MEROMORPHIC MULTIVALENT CLOSE-TO-CONVEX FUNCTIONS

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1. **Introduction.** Recently the author [5] has discussed some properties of the class of p-valent regular close-to-convex functions, called $\mathscr{K}(p)$. It is the purpose of this paper to generalize some of these results to the meromorphic case.

Let $f(z)$ be meromorphic for $|z| < 1$ with q $(1 \leqq q \leqq p)$ poles at the origin and $f(z) \neq 0$ for $|z| < 1$. We shall say that $f(z)$ is in $S_1^*(p)$ if there exists a ρ $(0 < \rho < 1)$ such that for $z = re^{i\theta}$ $(\rho < r < 1)$,

$$\text{(1.1)} \qquad \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] < 0$$

and

$$\text{(1.2)} \qquad \int_0^{2\pi} d \arg f(z) = \int_0^{2\pi} \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] d\theta = -2p\pi .$$

We shall say that $f(z)$ is in $S_2^*(p)$ if it is regular on $|z| = 1$ and if (1.1) and (1.2) hold for $|z| = 1$. If $f(z)$ is in $S_1^*(p)$, there exists a δ $(0 < \delta < 1)$ such that $f(rz)$ is in $S_2^*(p)$ if $\delta < r < 1$.

We set $S^*(p) = S_1^*(p) \cup S_2^*(p)$ and say that a function in $S^*(p)$ is starlike of order p.

Condition (1.2) along with the argument principle implies that a function in $S^*(p)$ has exactly p poles in $|z| < 1$. It is easily seen that a function $f(z)$, meromorphic in $|z| < 1$, is in $S^*(p)$ if and only if the function $[f(z)]^{-1}$ is regular and p-valently starlike in $|z| < 1$. Since the reciprocal of a p-valent function is p-valent, a function in $S^*(p)$ is p-valent in $|z| < 1$. Also, using the fact that a regular p-valent starlike function can be written as the pth power of a regular univalent starlike function, it is easily seen that a function in $S^*(p)$ with p poles at the origin can be written as the pth power of a meromorphic univalent starlike function.

Let $F(z)$ be meromorphic in $|z| < 1$ with q $(1 \leqq q \leqq p)$ poles at the origin and with at most p poles in $|z| < 1$. We shall say that $F(z)$ is in $\mathscr{K}_1^*(p)$ if there exists a function in $S^*(p)$ and a ρ $(0 < \rho < 1)$ such that for $z = re^{i\theta}$ $(\rho < r < 1)$

$$\text{(1.3)} \qquad \operatorname{Re}\left[\frac{zF'(z)}{f(z)}\right] > 0.$$

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We shall say that $F(z)$ is in $\mathscr{K}_2^*(p)$ if $F(z)$ is regular on $|z|=1$ and if there exists a function $f(z)$ in $S_2^*(p)$ such that (1.3) is satisfied for $|z|=1$. If $F(z)$ is in $\mathscr{K}_1^*(p)$, there exists a δ $(0<\delta<1)$ such that $F(rz)$ is in $\mathscr{K}_2^*(p)$ if $\delta<r<1$.

We set $\mathscr{K}^*(p)=\mathscr{K}_1^*(p)\cup\mathscr{K}_2^*(p)$ and say that a function in $\mathscr{K}^*(p)$ is close-to-convex of order p.

The class $\mathscr{K}^*(1)$ was defined by Libera and Robertson [4] and Pommerenke [7]. It was shown in both papers that a function in $\mathscr{K}^*(1)$ need not be univalent. To show that a function in $\mathscr{K}^*(p)$ need not be p-valent, let $F(z)$ be such that

$$\frac{zF'(z)}{z^{-p}}=\frac{1+z^{2p}}{1-z^{2p}} \qquad (|z|<1).$$

Then

$$F(z)=-\frac{1}{pz^p}+\frac{2}{p}z^{2p}+\frac{2}{3p}z^{3p}+\cdots \qquad (0<|z|<1).$$

If $F(z)$ was p-valent, then

$$F(z^{1/p})=-\frac{1}{pz}+\frac{2}{p}z+\cdots \qquad (0<|z|<1)$$

would be univalent, and so would

$$-pF(z^{1/p})=\frac{1}{z}-2z+\cdots \qquad (0<|z|<1).$$

But this is impossible, since the coefficient of z has modulus greater than 1. Thus $F(z)$ is at least $2p$-valent.

