SIGN-INVARIANT RANDOM VARIABLES AND STOCHASTIC PROCESSES WITH SIGN-INVARIANT INCREMENTS(1)

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0. **Introduction and summary.** The random variables $X_1,\cdots,X_k$ are called sign-invariant if the 2^k joint distributions corresponding to the sets $(\varepsilon_1 X_1,\cdots,\varepsilon_k X_k)$, $\varepsilon_1 = \pm 1,\cdots,\varepsilon_k = \pm 1$, are all the same. An arbitrary family of random variables $\{X_t, t \in T\}$, where T is some index set, is called sign-invariant if every finite subfamily of the family consists of sign-invariant variables. We shall also describe such a family as a "family of sign-invariant random variables." In the special case where the X_t are mutually independent, they are also sign-invariant if and only if their marginal distribution functions are symmetric. The sign-invariance of exchangeable random variables is discussed in [1].

In §1, basic properties of sign-invariant families are given. A fundamental property is that sign-invariant random variables are conditionally independent given their absolute values. A series of sign-invariant random variables converges in distribution if and only if it converges with probability 1.

In §2, the real-valued stochastic process with sign-invariant increments is presented. The sample functions behave like those of a continuous parameter martingale: they are bounded over every interval and have finite left- and right-hand limits at each point. The process is characterized as consisting of a pure jump function composed with a Brownian motion process having a random time parameter. The sum of the squares of the increments of the process over a sequence of subdivisions of the time parameter interval converges with probability 1 to a limit consisting of the sum of the squares of the jumps of the process plus the random time parameter of the Brownian part of the process.

A limit theorem on the sum of the γth powers of the increments, $0 < \gamma < 2$, is in §3. This sum converges to $+\infty$ if the process has a Brownian component. If there is no Brownian component, then, under certain conditions, the sum converges to the sum of the γth powers of the magnitudes of the jumps.

We remark that the formulation of sign-invariance was motivated by the search for a "natural" setting for these theorems on increments. The result on the sum of the squares of the increments is extended in §4 to a general diffusion

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process (Ito process). In §5, the result for the γth powers is extended to processes with independent increments. In the latter case, we show that the property under discussion is invariant under a transformation of the process to another process of the same type having sign-invariant increments.

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1. **General properties of sign-invariant families.** A single random variable X is sign-invariant if and only if its characteristic function is equal to $E[\cos uX]$. A natural generalization of this property to a family of sign-invariant random variables is

LEMMA 1.1. *$X_1, \cdots, X_k$ are sign-invariant if and only if their joint characteristic function is of the form*

$$E[\exp i(u_1X_1 + \cdots + u_kX_k)] = E\left[\prod_{j=1}^{k} \cos u_jX_j\right]. \tag{1.1}$$

Proof. The sufficiency of the form (1.1) follows from the status of $\cos x$ as an even function of x. For the proof of necessity, take $k = 2$:

$$\begin{aligned} E[\exp i(u_1X_1 + u_2X_2)] &= E[\cos(u_1X_1 + u_2X_2)] \\ &= E[\cos u_1 X_1 \cos u_2X_2] - E[\sin u_1X_1 \sin u_2X_2] \\ &= E[\cos u_1X_1 \cos u_2X_2]. \end{aligned}$$

Here the first equality follows from the fact that every linear combination of sign-invariant random variables is sign-invariant; the second equality is the trigonometric identity $\cos(x + y) = \cos x \cos y - \sin x \sin y$; and the third equality holds because $\sin x$ is an odd function of x. The proof for arbitrary k is similarly obtained by successive application of these arguments.

In connection with the argument used in this proof, we mention that it is not true that the sign-invariance of each *linear combination of a set* of random variables implies the sign-invariance of the *set*; for example, every linear combination of jointly normally distributed random variables with expectations 0 has a sign-invariant (normal) distribution, while every such set of jointly distributed variables is not sign-invariant.

A single sign-invariant random variable X has the property that the conditional distribution of X, given $|X|$, assigns probability 1/2 to the values X and $-X$. For a sign-invariant set, we have

LEMMA 1.2. *$X_1, \cdots, X_k$ are sign-invariant if and only if they are conditionally independent, given $|X_1|, \cdots, |X_k|$, with the conditional joint characteristic function $\prod_{j=1}^{k} \cos u_jX_j$.*

Proof. Since $\cos x = \cos|x|$, the expression $\prod_{j=1}^{k} \cos u_j X_j$ is measurable with respect to $|X_1|, \cdots, |X_k|$; therefore, the form of the conditional characteristic function is obtained from Lemma 1.1.

