.17) was deduced from (1.16).

In order to get the corresponding results for the nonprincipal characters we proceed along different lines. Moreover we need a result that can be proved by elementary methods (Shapiro and Forman [1]).

(1.29) If χ is a nonprincipal character of G_m then

$$\sum_{N\mathfrak{a} \leq x} \frac{\chi(\mathfrak{a})\mu(\mathfrak{a})}{N\mathfrak{a}} = O(1).$$

We consider the identity $(\alpha'/\alpha)' + (\alpha'/\alpha) = \alpha''/\alpha$ and let α be defined by $\alpha(\mathfrak{a}) = \chi(\mathfrak{a})/N\mathfrak{a}$. Note that

$$C\left(\left(\frac{\alpha'}{\alpha}\right)', x\right) = \sum_{N\mathfrak{a} \leq x} \frac{\chi(\mathfrak{a})\Lambda(\mathfrak{a}) \log N\mathfrak{a}}{N\mathfrak{a}}$$

and

$$C\left(\left(\frac{\alpha'}{\alpha}\right)^2, x\right) = \sum_{N\mathfrak{a}\mathfrak{b} \leq x} \frac{\chi(\mathfrak{a}\mathfrak{b})\Lambda(\mathfrak{a})\Lambda(\mathfrak{b})}{N\mathfrak{a}\mathfrak{b}}.$$

By summation by parts we see that

$$C(\alpha'', x) = \sum_{N\mathfrak{a} \leq x} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}} \log^2 N\mathfrak{a} = b_0 + O(x^{-\epsilon})$$

for some constant b_0 and sufficiently small $\epsilon > 0$. Hence

$$\begin{aligned} C\left(\frac{\alpha''}{\alpha}, x\right) &= \sum_{N\mathfrak{a} \leq x} \frac{\chi(\mathfrak{a})\mu(\mathfrak{a})}{N\mathfrak{a}} \left\{ b_0 + O\left(\left(\frac{x}{N\mathfrak{a}}\right)^{-\epsilon}\right) \right\} \\ &= b_0 \sum_{N\mathfrak{a} \leq x} \frac{\chi(\mathfrak{a})\mu(\mathfrak{a})}{N\mathfrak{a}} + O(x^{-\epsilon}) \sum_{N\mathfrak{a} \leq x} N\mathfrak{a}^{\epsilon-1} = O(1). \end{aligned}$$

It follows that

$$(1.30) \quad \sum_{N\mathfrak{a} \leq x} \frac{\chi(\mathfrak{a})\Lambda(\mathfrak{a}) \log N\mathfrak{a}}{N\mathfrak{a}} + \sum_{N\mathfrak{a}\mathfrak{b} \leq x} \frac{\chi(\mathfrak{a}\mathfrak{b})\Lambda(\mathfrak{a})\Lambda(\mathfrak{b})}{N\mathfrak{a}\mathfrak{b}} = \eta_\chi(x)$$

where $\eta_\chi(x) = \log^2 x - 2c_0 \log x + O(1)$ if $\chi = \epsilon$ and $\eta_\chi(x) = O(1)$ if $\chi \neq \epsilon$. We can rewrite this as

$$\sum_{\mathfrak{g} \in G_m} \chi(\mathfrak{g}) \left\{ \sum_{N\mathfrak{a} \leq x, \mathfrak{a} \in \mathfrak{g}} \frac{\Lambda(\mathfrak{a}) \log N\mathfrak{a}}{N\mathfrak{a}} + \sum_{N\mathfrak{a}\mathfrak{b} \leq x, \mathfrak{a}\mathfrak{b} \in \mathfrak{g}} \frac{\Lambda(\mathfrak{a})\Lambda(\mathfrak{b})}{N\mathfrak{a}\mathfrak{b}} \right\} = \eta_\chi(x)$$

and solving we get, for each $\mathfrak{g} \in G_m$

$$\sum_{N\mathfrak{a} \leq x, \mathfrak{a} \in \mathfrak{g}} \frac{\Lambda(\mathfrak{a}) \log N\mathfrak{a}}{N\mathfrak{a}} + \sum_{N\mathfrak{a}\mathfrak{b} \leq x, \mathfrak{a}\mathfrak{b} \in \mathfrak{g}} \frac{\Lambda(\mathfrak{a})\Lambda(\mathfrak{b})}{N\mathfrak{a}\mathfrak{b}} = \frac{1}{h} \{ \log^2 x - 2c_0 \log x \} + O(1).$$

II. Tauberian methods. Now let m be an integral ideal of k and let χ be a nonprincipal character of G_m . Then by (1.23) we have:

$$(2.1) \quad f(x) + \frac{1}{\log x} \sum_{n \leq x} \alpha(n) f\left(\frac{x}{n}\right) \rightarrow 0 \text{ as } x \rightarrow \infty,$$

where

$$(2.2) \quad f(x) \frac{1}{x} = \sum_{N\mathfrak{a} \leq x} \chi(\mathfrak{a}) \Lambda(\mathfrak{a}),$$

and

$$(2.3) \quad \alpha(n) = \sum_{N\mathfrak{a}=n} \frac{\chi(\mathfrak{a}) \Lambda(\mathfrak{a})}{N\mathfrak{a}}.$$

Furthermore if ε is the principal character of $G_{\mathfrak{m}}$ then by (1.23), formula (2.2) also holds for

$$(2.4) \quad f(x) = \frac{1}{x} \left[\sum_{N\mathfrak{a} \leq x} \varepsilon(\mathfrak{a}) \Lambda(\mathfrak{a}) - x \right],$$

and

$$(2.5) \quad \alpha(n) = \sum_{N\mathfrak{a} \leq x} \frac{\varepsilon(\mathfrak{a}) \Lambda(\mathfrak{a})}{N\mathfrak{a}}.$$

We would like a result stating that with appropriate hypotheses on f and α , the condition (2.1) implies $f(x) = o(1)$ as $x \rightarrow \infty$. The condition

$$(2.6) \quad \sum_{N\mathfrak{a} \leq x} \chi(\mathfrak{a}) \Lambda(\mathfrak{a}) = o(x) \text{ for } \chi \neq \varepsilon \text{ and } x + o(x) \text{ for } \chi = \varepsilon,$$

is a simple equivalent of the theorem on equidistribution of prime ideals in ideal classes.