Necessary and sufficient conditions for a function to be in $\mathscr{K}^*(1)$ have been given in [4] and [7]. In §2 we obtain necessary conditions for a function $F(z)$ to be in $\mathscr{K}^*(p)$ and show that these conditions with the added assumptions of regularity on $|z|=1$ and $F'(z)\neq 0$ in $|z|\leqq 1$ are sufficient.

Recently, Royster [8] has shown that if

$$f(z)=\sum_{n=-p}^{\infty}a_nz^n \qquad (0<|z|<1)$$

is in $S^*(p)$ then $|a_n|=O(1/n)$. In §3 we will extend this result to functions in $\mathscr{K}^*(p)$ with p poles at the origin. This result was obtained for $\mathscr{K}^*(1)$ by Libera and Robertson [4] and Pommerenke [7].

2. The class $\mathscr{K}^*(p)$.

THEOREM 1. *If $F(z)$ is in $\mathscr{K}^*(p)$, then there exists ρ $(0<\rho<1)$ such that for $z=re^{i\theta}$ $(\rho<r<1)$*

$$(2.1)\qquad \int_0^{2\pi} d \arg d\, F(z) = \int_0^{2\pi} \frac{d}{d\theta} \arg \left[re^{i\theta}F'(re^{i\theta})\right] d\theta = -2p\pi$$

and for any θ_1 *and* θ_2 *with* $0 \leqq \theta_1 < \theta_2 \leqq 2\pi$

$$(2.2)\qquad \int_{\theta_1}^{\theta_2} d \arg d\, F(z) < \pi.$$

Proof. Suppose $F(z)$ is in $\mathscr{K}_1^*(p)$. Then there exists $f(z)$ in $S^*(p)$ and ρ $(0 < \rho < 1)$ such that (1.1), (1.2) and (1.3) hold for $\rho < |z| < 1$.

Since $\operatorname{Re}[zF'(z)/f(z)] > 0$ for $|z| = r$ $(\rho < r < 1)$, we may define

$$\arg [zF'(z)/f(z)]$$

to be single valued and continuous for $|z| = r$ and such that

$$(2.3)\qquad \left| \arg \frac{zF'(z)}{f(z)} \right| < \frac{\pi}{2} \qquad (|z| = r).$$

Furthermore, since $zF'(z) \neq 0$ for $|z| = r$, we may define $\arg [zF'(z)]$ to be single valued and continuous for $|z| = r$. Since $f(z) = [f(z)/zF'(z)]\,[zF'(z)]$, we may define $\arg [f(z)] = \arg [zF'(z)] - \arg [zF'(z)/f(z)]$ to be a single valued and continuous determination of $\arg [f(z)]$ for $|z| = r$. Then

$$(2.4)\qquad |\arg zF'(z) - \arg f(z)| = \left| \arg \frac{zF'(z)}{f(z)} \right| < \frac{\pi}{2} \quad (|z| = r).$$