Lemma 1.2 shows the relation between independence and sign-invariance. Important features are shared by both concepts. On one hand, properties of independence are sometimes consequences of sign-invariance, and, on the other hand, properties of sign-invariance are sometimes deducible from independence and the application of Lemma 1.2. An example of the first possibility is

LEMMA 1.3. *If X and Y are sign-invariant, then the finiteness of $E|X+Y|^r$ implies that of $E|X|^r$ and $E|Y|^r$, for every $r > 0$.*

Proof. By sign-invariance, $|X - Y|$ has the same distribution as $|X + Y|$ so that $E|X - Y|^r$ is finite if $E|X + Y|^r$ is; hence,

$$E|\max(X,Y)|^r = 2^{-r} E\big|X + Y + |X - Y|\big|^r < \infty.$$

By sign-invariance, $E|\min(X,Y)|^r$ is also finite; therefore, the assertion of the lemma now follows from the inequalities $(x)^+ \leqq (\max(x,y))^+, (x)^- \leqq (\min(x,y))^-$.

The same assertion for independent X and Y can be deduced from Lemma 1.3 by symmetrizing the variables and using the fact that the rth moment of a random variable exists if and only if rth moment of the symmetrized random variable exists [9, p. 246]. This result is apparently known for the case of independence only when $r \geqq 1$ [9, p. 263].

An example of the deduction of properties of sign-invariant random variables from independence via Lemma 1.2 is

LEMMA 1.4. *Let $X_1, \cdots, X_n$ be sign-invariant; then, for every $x \geqq 0$,*

$$P\left\{ \max_{1 \leqq k \leqq n} (X_1 + \cdots + X_k) \geqq x \right\} \leqq 2P\{X_1 + \cdots + X_n \geqq x\}. \tag{1.2}$$

Proof. (1.2) holds when the probability statement is repaced by a "conditional probability given $|X_1|, \cdots, |X_n|$" statement; for in the latter case, Lemma 1.2 asserts that the X's are independent and with marginal symmetrical distributions, and then (1.2) is already known [5, p. 106]. The assertion for the unconditional probabilities follows by taking expectations on both sides of the conditional probability inequality.

It is not a surprise that (1.2) holds for sign-invariant random variables because at the heart of the proof of (1.2) for independent random variables is the "reflection principle," which is a property of sign-invariance.

The similarity between independence and sign-invariance continues in

THEOREM 1.1. *For a sequence $X_1, X_2, \cdots$, of sign-invariant random variables the following three conditions are equivalent:*

(i) $\sum_{n=1}^{\infty} X_n$ *converges in distribution,*
(ii) $\sum_{n=1}^{\infty} X_n^2 < \infty$ *with probability* 1,
(iii) $\sum_{n=1}^{\infty} X_n$ *converges with probability* 1.

Proof. In general, the convergence (iii) implies the convergence (i). We shall show that (i) implies (ii) and then that (ii) implies (iii).

Being bounded and nonincreasing, the sequence of partial products $\prod_{k=1}^{n} |\cos uX_k|$ converges to a limit $\prod_{n=1}^{\infty} |\cos uX_n|$, which may be 0. By the continuity theorem for characteristic functions, we know that (i) implies that the characteristic function of $X_1 + \cdots + X_n$, namely $E(\prod_{k=1}^{n} \cos uX_k)$, converges to a characteristic function $\phi(u)$ uniformly on every bounded u-interval. Applying the bounded convergence theorem to $\prod_{k=1}^{n} |\cos uX_k|$, we obtain

$$E\left(\prod_{n=1}^{\infty} |\cos uX_n|\right) \geqq \lim_{n\to\infty} E\left(\prod_{k=1}^{n} \cos uX_k\right) = \phi(u). \tag{1.3}$$

Let A denote the event $\{\prod_{n=1}^{\infty} |\cos uX_n| = 0$ for almost all u in the sense of Lebesgue measure$\}$, and let I_A be the indicator function of A; then, by (1.3),

$$\lim_{u\to 0} \left| P(A) - E\left(I_A \prod_{n=1}^{\infty} |\cos uX_n|\right)\right|$$
$$= \lim_{u\to 0} E\left(I_A\left[1 - \prod_{u=1}^{\infty} |\cos uX_n|\right]\right) \leqq \lim_{u\to 0} [1 - \phi(u)] = 0.$$

On the other hand, $\prod_{n=1}^{\infty} |\cos uX_n| = 0$ on a dense u-set on the set A, so that $\lim_{u\to 0} EI_A \prod_{n=1}^{\infty} |\cos uX_n| = 0$; hence, $P(A) = 0$.