In his paper, *Some results on arithmetic functions*, Amitsur [2] states the following:

Let $g(n)$ be a non-negative function such that

$$(A.1) \quad \sum_{n \leq x} g(n) = a \log^m x + b \log^{m-1} x + o(\log^{m-1} x), \quad m \geq 1, a > 0.$$

“Let $f(x)$ be a complex-valued function such that

$$(A.2) \quad f(x) = O(1),$$

$$(A.3) \quad \sum_{n \leq x} \frac{f(n)}{n} = O(1),$$

$$(A.4) \quad f(tx) - f(x) = o(1) \text{ as } (t, x) \rightarrow (1, \infty).$$

“Then the condition

$$(A.5) \quad |f(x)| \log^m x \leq \frac{1}{a} \sum_{n \leq x} g(n) \left| f\left(\frac{x}{n}\right) \right| + o(\log^m x)$$

implies that $f(x) = o(1)$.”

By (2.1) we have

$$(2.7) \quad |f(x)| \log x \leq \sum_{n \leq x} g(n) \left| f\left(\frac{x}{n}\right) \right| + o(\log x),$$

where f is given either by (2.2) or (2.4) and $g(n) = \sum_{N\mathfrak{a}=n} (\Lambda(\mathfrak{a})/N\mathfrak{a})$.

(A. 1) follows from the prime ideal theorem, (A. 5) is just (2.7) and (A. 2)–(A. 4) can be verified by elementary methods. However, the proposition is not true as it stands. In fact, one easily checks that the functions $f(x) = e^{i \log x}$ and $g(n) = 1/n$, provide a counterexample.

It should be pointed out that if we have (1.23) for some nonprincipal character χ of G_m , and we know (1.23a) for all $h \in G_m$ then it is possible to give a proof very similar to the one in Selberg [2]. However we will take a slightly more general approach.

Suppose f is a complex-valued function defined on $(0, \infty)$ and α is a non-negative function defined on integral ideals; suppose we have:

$$(2.8) \quad f(x) + \frac{1}{\log x} \sum_{N\mathfrak{a} \leq x} \chi(\mathfrak{a})\alpha(\mathfrak{a})f\left(\frac{x}{N\mathfrak{a}}\right) \rightarrow 0.$$

For $h \in G_m$ we multiply by $\chi(h)$ and rewrite as follows:

$$\chi(h)f(x) + \sum_{g \in G_m} \frac{1}{\log x} \sum_{N\mathfrak{a} \leq x, \mathfrak{a} \in g} \alpha(\mathfrak{a})\chi(hg)f\left(\frac{x}{N\mathfrak{a}}\right) \rightarrow 0.$$

Or, with $f_h(x) = \chi(h)f(x)$, we have:

$$f_h(x) + \sum_{g \in G_m} \frac{1}{\log x} \sum_{N\mathfrak{a} \leq x, \mathfrak{a} \in gh^{-1}} \alpha(\mathfrak{a})f_g\left(\frac{x}{N\mathfrak{a}}\right) \rightarrow 0$$

for all $h \in G_m$. If the f_g are replaced by either their real or imaginary parts the relation still holds. Finally, for each $h \in G_m$ we have:

$$(2.9) \quad |f_h(x)| \leq \sum_{g \in G_m} \frac{1}{\log x} \sum_{n \leq x} \beta_{h,g}(n) \left| f_g\left(\frac{x}{n}\right) \right| + o(1),$$

where we can assume the f_g are real-valued, and $\beta_{h,g}(n) = \sum_{N\mathfrak{a} = n, \mathfrak{a} \in gh^{-1}} \alpha(\mathfrak{a})/N\mathfrak{a}$. Starting with the single relation, (2.8), involving complex-valued functions we arrive at the system, (2.9), of relations involving real-valued functions and where the coefficients $\beta_{h,g}$ are even non-negative. It is from this point of view that we approach the problem.

It is very convenient, although not necessary, to discuss the Tauberian theorem in terms of Lebesgue-Stieltjes integrals rather than finite sums.

Let $k(x)$ be monotone increasing, defined on $(0, \infty)$, we assume that k is normalized to be continuous from the left at each point. For any interval $[a, b)$ we define its measure to be $k(b) - k(a)$, this measure has a unique extension to the Borel sets. If $f(x)$ is integrable for this measure we denote its integral by $\int_0^\infty f(x) dk(x)$. We will also need the following results which can be found in Pitt [1], [2].

(2.10) Suppose h, k are monotone increasing on $(0, \infty)$ then

$$\int_0^x h(x-y) dk(y) = \int_0^x k(x-y) dh(y).$$

(2.11) If $k(x)$ is monotone increasing and absolutely continuous and f is integrable with respect to k , then $\int_0^\infty f(x) dk(x) = \int_0^\infty f(x) k'(x) dx$.