It is easily seen that (2.4) implies

$$(2.5)\qquad -\pi + \int_{\theta_1}^{\theta_2} d \arg f(z) < \int_{\theta_1}^{\theta_2} d \arg d\, F(z) < \pi + \int_{\theta_1}^{\theta_2} d \arg f(z)$$

for $\theta_1 < \theta_2$ and $|z| = r$. Since $f(z)$ is in $S^*(p)$,

$$\int_{\theta_1}^{\theta_2} d \arg f(z) < 0 \qquad (|z| = r).$$

Thus we obtain (2.2) for $|z| = r$ from the right side of (2.5). Letting $\theta_1 = 0$ and $\theta_2 = 2\pi$ in (2.5) and noting that

$$\int_0^{2\pi} d \arg f(z) = -2p\pi$$

we obtain

$$(2.6)\qquad -(2p+1)\pi < \int_0^{2\pi} d \arg d\, F(z) < -(2p-1)\pi.$$

However, the integral appearing in (2.6) is an integral multiple of 2π. Thus (2.1) holds for $|z| = r$. Since r was arbitrary $(\rho < r < 1)$ (2.1) and (2.2) hold for $\rho < |z| < 1$.

If $F(z)$ is in $\mathscr{K}_2^*(p)$, then the preceding argument with $r = 1$ shows that (2.1) and (2.2) hold for $|z| = 1$. But since $F(z)$ is regular near $|z| = 1$, we can show the existence of a ρ $(0 < \rho < 1)$ such that (2.1) and (2.2) hold for $\rho < |z| \leqq 1$.

Using (2.1) and the argument principle we immediately obtain the following corollary.

COROLLARY 1. *If $F(z)$ is in $\mathscr{K}^*(p)$, then $F'(z)$ has at least $(p+1)$ poles in $|z| < 1$ and if $F'(z) \neq 0$ for $|z| < 1$, then $F'(z)$ has exactly $(p+1)$ poles in $|z| < 1$.*

THEOREM 2. *Let $F(z)$ be meromorphic in $|z| < 1$ with q $(1 \leqq q \leqq p)$ poles at the origin. If $F'(z) \neq 0$ for $0 < |z| \leqq 1$ and $F(z)$ is regular on $|z| = 1$ and if* (2.1) *and* (2.2) *hold for $|z| = 1$, then $F(z)$ is in $\mathscr{K}_2^*(p)$.*

Proof. Consider the function $G(z)$, regular for $|z| = 1$, given by

$$G(z) = \int_0^z \frac{dz}{z^2 F'(z)} = b_q z^q + \cdots .$$

Since $zF'(z) \neq 0$ for $|z| = 1$ we may define $\arg [zF'(z)]$ to be single valued and continuous for $|z| = 1$. Since $zG'(z) = [zF'(z)]^{-1}$, we may define $\arg zG'(z) = -\arg zF'(z)$. Thus, for $|z| = 1$

$$\int_0^{2\pi} d \arg d\, G(z) = 2p\pi$$

and

$$\int_{\theta_1}^{\theta_2} d \arg d\, G(z) > -\pi \qquad (\theta_1 < \theta_2).$$

The author has shown (Theorem 3 [5]) that under these conditions $G(z)$ is in $\mathscr{K}(p)$. That is, there exists $g(z)$, regular for $|z| \leqq 1$ such that

$$\operatorname{Re}\left[\frac{zg'(z)}{g(z)}\right] > 0 \qquad (|z| = 1)$$

and

$$\operatorname{Re}\left[\frac{zG'(z)}{g(z)}\right] > 0 \qquad (|z| = 1).$$

The function $f(z) = [g(z)]^{-1}$ is in $S^*(p)$ and

$$\frac{zG'(z)}{g(z)} = zG'(z)f(z) = \frac{f(z)}{zF'(z)}.$$

Thus

$$\operatorname{Re}\left[\frac{zF'(z)}{f(z)}\right] = \operatorname{Re}\left[\frac{g(z)}{zG'(z)}\right] > 0 \qquad (|z| = 1).$$

Therefore $F(z)$ is in $\mathscr{K}_2^*(p)$.

Using the same procedure as above and by appealing to Theorem 2 [5], we may remove the condition of regularity on $|z| = 1$, if $q = p$. We thus have the following theorem.