Let $x_1, x_2, \cdots$ be a sequence of fixed real numbers. If $\prod_{n=1}^{\infty} |\cos ux_n|$ is not 0 on a u-set of positive Lebesgue measure, then the series $\sum \pm x_n$, where the signs are chosen independently and with equal probabilities, converges with probability 1 [5, p. 115]; hence, $x_n \to 0$, and so the convergence of the infinite product implies that $\sum_{n=1}^{\infty} x_n^2 < \infty$. From this we conclude that the fact $P(A) = 0$ implies $\sum_{n=1}^{\infty} X_n^2 < \infty$ a.s.; hence, (i) implies (ii).

To prove (iii) as a consequence of (ii), we consider the conditional probability of convergence of $\sum_{n=1}^{\infty} X_n$ given $|X_1|, |X_2|, \cdots$. By Lemma 1.2, the X's are conditionally independent and each X_n assumes the values $\pm |X_n|$ with probabilities $(1/2, 1/2)$. (iii) is now a direct consequence of the Rademacher-Steinhaus theorem, which states that if $C_1, C_2, \cdots$ is a sequence of real numbers whose signs are selected independently and with probabilities $(1/2, 1/2)$, then $\sum_{n=1}^{\infty} C_n^2 < \infty$ implies the probability 1 convergence of $\sum_{n=1}^{\infty} C_n$ [10], [11]. This result can also be deduced from the more general theorem [5, p. 108].

The sign-invariance property is also very close to a martingale property.

Lemma 1.5. $X_1, \cdots, X_k$ *are sign-invariant if and only if the conditional distribution of any subset* $X_{i_1}, \cdots, X_{i_\alpha}$ *given any other subset* $X_{j_1}, \cdots, X_{j_\beta}$, *is invariant under all sign changes in the former subset.*

Proof. This is established by expressing the joint distribution of the two subsets as the integral of the conditional distribution of the first subset given the second, and then using the elementary properties of conditional probabilities.

COROLLARY 1.1. *If $X_1, X_2, \cdots$ are sign-invariant, then the conditional distribution of X_n, given $X_1, \cdots, X_{n-1}$, is symmetric about the origin, $n > 1$. If, in addition, $E|X_n| < \infty$, $n \geqq 1$, then, in particular $E(X_n | X_1, \cdots, X_{n-1}) = 0$ with probability 1, $n > 1$; thus, the concsecutive partial sums, $X_1, X_1 + X_2, \cdots$ form a martingale.*

We could go on to point out many analogies between sign-invariance and martingale dependence, but we do not need these results for our pending study, and so we shall just mention one point and leave the subject. Let $X_1, X_2, \cdots$ be a sign-invariant sequence, and let N be a nonnegative, integer-valued random variable which is measurable with respect to the σ-field generated by the absolute values of the X's. Define a new sequence $\tilde{X}_1, \tilde{X}_2, \cdots$ as $\tilde{X}_k = X_k$ if $N \geqq k$ and $\tilde{X}_k = 0$ if $N < k$; then, it can be shown with the help of Lemma 1.5 that the sequence $\tilde{X}_1, \tilde{X}_2, \cdots$ is also sign-invariant. This transformation of the X-sequence corresponds to "optional stopping" [5, p. 300] on the sequence of consecutive partial sums of the X's.

2. **The stochastic process with sign-invariant increments.** Let $X(t), a \leqq t \leqq b$, be a separable stochastic process on a probability space $(\Omega, \mathscr{F}, P)$. We say that $X(t)$ has sign-invariant increments if for every finite set of mutually disjoint intervals (a_i, b_i), $i = 1, \cdots, k$, lying in $[a, b]$, the random variables $X(b_1) - X(a_1), \cdots, X(b_k) - X(a_k)$ are sign-invariant. An example of such a process is one with symmetric, independent increments. If a process $X(t)$ with sign-invariant increments satisfies $E|X(b) - X(a)| < \infty$, then, by Lemma 1.3, the expectations of all the increments exist, and, by Corollary 1.1, $X(t)$ is a martingale. If $E|X(b) - X(a)|^2 < \infty$, then, by Lemma 1.3, all increments have finite second moments, and, by sign-invariance, are orthogonal; hence, $X(t)$ has orthogonal increments.