(2.12) Suppose h, k are monotone increasing and f is Borel measurable and bounded in any finite interval then

$$\int_0^x \left\{ \int_0^{x-y} f(x-y-t) dh(t) \right\} dk(y) = \int_0^x f(x-y) dl(y),$$

where

$$l(y) = \int_0^y h(y-t) dk(t).$$

From now on we assume that all functions mentioned are Borel measurable and defined on $(0, \infty)$, and all measures $k(x)$ are continuous from the left.

LEMMA 2.1. *Suppose $f(x)$ is real-valued and bounded. Suppose $k(x)$ is monotone, $k(0) = 0$, and k is non-negative and $k(x) \sim aF(x)$ for some function $F(x)$, and $k(x) - k(x - x_0) = o(F(x))$ for each fixed x_0 . Let $S = \limsup_{x \rightarrow \infty} f(x)$, $s = \liminf_{x \rightarrow \infty} f(x)$ then:*

$$\begin{aligned} as &\leq \liminf_{x \rightarrow \infty} \frac{1}{F(x)} \int_0^x f(x-y) dk(y) \\ &\leq \limsup_{x \rightarrow \infty} \frac{1}{F(x)} \int_0^x f(x-y) dk(y) \leq aS. \end{aligned}$$

Proof. The middle inequality is clear. We prove only that

$$\limsup_{x \rightarrow \infty} \frac{1}{F(x)} \int_0^x f(x-y) dk(y) \leq aS$$

since the other inequality is similar. Let $\varepsilon > 0$ be given, then choose x_0 such that $f(t) < S + \varepsilon$ for all $t \geq x_0$. We split the integral as follows:

$$\frac{1}{F(x)} \int_0^x f(x-y) dk(y) = \frac{1}{F(x)} \int_0^{x-x_0} f(x-y) dk(y) + \frac{1}{F(x)} \int_{x-x_0}^x f(x-y) dk(y).$$

The second term goes to zero, since it is bounded by $(k(x) - k(x - x_0))M/F(x)$ where M is a bound for f . Noting that $x - y \geq x_0$ when $y \leq x - x_0$ we see that the first term is bounded by

$$(S + \varepsilon) \frac{k(x - x_0)}{F(x)} = (S + \varepsilon) \left\{ \frac{k(x)}{F(x)} - \frac{k(x) - k(x - x_0)}{F(x)} \right\}$$

which has the limit $a(S + \varepsilon)$ as $x \rightarrow \infty$. Since ε was arbitrary the result follows.

The next lemma is essentially the same as a result of Selberg [1]. We include a proof since we use integrals instead of finite sums.

LEMMA 2.2. *Suppose $f(x)$ is real-valued, that $f(x) - f(x - d) \rightarrow 0$ as $(x, d) \rightarrow (\infty, 0)$, and that $\int_0^x f(t) dt = O(1)$, then given $\varepsilon > 0$ there exist x_0, T , and d all greater than zero such that for all $x \geq x_0$ the interval $(x, x + T)$ contains a subinterval of length d , throughout which $|f(t)| < \varepsilon$.*

Proof. By hypothesis, there exists a constant M such that $|\int_x^{x'} f(t) dt| \leq M$ for any x, x' . We first choose T so large that $M/T < \varepsilon/2$. Next choose x_0 and d so that $|f(x) - f(x - \delta)| < \varepsilon/2$ for all $x \geq x_0$ and $0 < \delta < d$. We may also assume that $d < T/2$. Consider the interval $(x, x + T)$ for some $x \geq x_0$. If $f(t)$ does not change sign in this interval then $\inf_{t \in (x, x+T)} |f(t)| T \leq |\int_x^{x+T} f(t) dt| \leq M$ and hence $\inf_{t \in (x, x+T)} |f(t)| \leq M/T < \varepsilon/2$ and it follows that $|f(\tau)| < \varepsilon/2$, for some $\tau \in (x, x + T)$. Now one of the intervals $(\tau, \tau + d)$ or $(\tau - d, \tau)$ is contained in $(x, x + T)$ since $d < T/2$. If t is in $(\tau, \tau + d)$ say, then $|f(t)| = |f(\tau + \delta)| = |f(\tau) - (f(\tau) - f(\tau + \delta))| \leq |f(\tau)| + |f(\tau) - f(\tau + \delta)|$ for some $0 < \delta < d$. Hence $|f(t)| < \varepsilon$ in this interval. On the other hand if f does change sign in this interval then there exists $\tau \in (x, x + T)$ such that $f(\tau)$ and $f(\tau + d)$ have opposite sign. In this case $|f(\tau)| \leq |f(\tau)| + |f(\tau + d)| = |f(\tau) - f(\tau + d)| < \varepsilon/2$. As before we can find an interval contained in $(x, x + T)$ with one end point at τ throughout which $|f(t)| < \varepsilon$.

From now on we assume that we have N real-valued functions f_1, \dots, f_N defined on $(0, \infty)$ and N^2 monotone increasing non-negative functions k_{ij} , subject to the following conditions:

$$(2.13) \quad f_i(x) = O(1); \quad i = 1, \dots, N.$$

$$(2.14) \quad \int_0^x f_i(t) dt = O(1); \quad i = 1, \dots, N.$$

$$(2.15) \quad |f_i(x)| \leq \sum_{j=1}^N \frac{1}{x} \int_0^x |f_j(x - y)| dk_{ij}(y) + o(1); \quad i = 1, \dots, N.$$

$$(2.16) \quad f_i(x - d) - f_i(x) \rightarrow 0 \text{ as } (x, d) \rightarrow (\infty, 0).$$

$$(2.17) \quad k_{ij}(x) \sim \frac{ix}{N}; \quad i, j = 1, \dots, N, k_{ij}(0+) = 0.$$

$$(2.18) \quad k_{ij}(x) - k_{ij}(x - d) = O(1) \text{ for each fixed } d > 0.$$

$$(2.19) \quad \text{There exists } d > 0 \text{ such that}$$

$$(2.20) \quad \liminf_{x \rightarrow \infty} [k_{ij}(x) - k_{ij}(x - d)] > 0; \quad i, j = 1, \dots, N.$$

$$\eta_{ij}(\delta) = \limsup_{x \rightarrow \infty} [k_{ij}(x) - k_{ij}(x - \delta)] \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

DEFINITION 2.1. We let $k_{ij}^{(1)}(x) = k_{ij}(x)$ and

$$k_{ij}^{(n+1)}(x) = \sum_{m=1}^N \int_0^x k_{im}^{(n)}(x-y) dk_{mj}(y).$$

The next three results can be found in Pitt [1].