THEOREM 3. *Let*

$$F(z) = \sum_{n=-p}^{\infty} a_n z^n \qquad (0 < |z| < 1)$$

be meromorphic for $|z| < 1$ *and* $F'(z) \neq 0$. *If there exists a* ρ $(0 < \rho < 1)$ *such that* (2.1) *and* (2.2) *hold for* $\rho < |z| < 1$, *then* $F(z)$ *is in* $\mathscr{K}^*(p)$.

We will have need of the next lemma in what follows.

LEMMA 1. *Let* $F(z)$ *be in* $\mathscr{K}_2^*(p)$. *Then, there exists a function*

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \qquad (0 < |z| < 1)\ (|b_{-p}| = 1),$$

in $S_2^*(p)$, *such that*

$$\operatorname{Re}\left[\frac{zF'(z)}{f(z)}\right] > 0 \qquad (|z| = 1).$$

Proof. There exists a function $g(z)$ in $S_2^*(p)$ with s poles $(1 \leqq s \leqq p)$ at the origin such that,

$$\operatorname{Re}\left[\frac{zF'(z)}{g(z)}\right] > 0 \qquad (|z| = 1).$$

The function $g(z)$ has $(p-s)$ nonzero poles in $|z| < 1$. Let $\alpha_1, \alpha_2, \cdots, \alpha_{p-s}$ be these poles and let

$$h(z) = z^{s-p} \prod_{i=1}^{p-s} (z - \alpha_i)(1 - \bar{\alpha}_i z)$$

and

$$f(z) = h(z)g(z) = \sum_{n=-p}^{\infty} c_n z^n \qquad (0 < |z| < 1).$$

Since $[zh'(z)/h(z)]$ is purely imaginary on $|z| = 1$ and $\operatorname{Re}[zg'(z)/g(z)] < 0$ for $|z| = 1$, then $\operatorname{Re}[zf'(z)/f(z)] < 0$ for $|z| = 1$. Furthermore, since $f(z)$ has p poles in $|z| \leqq 1$, all of them at the origin, and since $f(z) \neq 0$ in $|z| \leqq 1$,

$$\int_0^{2\pi} \operatorname{Re}\left[\frac{e^{i\theta} f'(e^{i\theta})}{f(e^{i\theta})}\right] d\theta = -2p\pi.$$

Thus, $f(z)$ in $S_2^*(p)$. Furthermore,

$$\frac{zF'(z)}{f(z)} = \frac{z^{p-s}zF'(z)}{\prod_{i=1}^{p-s}(z-\alpha_i)(1-\bar{\alpha}_i z)g(z)}.$$

But $z^{p-s}[\prod_{i=1}^{p-s}(z-\alpha_i)(1-\bar{\alpha}_i z)]^{-1}$ is real and positive on $|z|=1$. Therefore

$$\operatorname{Re}\left[\frac{zF'(z)}{f(z)}\right] > 0 \qquad (|z|=1).$$

Replacing $f(z)$ by $1/|c_{-p}|f(z)$, the proof of the lemma is completed.

THEOREM 4. *If $F(z)$ is in $\mathscr{K}^*(p)$ and has all its poles at the origin, then necessarily it has p poles there and $F'(z) \neq 0$ for $|z| < 1$.*

Proof. Suppose

$$F(z) = \sum_{n=-q}^{\infty} a_n z^n \quad (0 < |z| < 1) \ (1 \leqq q \leqq p).$$

There exists a ρ $(0 < \rho < 1)$ such that $F(rz)$ is in $\mathscr{K}_2^*(p)$ if $\rho < r < 1$. By Lemma 1, there exists

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \qquad (0 < |z| < 1)$$

in $S_2^*(p)$ such that

$$\operatorname{Re}\left[\frac{rzF'(rz)}{f(z)}\right] > 0 \qquad (|z|=1).$$

Since $f(z) \neq 0$ for $|z| \leqq 1$,

$$\frac{rzF'(rz)}{f(z)} = \sum_{n=p-q}^{\infty} c_n z^n$$

is regular for $|z| \leqq 1$. Thus,

$$\operatorname{Re}\left[\frac{rzF'(rz)}{f(z)}\right] > 0 \qquad (|z| \leqq 1).$$

Therefore, we must necessarily have $q = p$ and $F'(rz) \neq 0$ for $|z| \leqq 1$. Thus, $F'(z) \neq 0$ for $|z| \leqq r$. Since r was arbitrary $(\rho < r < 1)$, $F'(z) \neq 0$ for $|z| < 1$.