In general, we do not assume the existence of any moments of $X(t)$, so that our process does not fit into any previously known category. We shall study the nature of the sample functions of the process: just as sign-invariant sequences resemble independent and martingale-dependent sequences, so $X(t)$ resembles processes with independent increments and continuous parameter martingales. The theory of processes with sign-invariant increments is related to that of these other processes, but is independent of it. The martingale sequence convergence theorems and the "upcrossing inequality" [5, p. 316] are basic to our work.

In what follows, we use the expression "for almost all sample functions" in the usual sense, e.g., as in [5, p. 11]. A property holding with probability 1 will be said to hold "almost surely," denoted by a.s.

LEMMA 2.1. *Almost all sample functions $X(t)$ are bounded on $[a,b]$.*

The proof is identical with the proof for centered processes with independent increments [5, p. 411]; in fact, all that is needed is the conversion of the inequality (1.2) into the inequality $P\{\sup_{a\leqq s\leqq t}[X(s)-X(a)]\geqq x\}\leqq 2P\{X(t)-X(a)\geqq x\}$ by means of the separability argument.

Let $t_0, t_1, \cdots, t_n, \cdots$ be a sequence of distinct points in $[a,b]$ and dense in it, and assume $\{t_n\}$ to be a separability sequence for $X(t)$. For each $n\geqq 1$, let $t_{n,0},\cdots,t_{n,n}$ be the first $n+1$ elements of $\{t_n\}$ arranged so that $t_{n,0}<t_{n,1}<\cdots<t_{n,n}$ and with $t_{n,0}=a, t_{n,n}=b$; put

$$X_{n,k} = X(t_{n,k})-X(t_{n,k-1}), \qquad k=1,\cdots,n;$$

$$\mathscr{B}_n = \sigma\text{-field generated by } |X_{n,1}|,\cdots,|X_{n,n}|,$$

$$\mathscr{B}^{(n)} = \sigma\text{-field generated by the field } \mathscr{B}_n\cup\mathscr{B}_{n+1}\cup\cdots,\ n=1,2,\cdots;$$

$$\mathscr{B} = \bigcap_{n=1}^{\infty}\mathscr{B}^{(n)}.$$

These σ-fields are contained in the basic σ-field $\mathscr{F}$.

LEMMA 2.2. *For every u and every $n\geqq 1$, we have*

$$\begin{aligned} &E[\cos u(X(b)-X(a))\,|\,\mathscr{B}_n] \\ (2.1)\qquad &= E[\cos u(X(b)-X(a))\,|\,\mathscr{B}^{(n)}] \\ &= \textstyle\prod_{k=1}^n \cos uX_{n,k},\ a.s. \end{aligned}$$

Proof. By Lemma 2.1, the increments $X_{n,k}$ are conditionally independent, given $\mathscr{B}_n$, with the conditional joint characteristic function given by the third member of (2.1); thus the first and third members are equal.

We shall show that

$$\begin{aligned} &E[\cos u(X(b)-X(a))\,|\,\mathscr{B}_n,\cdots,\mathscr{B}_{n+m}] \\ (2.2)\qquad &= \prod_{k=1}^{n}\cos uX_{n,k},\ \text{a.s.,} \qquad m\geqq 1. \end{aligned}$$

The second equality in (2.1) then follows by letting $m\to\infty$ in (2.2) and applying the martingale convergence theorem for conditional expectations with respect to a monotone sequence of σ-fields [5, p. 331].

Writing $X(b)-X(a)$ as $X_{n,1}+\cdots+X_{n,n}$, and employing the same trigonometric identity used in the proof of Lemma 1.1, we express the left-hand side of (2.2) as

$$\begin{aligned} (2.3)\qquad &E[\cos u(X_{n,1}+\cdots+X_{n,n-1})\cos uX_{n,n}\,|\,\mathscr{B}_n,\cdots,\mathscr{B}_{n+m}] \\ &-E[\sin u(X_{n,1}+\cdots+X_{n,n-1})\sin uX_{n,n}\,|\,\mathscr{B}_n,\cdots,\mathscr{B}_{n+m}]. \end{aligned}$$