(2.21) Suppose $g(x) \sim ax^n$, then

$$\int_0^x g(x-y) dk_{ij}(y) \sim \frac{ax^{n+1}}{N(n+1)} \quad \text{for all } i, j = 1, \dots, N.$$

(2.22) For all pairs $(i, j); i, j = 1, \dots, N$ we have $k_{ij}^{(n)}(x) \sim x^n / Nn!$.

(2.23) For all pairs $(i, j); i, j = 1, \dots, N$

$$k_{ij}^{(n)}(x) - k_{ij}^{(n)}(x-d) = O(x^{n-1}) \quad \text{for each fixed } d.$$

LEMMA 2.3. For all $i, j = 1, \dots, N$ we have

$$\limsup_{x \rightarrow \infty} |f_i(x)| = \limsup_{x \rightarrow \infty} |f_j(x)|.$$

Proof. We let $S_i = \limsup_{x \rightarrow \infty} |f_i(x)|$ and note that by Lemma 2.1 we have $\limsup_{x \rightarrow \infty} (1/x) \int_0^x |f_j(x-y)| dk_{ij}(y) \leq S_j / N$. From (2.15) it follows that $S_i \leq N^{-1} \sum_{j=1}^N S_j$ for each $i = 1, \dots, N$. Suppose that for some i_0 we had $S_{i_0} < N^{-1} \sum_{j=1}^N S_j$, then summing on i would give $\sum_{j=1}^N S_j < \sum_{j=1}^N S_j$. Hence $S_i = S_j = N^{-1} \sum_{m=1}^N S_m = S$.

LEMMA 2.4. For each n we have

$$\frac{S}{n!} \leq \limsup_{x \rightarrow \infty} \sum_{j=1}^N \frac{1}{x^n} \int_0^x |f_j(x-y)| dk_{ij}^{(n)}(y).$$

Proof. By (2.15) the proposition is true for $n = 1$. Suppose it is true for some $n \geq 1$, i.e.,

$$\frac{S}{n!} \leq \limsup_{x \rightarrow \infty} \sum_{j=1}^N \frac{1}{x^n} \int_0^x |f_j(x-y)| dk_{ij}^{(n)}(y).$$

By (2.15) again,

$$\begin{aligned} & \sum_{j=1}^N \frac{1}{x^n} \int_0^x |f_j(x-y)| dk_{ij}^{(n)}(y) \\ & \leq \sum_{j=1}^N \frac{1}{x^n} \int_0^x \left\{ \sum_{m=1}^N \frac{1}{x-y} \int_0^{x-y} |f_m(x-y-t)| dk_{im}(t) + o(1) \right\} dk_{ij}^{(n)}(y) \\ & \leq \sum_{j,m=1}^N \frac{1}{x^n} \int_0^x \frac{1}{x-y} \int_0^{x-y} |f_m(x-y-t)| dk_{im}(t) dk_{ij}^{(n)}(y) + o(1) \end{aligned}$$

by Lemma 2.1.

The first term equals

$$\begin{aligned} & \sum_{j,m=1}^N \frac{1}{x^{n+1}} \int_0^x \frac{x dk_j^{(n)}(y)}{x-y} \int_0^{x-y} |f_m(x-y-t)| dk_{jm}(t) \\ &= \sum_{j,m=1}^N \frac{1}{x^{n+1}} \int_0^x dk_{ij}^{(n)}(y) \int_0^{x-y} |f_m(x-y-t)| dk_{jm}(t) \\ & \quad + \sum_{j,m=1}^N \frac{1}{x^{n+1}} \int_0^x \frac{y dk_{ij}^{(n)}(y)}{x-y} \int_0^{x-y} |f_m(x-y-t)| dk_{jm}(t). \end{aligned}$$

The first term here is just $\sum_{m=1}^N (1/x^{n+1}) \int_0^x |f_m(x-y)| dk_{im}^{(n+1)}(y)$ by 2.12. The lemma will be proved if we show that

$$\limsup_{x \rightarrow \infty} \sum_{j,m=1}^N \frac{1}{x^{n+1}} \int_0^x \frac{y dk_{ij}^{(n)}(y)}{x-y} \int_0^{x-y} |f_m(x-y-t)| dk_{jm}(t) \leq \frac{S}{n!} - \frac{S}{(n+1)!}.$$

We let $g_m(x) = (1/x) \int_0^x |f_m(x-y)| dk_{jm}(y)$ and note that $g_m(x)$ is bounded and $\limsup_{x \rightarrow \infty} g_m(x) \leq S/N$.