If $F(z)$ has all its poles at the origin we may improve Lemma 1 by removing the condition of regularity on $|z| = 1$.

LEMMA 2. *Let*

$$F(z) = \sum_{n=-p}^{\infty} a_n z^n \qquad (0 < |z| < 1)$$

be in $\mathscr{K}^*(p)$. *Then there exists*

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \qquad (0 < |z| < 1) \quad (|b_{-p}| = 1)$$

in $S^*(p)$ *such that*

$$\operatorname{Re} \frac{zF'(z)}{f(z)} > 0 \qquad (|z| < 1).$$

Proof. There exists a ρ $(0 < \rho < 1)$ such that the function $F_r(z) = F(rz)$ is in $\mathscr{K}_2^*(p)$ if $\rho < r < 1$. Then by Lemma 1 there exists

$$f_r(z) = \sum_{n=-p}^{\infty} c_n z^n \qquad (0 < |z| < 1) \, (|c_{-p}| = 1)$$

in $S_2^*(p)$, such that

$$\operatorname{Re} \left[\frac{zF_r'(z)}{f_r(z)}\right] > 0 \qquad (|z| \leqq 1).$$

Let r_i $(\rho < r_i < 1)$ be an increasing sequence tending to 1. The functions $[f_{r_i}(z)]^{-1}$ are regular and p-valently starlike and have the moduli of their first p coefficients fixed. The class of regular and p-valently starlike functions with the moduli of their first p coefficients fixed forms a normal family of functions [1]. Thus, we can obtain a subsequence $[f_{r_{i_k}}(z)]^{-1}$ tending uniformly in every closed subset of $|z| < 1$ to a function $f(z)$ regular and p-valently starlike and such that

$$f(z) = \sum_{n=p}^{\infty} d_n z^n \qquad (|z| < 1) \, (|d_p| = 1).$$

Since $F_{r_{i_k}}(z)$ tends to $F(z)$ as r_{i_k} tends to 1 and since

$$\operatorname{Re}[zF_{r_{i_k}}(z)[f_{r_{i_k}}(z)]^{-1}] > 0 \qquad \text{for } |z| < 1$$

we have

$$\operatorname{Re}\,[zF'(z)f(z)] > 0 \qquad \text{for } |z| < 1.$$

But

$$g(z) = [f(z)]^{-1} = \sum_{n=-p}^{\infty} b_n z^n \qquad (0 < |z| < 1) \, (|b_{-p}| = 1)$$

is in $S^*(p)$ and

$$\operatorname{Re}\left[\frac{zF'(z)}{g(z)}\right] = \operatorname{Re}\,[zF'(z)f(z)] > 0 \qquad \text{for } |z| < 1.$$

3. **The coefficients of a function in $\mathscr{K}^*(p)$.** We will make use of the following lemma, proven by Royster [8] and the author [6].

LEMMA 3. *Let*

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \qquad (0 < |z| < 1) \quad (|b_{-p}| = 1)$$

be in $S^*(p)$, *then for* $n \geqq 1$

$$|b_n| \leqq \frac{2p}{(n+p)\sqrt{p}} \left(\sum_{k=-p}^{-1} |k|\,|b_k|^2 \right)^{1/2}.$$

The following lemma was proven for $p = 1$ by Pommerenke [7].

