DISKS IN E^3.
II. DISKS WHICH "ALMOST" LIE ON A 2-SPHERE

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Introduction. The results of this paper were first proven in the author's thesis [1]. The proofs contained in that thesis are long and laborious. However, since that work was done, results of R. H. Bing [2] and J. P. Hempel [4] have made it possible to greatly shorten the proofs of the following theorems. Thus, not only appreciation but condolences are offered to the author's advisor, Guydo Lehner, who checked the original work.

Statement of results. Let D be a disk in E^3. We will say that D *"almost" lies on a 2-sphere in* E^3 if given any $\varepsilon > 0$ there exist a finite number of mutally disjoint subdisks $\{B_i\}_{i=1}^n$ of the interior of D with $\operatorname{diam} B_i < \varepsilon$ for $i = 1, 2, \cdots, n$, and such that $D - \sum_{i=1}^n B_i$ lies on a 2-sphere in E^3.

THEOREM 1. *Let D be a disk in E^3 which contains all its wild points in its interior. Then D "almost" lies on a 2-sphere in E^3.*

In the author's thesis Theorem 1 was used to show

THEOREM 2. *If D is a disk in E^3 which contains all its wild points in its interior and $\varepsilon > 0$ then D contains a Sierpiński curve K such that*

(1) $\operatorname{Bd} D \subset K$,

(2) *K lies on a tame 2-sphere in E^3, and*

(3) *The components of $D - K$ all have diameter $< \varepsilon$.*

In the shortened proof of Theorem 1 given here, Theorem 2 will be proven and used.

The result of Hempel mentioned above states that each compact orientable 2-manifold with boundary in E^3 which is locally tame at each point of its boundary lies on a closed 2-manifold in E^3. We prove

THEOREM 3. *Each compact orientable 2-manifold with boundary in E^3 which is locally tame at each point of its boundary "almost" lies on a closed 2-manifold in E^3 whose genus is $2g + h - 1$, where g is the genus and h is the number of components in the boundary of the original 2-manifold with boundary.*

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Proof of Theorem 1. The essential idea of this proof was pointed out to the author by Joseph Martin and used by him in [5]. Let D be the disk of our theorem and let D' be a subdisk of $\operatorname{Int} D$ such that all the wild points of D are in $\operatorname{Int} D'$. With the same technique developed by Martin [5] we can find a new disk D'' with the following properties.

(1) D'' is locally tame at each point of its interior.

(2) $\operatorname{Bd} D'' = \operatorname{Bd} D'$.

(3) $D'' \cap (D - D') = \varnothing$.

(4) D'' is obtained from D' by replacing a null sequence of mutually disjoint subdisks $\{D_i'\}_{i=1}^{\infty}$ of Int D' by another sequence $\{D_i''\}_{i=1}^{\infty}$. We do this so that none of the replaced subdisks has diameter as great as ε. Note that $D' \cap D''$ contains a tame Sierpiński curve containing Bd D', thus we immediately have a proof of Theorem 2.

Since Bd D'' is tame, D'' is tame [3] so $E = D'' \cup (D - D')$ is tame and lies on a tame 2-sphere in E^3. This 2-sphere has a cartesian product neighborhood, so we have an imbedding h of $E \times [0,1]$ into E^3 such that $h|_{E \times [0]} = \text{id}$. Let $S = \operatorname{Bd}\,(h(E \times [0,1]))$. S is a 2-sphere.

Consider $D' \cap (S - E)$. Since $S - E$ is bounded away from D'' which was found by replacing a null sequence $\{D_i'\}_{i=1}^{\infty}$ of subdisks of D' we see that $D' \cap (S - E)$ is contained in a finite number of the D_i''s. We will name these subdisks $B_1, B_2, \cdots, B_n$. These are the subdisks promised in the theorem. By a standard technique of 3-space topology we can replace the B_i's by new disks B_i''s which miss $S - \sum B_i$ and are such that $B_i' \cap B_j' = \varnothing$ if $i \neq j$. $(S - \sum B_i) \cup \sum B_i'$ is the 2-sphere we were promised by the theorem.

Proof of Theorem 3. The proof of Theorem 3 is parallel to that of Theorem 1 so we will omit the details. The number $2g + h - 1$ is natural in this theorem since if M is a compact 2-manifold with boundary, whose genus is g and which has h boundary components, then Bd $(M \times [0,1])$ will be a closed 2-manifold of genus $2g + h - 1$.

References

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5. Joseph Martin, *Tame arcs on disks*, Proc. Amer. Math. Soc. **16** (1965), 131–133.

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