The expression to be estimated is $\sum_{j,m=1}^N (1/x^{n+1}) \int_0^x y g_m(x-y) dk_{jm}^{(n)}(y)$. Let $\varepsilon > 0$ be given, then we choose x_0 so large that $g_m(t) \leq S/N + \varepsilon$ for $t \geq x_0$, then

$$\begin{aligned} & \frac{1}{x^{n+1}} \int_0^x y g_m(x-y) dk_{jm}^{(n)}(y) \\ &= \frac{1}{x^{n+1}} \int_0^{x-x_0} y g_m(x-y) dk_{jm}^{(n)}(y) + \frac{1}{x^{n+1}} \int_{x-x_0}^x y g_m(x-y) dk_{jm}^{(n)}(y). \end{aligned}$$

The second term is bounded by

$$\frac{M}{x^n} [k_{ij}^{(n)}(x) - k_{ij}^{(n)}(x-x_0)] = O\left(\frac{1}{x}\right) \rightarrow 0,$$

where M is a bound for g_1 , here we use (2.23).

The first term is bounded by

$$\begin{aligned} & \left(\frac{S}{N} + \varepsilon\right) \frac{1}{x^{n+1}} \int_0^x y dk_{jm}^{(n)}(y) \\ &= \left(\frac{S}{N} + \varepsilon\right) \left[\frac{1}{x^n} \int_0^x dk_{jm}^{(n)}(y) - \frac{1}{x^{n+1}} \int_0^x (x-y) dk_{jm}^{(n)}(y) \right] \\ &= \left(\frac{S}{N} + \varepsilon\right) \left[\frac{k_{jm}^{(n)}(x)}{x^n} - \frac{1}{N(n+1)!} \right] + o(1) \\ &= \frac{1}{N} \left(\frac{S}{N} + \varepsilon\right) \left[\frac{1}{n!} - \frac{1}{(n+1)!} \right] + o(1). \end{aligned}$$

Since ε was arbitrary it follows that

$$\limsup_{x \rightarrow \infty} \sum_{j,m} \frac{1}{x^{n+1}} \int_0^x y g_m(x-j) dk_{jm}(y) \leq \frac{S}{n!} - \frac{S}{(n+1)!}$$

which was to be proved.

We quote without proof the following elementary lemma, which can be found in Pitt [2].

(2.24) Suppose for $j = 0, 1, \dots, J$ we have $q_j(x) \geq 0$ for all x and $q_j(x) \geq 1$ in an interval I_j of length 2α with center in a fixed interval I of length $2d$. Then there is an interval of length 2α with center in I , throughout which

$$\sum_{j=0}^J q_j(x) \geq \frac{J\alpha}{2(2d + \alpha)} - 1.$$

The next lemma is crucial and is based on an idea of Pitt.

LEMMA 2.5. *If $\delta > 0$ is given then there exist x_0, n , and $\beta > 0$ such that for all $x \geq x_0$, $k_{ij}^{(n)}(x) - k_{ij}^{(n)}(x - \delta) \geq \beta x^{n-1}$, this holding for all pairs (i, j) .*

Proof. Let $\phi_{ij}^{(n)}(x) = k_{ij}^{(n)}(x) - k_{ij}^{(n)}(x - \delta)$, then

$$\begin{aligned} \phi_{ij}^{(n+1)}(x) &= \sum_{m=1}^N \int_0^x k_{im}^{(n)}(x-y) dk_{mj}(y) - \sum_{m=1}^N \int_0^{x-\delta} k_{im}^{(n)}(x-\delta-y) dk_{mj}(y) \\ &= \sum_{m=1}^N \int_0^x \phi_{im}^{(n)}(x-y) dk_{mj}(y). \end{aligned}$$

The proof will be divided into several parts.

(2.5a) There exist $\alpha_1, \beta_1 > 0$ and x_1 such that for each pair (i, j) there is a sequence of intervals $\{I_m(i, j)\}_{m=0}^\infty$ of length $2\alpha_1$, such that every interval $(x, x + 2d)$ for $x \geq x_1$ contains the center of one of these intervals and $\phi_{ij}^{(1)}(x) \geq \beta_1$ for $x \in I_m(i, j)$. First by (2.19) choose x'_1 so that $k_{ij}(x) - k_{ij}(x - d) \geq \gamma > 0$ for all $x \geq x'_1$. Then choose α so small that by (2.20) $\limsup_{x \rightarrow \infty} [k_{ij}(x) - k_{ij}(x - \alpha)] < \gamma\delta/4d$, then choose $x_1 \geq x'_1$ so that $k_{ij}(x) - k_{ij}(x - \alpha) < \gamma\delta/4d$ for all $x \geq x_1$. Then it follows that for any m such that $md \geq x_1$, the interval $(md, (m + 1)d)$ contains a subinterval of length α , throughout which $\phi_{ij}^{(1)}(x) \geq \gamma\delta/4d$. Clearly every interval of the type $(x, x + 2d)$ for $x \geq x_1$ contains the center of such an interval. We can do this for each pair (i, j) and hence can assume that α_1, β_1 , and x_1 are sufficient for all (i, j) . However the sequence of intervals $\{I_m(i, j)\}_{m=0}^\infty$ may vary with (i, j) .