LEMMA 4. *Let*

$$F(z) = \frac{1}{z^p} + \sum_{n=-(p-1)}^{\infty} a_n z^n \text{ and } f(z) = \frac{e^{i\beta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n, \quad (0 < |z| < 1)$$

and let $U(z) = \operatorname{Re}[zF'(z)/f(z)]$, *then for* $r < 1$

$$na_n = -pe^{-i\beta} b_n + \frac{1}{\pi} \int_0^{2\pi} \left[U(re^{i\theta}) \frac{e^{-in\theta}}{r^n} \right] \times \left[f(re^{i\theta}) - \sum_{k=n}^{\infty} b_k (re^{i\theta})^k \right] d\theta. \tag{3.1}$$

Proof. Let

$$\frac{zF'(z)}{f(z)} = -pe^{-i\beta} + \sum_{k=1}^{\infty} C_k z^k \qquad (|z| < 1).$$

Then

$$\begin{aligned} \frac{-p}{z^p} + \sum_{n=-(p-1)}^{\infty} na_n z^n &= \left[-pe^{-i\beta} + \sum_{k=1}^{\infty} C_k z^k \right] \left[\frac{e^{i\beta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n \right] \\ &= \frac{-p}{z^p} - pe^{-i\beta} \left[\sum_{n=-(p-1)}^{\infty} b_n z^n \right] + e^{i\beta} \left[\sum_{k=1}^{\infty} C_k z^{k-p} \right] \\ &\quad + \sum_{n=-(p-2)}^{\infty} \left[\sum_{k=1}^{n+p-1} C_k b_{n-k} \right] z^n. \end{aligned}$$

Thus, for $n \geqq 1$

$$na_n = -pe^{-i\beta} b_n + e^{i\beta} C_{p+n} + \sum_{k=1}^{n+p-1} C_k b_{n-k}. \tag{3.2}$$

Now

$$C_k = \frac{1}{r^k \pi} \int_0^{2\pi} U(re^{i\theta}) e^{-ik\theta} d\theta.$$

Substituting into (3.2), we obtain

$$\begin{aligned} na_n = & -pe^{-i\beta}b_n + \frac{1}{\pi}\int_0^{2\pi}\left[U(re^{i\theta})\,\frac{e^{-in\theta}}{r^n}\right] \\ & \times\left[\frac{e^{i\beta}}{r^p e^{ip\theta}} + \sum_{k=1}^{n+p-1} r^{n-k}e^{i(n-k)\theta}b_{n-k}\right]d\theta \\ = & -pe^{-i\beta}b_n + \frac{1}{\pi}\int_0^{2\pi}\left[U(re^{i\theta})\,\frac{e^{-in\theta}}{r^n}\right] \\ & \times\left[f(re^{i\theta}) - \sum_{k=n}^{\infty} b_k(re^{i\theta})^k\right]d\theta. \end{aligned}$$

THEOREM 5. *Let*

$$F(z) = \sum_{n=-p}^{\infty} a_n z^n \qquad (0<|z|<1)\ (a_{-p}\neq 0)$$

be in $\mathscr{K}^*(p)$, *then* $|a_n| = O(n^{-1})$.

Proof. We may assume without loss of generality that $a_{-p}=1$. There exists, by Lemma 2,

$$f(z) = \frac{e^{i\beta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n \qquad (0<|z|<1)$$

in $S^*(p)$ such that

$$\left[\operatorname{Re}\frac{zF'(z)}{f(z)}\right] > 0 \qquad (|z|<1).$$

Let $U(z) = \operatorname{Re}\,[zF'(z)/f(z)]$, then by a well-known result on harmonic functions,

$$\frac{1}{\pi}\int_0^{2\pi} U(re^{i\theta})\,d\theta = 2\,U(0) = -2p\cos\beta \leqq 2p.$$