(2.5b) There exists $h, \eta > 0$ and x_2 such that for all m such that $md \geq x_2$, and for each (i, j) the interval $(md, (m + 1)d)$ contains 2 subintervals $(md + a_m(i, j), md + a_m(i, j) + h)$ and $(md + b_m(i, j), md + b_m(i, j) + h)$ such that k_{ij} increases by at least η in each of these intervals and:

$$(2.25) \quad b_m(i, j) - a_m(i, j) + h \leq 2\alpha_1$$

$$(2.26) \quad a_m(i, j) + 2h \leq b_m(i, j).$$

First choose x'_2 so that for all m such that $md \geq x'_2$, k_{ij} increases by more than $\gamma' > 0$ in the interval from md to $(m + 1)d$. In any such interval clearly there is a subinterval of length $2\alpha_1$, throughout which k_{ij} increases by at least $\gamma > 0$, we may take $\gamma = \gamma'2\alpha_1/d$ and we see that γ is independent of m . Now choose h so small that $\limsup_{x \rightarrow \infty} [k_{ij}(x) - k_{ij}(x - h)] < \gamma/4$ and then $x_2 \geq x'_2$ so that

$$k_{ij}(x) - k_{ij}(x - h) < \frac{\gamma}{4} \quad \text{for all } x \geq x_2.$$

Now let $\eta = \gamma h/8d$. Let m be such that $md \geq x_2$, then $(md, (m + 1)d)$ contains a subinterval of length $2\alpha_1$ such that k_{ij} increases by at least γ in this subinterval. We divide this subinterval into subintervals of length h , in at least one of these k_{ij} increases by more than η , pick the first one, then skip one and there must be at least one more farther on where it increases by at least η . Let $a_m(i, j), b_m(i, j)$ be the left-hand end points of these intervals, then conditions (2.25), (2.26) follow by the construction.

(2.6c) Proceed by induction. We assume that for some $n \geq 1$, there exists $\beta_n > 0$, $\alpha_n \geq \alpha_1 + (n - 1/2)h$, and x_n such that for each (i, j) there is a sequence $\{I_m^{(n)}(i, j)\}_{m=0}^\infty$ of intervals of length $2\alpha_n$, such that for $x \geq x_n$ the interval $(x, x + 2d)$ contains the center of one of these intervals and for $x \in I_m^{(n)}(i, j)$, $\phi_{ij}^{(n)}(x) \geq \beta_n x^{n-1}$. We have already verified this for $n = 1$.

Fix (i, j) ; we have

$$\begin{aligned} \phi_{ij}^{(n+1)} &= \sum_{m=1}^N \int_0^x \phi_{im}^{(n)}(x - y) dk_{mj}(y) \\ &= \sum_{r=1}^N \sum_{m \geq 0} q_{r,m}(x), \end{aligned}$$

where

$$q_{r,m}(x) = \int_{md}^{(m+1)d} \phi_{ir}^{(n)}(x - y) dk_{rj}(y).$$

Now choose m_0 so that $md \geq x_2$ for $m \geq m_0$ and pick $X \geq x_n$, $X \geq (m + 1)d$. By the induction hypothesis, for each (r, j) there exists $\tau_r \in (X, X + 2d)$ such that $\phi_{ir}^{(n)}(y) \geq \beta_n y^{n-1}$ for $y \in (\tau_r - \alpha_n, \tau_r + \alpha_n)$.

Now suppose x is in one of the intervals

$$\begin{aligned} I_1 &= (\tau_r - \alpha_n + md + a_m(r, j) + h, \tau_r + \alpha_n + md + a_m(r, j)), \\ I_2 &= (\tau_r - \alpha_n + md + b_m(r, j) + h, \tau_r + \alpha_n + md + b_m(r, j)). \end{aligned}$$

If $x \in I_1$, then

$$\begin{aligned} q_{r,m}(x) &= \int_{md}^{(m+1)d} \phi_{ir}^{(n)}(x-y) dk_{rj}(y) \\ &\geq \int_{md+a_m(r,j)}^{md+a_m(r,j)+h} \phi_{ir}^{(n)}(x-y) dk_{rj}(y) \\ &\geq \int_{md+a_m(r,j)}^{md+a_m(r,j)+h} \beta_n(x-y)^{n-1} dk_{rj}(y) \\ &\geq \eta \beta_n(x-md-d)^{n-1}, \end{aligned}$$

because if $x \in I_1$ and $y \in (md + a_m(r,j), md + a_m(r,j) + h)$ then

$$\tau_r - \alpha_n \leq x - y \leq \tau_r + \alpha_n.$$

Similarly, if $x \in I_2$, then

$$\begin{aligned} q_{r,m}(x) &\geq \int_{md+b_m(r,j)}^{md+b_m(r,j)+h} \phi_{ir}^{(n)}(x-y) dk_{rj}(y) \\ &\geq \eta \beta_n(x-md-d)^{n-1}. \end{aligned}$$

By (2.25), (2.26) these intervals overlap and have total length of at least $2\alpha_n + h$.

Now consider some large $X \geq x_n$.

$$\begin{aligned} \phi_{ij}^{(n+1)}(x) &= \sum_{r=1}^N \sum_{m \geq 0} q_{r,m}(x) \\ &\geq \sum_{r=1}^N \sum_{m_0 \leq m \leq X/2d-1} q_{r,m}(x). \end{aligned}$$

We focus our attention on one of the $q_{r,m}$ appearing in this sum. Now $X - md > m$ so there exists $\tau_r \in (X - md, X - (m - 1)d)$ such that $\phi_{ir}^{(n)}(y) \geq \beta_n y^{n-1}$ for $y \in (\tau_r - \alpha_n, \tau_r + \alpha_n)$. One easily checks that the center of the corresponding interval $I_1 \cup I_2$ lies in $(X, X + 2d)$. So we see from the above that for each of these functions $q_{r,m}$ with $r = 1, \dots, N$ and $m_0 \leq m \leq X/2d - 1$, the interval $(X, X + 2d)$ contains the center of an interval of length $2\alpha_n + h$ throughout which

$$q_{r,m} \geq \eta \beta_n (x - md - d)^{n-1} \geq \eta \beta_n \left(\frac{X}{2}\right)^{n-1},$$

since $(m + 1)d \geq X/2$. Therefore by (2.24) this interval contains the center of an interval of length $2\alpha_n + h$, throughout which

$$\phi_{ij}^{(n+1)}(x) \geq \eta \beta_n \left(\frac{X}{2}\right)^{n-1} \left\{ \frac{N \left(\left(\frac{X}{2d}\right) - 1 - m_0\right) \left(\alpha_n + \frac{h}{2}\right)}{2 \left(2d + \alpha_n + \frac{h}{2}\right)} - 1 \right\} \geq \beta'_{n+1} X^n$$

if X is taken sufficiently large. But since $X \geq x - 4d$ for all x in this interval we have $\phi_{ij}^{(n+1)}(x) \geq \beta_n x^n$ for all X sufficiently large. So we have verified the proposition for $n + 1$. Now we just take n so large that $2\alpha + (n - 1)h > 4d$ and then we will have

$$\phi_{ij}^{(n)}(x) \geq \beta_n x^{n-1} \quad \text{for all large } x.$$

THEOREM 2.1. *For each $i = 1, \dots, N$, $f_i(x) = o(1)$.*

Proof. We want to show $S = 0$. Suppose $S > 0$. Then by Lemma 2.2 there exist x_0, T, δ such that, for all $x \geq x_0$, the interval $(x, x + T)$ contains a sub-interval of length δ , throughout which $|f_i(t)| < S/2$ for all $i = 1, \dots, N$. Now choose x_1 and n so that for all (i, j)

$$k_{ij}^{(n)}(x) - k_{ij}^{(n)}(x - \delta) \geq \beta x^{n-1}$$

for some $\beta > 0$ and all $x \geq x_1$. This is possible by Lemma 2.5. Let $x_2 = \max(x_1, x_2)$. Suppose $x - T \geq x_2$, and $(y - \delta, y) \subseteq (x - T, x)$. Then $k_{ij}^{(n)}(y) - k_{ij}^{(n)}(y - \delta) \geq \beta x^{n-1}$ and $k_{ij}^{(n)}(x) - k_{ij}^{(n)}(x - T) \leq Ax^{n-1}$ by (2.23). Hence

$$\frac{k_{ij}^{(n)}(y) - k_{ij}^{(n)}(y - \delta)}{k_{ij}^{(n)}(x) - k_{ij}^{(n)}(x - T)} \geq \frac{\beta}{A},$$

or $k_{ij}^{(n)}(y) - k_{ij}^{(n)}(y - \delta) > \gamma[k_{ij}^{(n)}(x) - k_{ij}^{(n)}(x - T)]$ for some $\gamma > 0$. Now Lemma 2.4 gives

$$\frac{S}{n!} \leq \limsup_{x \rightarrow \infty} \sum_{j=1}^N \frac{1}{x^n} \int_0^x |f_j(x - y)| dk_{ij}^{(n)}(y)$$

for each $i = 1, \dots, N$. In order to get a contradiction it is enough to show that

$$\limsup_{x \rightarrow \infty} \frac{1}{x^n} \int_0^x |f_j(x - y)| dk_{ij}^{(n)}(y) < \frac{S}{Nn!}.$$

Let $\epsilon > 0$ be given, we assume that x_2 chosen before is also so large that $|f_j(t)| < S + \epsilon$ for all $t \geq x_2$. Now

$$\begin{aligned} & \frac{1}{x^n} \int_0^x |f_j(x - y)| dk_{ij}^{(n)}(y) \\ &= \frac{1}{x^n} \int_0^{x-x_2} |f_j(x - y)| dk_{ij}^{(n)}(y) + \frac{1}{x^n} \int_{x-x_2}^x |f_j(x - y)| dk_{ij}^{(n)}(y). \end{aligned}$$

The second integral goes to zero by (2.23) and the fact that f_j is bounded. The first term is broken up as follows:

$$\begin{aligned} & \sum_{m=0}^{[(x-x_2)/T-1]} \frac{1}{x^n} \int_{mT}^{(m+1)T} |f_j(x-y)| dk_{ij}^{(n)}(y) \\ & \leq \sum_{m=0}^{[(x-x_2)/T-1]} \frac{1}{x^n} \left\{ (S + \varepsilon) [k_{ij}^{(n)}((m+1)T) - k_{ij}^{(n)}(mT)] - \frac{S}{2} [k_{ij}^{(n)}(y_m + \delta) - k_{ij}^{(n)}(y_m)] \right\} \\ & \leq \sum_{m=0}^{[(x-x_2)/T-1]} \frac{1}{x^n} \left\{ (S + \varepsilon) - \gamma \frac{S}{2} \right\} [k_{ij}^{(n)}((m+1)T) - k_{ij}^{(n)}(mT)] \\ & = \frac{1}{x^n} \left\{ (S + \varepsilon) - \gamma \frac{S}{2} \right\} k_{ij}^{(n)}(x - x_2). \end{aligned}$$

Hence $\limsup_{x \rightarrow \infty} (1/x^n) \int_0^x |f_j(x-y)| dk_{ij}^{(n)}(y) \leq S(1-\gamma/2)/Nn!$ since ε was arbitrary. This gives a contradiction and hence $S = 0$.