By Lemma 4, we have for $n \geqq 1$

$$\begin{aligned} n|a_n| \leqq\ & p|b_n| + \frac{1}{\pi r^n}\left|\int_0^{2\pi} U(re^{i\theta})\,e^{-in\theta}\sum_{k=n}^{\infty} b_k(re^{i\theta})^k\,d\theta\right| \\ & + \frac{1}{\pi r^n}\left|\int_0^{2\pi} U(re^{i\theta})\,f(re^{i\theta})\,e^{-in\theta}\,d\theta\right| \qquad (3.3) \\ \leqq\ & p|b_n| + \frac{2p}{r^n}\sum_{k=n}^{\infty}|b_k|\,r^k + \frac{1}{\pi r^n}\int_0^{2\pi} U(re^{i\theta})\,|f(re^{i\theta})|\,d\theta. \end{aligned}$$

The Area Theorems of Golusin [2] and Kobori [3] give for $n \geqq 1$

$$\sum_{k=n}^{\infty} k|b_k|^2 \leqq \sum_{k=1}^{\infty} k|b_k|^2 \leqq \sum_{k=-p}^{-1} |k|\ |b_k|^2.$$

We thus have,

$$\begin{aligned} \frac{2p}{r^n} \sum_{k=n}^{\infty} |b_k|\, r^k &\leqq \frac{2p}{r^n} \left[\sum_{k=n}^{\infty} k|b_k|^2\right]^{1/2} \left[\sum_{k=n}^{\infty} \frac{r^{2k}}{k}\right]^{1/2} \\ &\leqq \frac{2p}{r^n} \left[\sum_{k=-p}^{-1} |k|\ |b_k|^2\right]^{1/2} \left[\frac{1}{n} \sum_{k=n}^{\infty} r^{2k}\right]^{1/2} \qquad (3.4) \\ &= 2p \left[\sum_{k=-p}^{-1} |k|\ |b_k|^2\right]^{1/2} [n(1-r^2)]^{-1/2}. \end{aligned}$$

Also for $n \geqq p$, by Lemma 3

$$\begin{aligned} |b_n| &\leqq \frac{2p}{(p+n)\sqrt{p}} \left[\sum_{k=-p}^{-1} |k|\ |b_k|^2\right]^{1/2} \qquad (3.5) \\ &\leqq \frac{1}{\sqrt{p}} \left[\sum_{k=-p}^{-1} |k|\ |b_k|^2\right]^{1/2}. \end{aligned}$$

Since $[f(z)]^{-1}$ is p-valently star like we have

$$|f(re^{i\theta})|^{-1} \geqq \frac{r^p}{1+r)^{2p}}$$

or

$$|f(re^{i\theta})| \leqq \frac{(1+r)^{2p}}{r^p}.$$

Therefore, for $n \geqq p$

$$\begin{aligned} &\frac{1}{\pi r^n} \int_0^{2\pi} U(re^{i\theta})\, |f(re^{i\theta})|\, d\theta \\ &\qquad \leqq \frac{(1+r)^{2p}}{r^{p+n}}\ \frac{1}{\pi} \int_0^{2\pi} U(re^{i\theta})\, d\theta \qquad (3.6) \\ &\qquad \leqq \frac{2p(1+r)^{2p}}{r^{p+n}} \leqq \frac{2p\, 4^p}{r^{2n}}. \end{aligned}$$

From (3.3), (3.4), (3.5) and (3.6) we have for $n \geqq p$ and any $r < 1$

$$n|a_n| \leqq \left[\sqrt{p} + 2p\, [n(1-r^2)]^{-1/2}\right] \left[\sum_{k=-p}^{-1} |k|\ |b_k|^2\right]^{1/2} + 2p\, 4^p r^{-2n}.$$

Let $r^2 = (1 - 1/n)$, then for $n \geqq p + 1$

$$n|a_n| \leqq (\sqrt{p} + 2p)\left[\sum_{k=-p}^{-1} |k|\ |b_k|^2\right]^{1/2} + 2p\, 4^p(1 + 1/(n-1))^n$$

$$\leqq (\sqrt{p} + 2p)\left[\sum_{k=-p}^{-1} |k|\ |b_k|^2\right]^{1/2} + 2p\, 4^p\frac{(p+1)}{p}e.$$

Thus, $|a_n| = O(n^{-1})$.

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