COROLLARY 2.1. *If the $f_i(x)$ are permitted to take on complex values with the restriction that there exist $\alpha_i, i = 1, \dots, N$ such that $\alpha_i \leq \arg f_i(x) \leq \alpha_i + \pi$ then Theorem 2.1 still holds.*

Proof. Noting that $e^{-i\alpha} f_j(x)$ satisfies (2.13)–(2.16) whenever $f_j(x)$ does we may assume that $0 \leq \arg f_j(x) \leq \pi$ for $j = 1, \dots, N$. Let $f_j(x) = f_j^{(1)}(x) + if_j^{(2)}(x)$. Then we see that $f_j^{(2)}(x) \geq 0$ for all x . By condition (2.14), $\int_0^x f_j^{(2)}(t) dt = O(1)$, hence $\int_0^\infty f_j^{(2)}(t) dt$ exists because $f_j^{(2)}(t) \geq 0$. Since f_j satisfies (2.16) so does $f_j^{(2)}$ and it follows easily from this and the fact that $\int_0^\infty f_j^{(2)}(t) dt < \infty$, that $f_j^{(2)}(x) = o(1)$. By (2.15) we have

$$|f_j(x)| \leq \sum_{m=1}^N \frac{1}{x} \int_0^x |f_m(x-y)| dk_{jm}(y) + o(1).$$

Now $|f_j(x)| = |f_j^{(1)}(x)| + o(1)$ and

$$\begin{aligned} & \frac{1}{x} \int_0^x |f_m^{(1)}(x-y) + if_m^{(2)}(x-y)| dk_{jm}(y) \\ & \leq \frac{1}{x} \int_0^x |f_m^{(1)}(x-y)| dk_{jm}(y) + \frac{1}{x} \int_0^x |f_m^{(2)}(x-y)| dk_{jm}(y). \end{aligned}$$

The second integral is $o(1)$, by Lemma 2.1. Hence

$$|f_j^{(1)}(x)| \leq \sum_{m=1}^N \frac{1}{x} \int_0^x |f_m^{(1)}(x-y)| dk_{jm}(y) + o(1)$$

for $j = 1, \dots, N$. Since $f_j^{(1)}(x)$, for $j = 1, \dots, N$ satisfy (2.13)–(2.16) it follows from Theorem 2.1 that $f_j^{(1)}(x) = o(1)$, hence $f_j(x) = o(1)$.

Next we show that given any angular sector in the plane: $\alpha \leq \theta \leq \pi + \alpha + \varepsilon$ for $\varepsilon > 0$, there exists a function f taking on values in this sector such that f satisfies conditions (2.13), (2.14), (2.15) and that for any monotone increasing function k , with $k(x) \sim x$ we have

$$|f(x)| \leq \frac{1}{x} \int_0^x |f(x-y)| dk(y) + o(1); \text{ yet } f(x) \neq o(1).$$

We may assume the angular sector is of the type $-\varepsilon \leq \theta \leq \pi + \varepsilon$, for some $\varepsilon > 0$. We define f as follows:

$$f(t) = e^{i(\pi/2-t)} \text{ for } 0 \leq t \leq \pi/2 + \varepsilon,$$

$$f(t) = e^{-i\varepsilon} \text{ for } \pi/2 + \varepsilon \leq t \leq T, \text{ where } T = \pi/2 + \varepsilon + \cot \varepsilon,$$

$$f(t) = f(2t - T) \text{ for } T \leq t \leq 2T,$$

$$f(t) = -\overline{f(t-2T)} \text{ for } 2T \leq t \leq 4T,$$

then define f periodically with period $4T$.

It is easily seen that f is continuous, and hence it is uniformly continuous since it is periodic. Therefore it satisfies (2.16). Since $|f(t)| = 1$, condition (2.13) is fulfilled.

To show that (2.14) is satisfied we show that $\int_0^{4T} f(t) dt = 0$. Let $f(t) = f_1(t) + if_2(t)$,

$$\begin{aligned} \int_{2T}^{4T} f_1(t) dt &= \int_{2T}^{4T} R\{-\overline{f(t-2T)}\} dt \\ &= -\int_{2T}^{4T} f_1(t-2T) dt = -\int_0^{2T} f_1(t) dt, \end{aligned}$$

hence $\int_0^{4T} f_1(t) dt = 0$. Next consider $f_2(t)$,

$$\begin{aligned} \int_0^T f_2(t) dt &= \int_0^{\pi/2+\varepsilon} \sin(\pi/2-t) dt - \sin \varepsilon \int_{\pi/2+\varepsilon}^T dt \\ &= \int_0^{\pi/2+\varepsilon} \cos t dt - \cot \varepsilon \sin \varepsilon \\ &= \sin(\pi/2 + \varepsilon) - \cos \varepsilon = 0. \end{aligned}$$

Furthermore,

$$\int_T^{2T} f_2(t) dt = \int_T^{2T} f_2(2T-t) dt = \int_0^T f_2(t) dt = 0,$$

hence

$$\int_0^{2T} f_2(t) dt = 0.$$

Next:

$$\int_{2T}^{4T} f_2(t) dt = \int_{2T}^{4T} \text{Im}\{-\overline{f(t-2T)}\} dt = \int_{2T}^{4T} f_2(t-2T) dt = \int_0^{2T} f_2(t) dt = 0,$$

so

$$\int_0^{4T} f_2(t) dt = 0.$$

Since $|f(t)| = 1$, the condition that $|f(x)| \leq (1/x) \int_0^x f(x-y) dk(y) + o(1)$ merely requires that $1 \leq k(x)/x + o(1)$ which is true for any k such that $k(x) \sim x$.

Finally, (2.9) can be written as:

$$|F_h(x)| \leq \sum_{g \in G_m} \frac{1}{x} \int_0^x |F_g(x-y)| dH_{h,g}(y) + o(1)$$

for all $h \in G_m$, where $F_h(x) = f_h(e^x)$ and $H_{h,g}(y) = \sum_{n \leq e^y} \beta_{h,g}(n)$. We point out that if $f(x)$ is given by (2.2) or (2.4) and $\alpha(a) = \Lambda(a)/Na$, it is possible to show by elementary methods that the conditions of Theorem 2.1 are satisfied.

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