Since $\cos uX_{n,n}$ is $\mathscr{B}_n$-measurable, the first term in (2.3) is

$$\cos uX_{n,n}E[\cos u(X_{n,1} + \cdots + X_{n,n-1}) \mid \mathscr{B}_n, \cdots, \mathscr{B}_{n+m}]. \tag{2.4}$$

In the second term of (2.3), we write the $X_{n,k}$'s as sums of $X_{n+m,k}$'s, and note that the conditional joint distribution of the two sums $X_{n,1} + \cdots + X_{n,n-1}$ and $X_{n,n}$, given the absolute values of all of their $X_{n+m,k}$-summands and partial summands, is invariant under sign changes. (For example, the joint conditional distribution of $X_1 + X_2 + X_3$ and $X_4 + X_5 + X_6$, given

$$|X_1|, \cdots, |X_6|, |X_1 + X_2|, \ |X_1 + X_3|, \ |X_2 + X_3|, \ |X_4 + X_5|, \ |X_4 + X_6|,$$
$$|X_5 + X_6|, \ |X_1 + X_2 + X_3|, \ |X_4 + X_5 + X_6|,$$

is invariant under sign changes. The proof is based on the fact that the joint distribution of the conditioning variables and the two sums, $X_1 + X_2 + X_3$ and $X_4 + X_5 + X_6$, is invariant under sign changes of the latter two sums.) From this we conclude that the second term in (2.3) vanishes; therefore, the expressions in (2.3) and (2.4) are equal a.s. We now apply the same reasoning, used above for (2.2), to the conditional expectation in (2.4), and obtain (2.1) by iteration.

LEMMA 2.3. *For each* $u, \lim_{n\to\infty} \prod_{k=1}^{n} \cos uX_{n,k}$ *exists a.s. and is equal to* $E[\cos u(X(b) - X(a)) \mid \mathscr{B}]$.

Proof. Since $\mathscr{B}^{(n)} \supset \mathscr{B}^{(n+1)}$, $n = 1, 2, \cdots$, this statement is a consequence of (2.1) and the martingale convergence theorem for conditional expectations with respect to a monotone sequence of σ-fields [5, p. 331].

COROLLARY 2.1. *The statement of Lemma* 2.3 *remains valid if* a, b *are replaced by any two elements* $t', t'', t' < t''$ *of the sequence* $\{t_n\}$ *and the cosine product is taken only over* $X_{n,k}$ *increments over intervals in* $[t', t'']$.

The proof is executed as that of Lemmas 2.2 and 2.3: we use the additional fact, implicit in Lemma 1.2, that the conditional distribution of a *subset* of the random variables $X_{n,k}, k = 1, \cdots, n$, given the absolute values of *all* of them, depends *only* on the absolute values of the *subset.*

LEMMA 2.4. $\limsup_{n\to\infty} \sum_{k=1}^{\infty} X_{n,k}^2 < \infty$ *a.s.*

Proof. By Lemma 2.1, one can determine a constant $M > 0$ sufficiently large so that the event

$$A = \left\{\limsup_{n\to\infty} \max_{1 \leq k \leq n} |X_{n,k}| \leq M\right\}$$

has probability arbitrarily close to 1. By Lemma 2.3, we have

$$\lim_{u\to 0} E\left(\lim_{n\to\infty} \prod_{k=1}^{n} \cos uX_{n,k}\right) = 1$$

so that the quantity under the expectation converges to 1 in probability; hence, if u is sufficiently small $\lim \prod_{k=1}^{n} \cos uX_{n,k}$ is not 0 with probability close to 1. This is equivalent to the finiteness of $\limsup \sum_{k=1}^{n} X_{n,k}^2$ on the set A defined above; therefore, this quantity is finite with probability arbitrarily close to 1; hence, it is finite a.s.

LEMMA 2.5. *For each t in (a,b), the limits of $X(s)$ for $s\uparrow t$ and $s\downarrow t$ exist a.s.; and the limits of $X(s)$ for $s\downarrow a$ and $s\uparrow b$ exist a.s.*

Proof. Let $s_0, s_1, \cdots$ be an increasing sequence in (a,b) such that $s_n\uparrow t$. By Lemma 2.4 the series $\sum_{n=0}^{\infty}[X(s_{n+1}) - X(s_n)]^2$ is finite a.s.; hence, by Theorem 1.1, $\sum_{n=0}^{\infty}[X(s_{n+1}) - X(s_n)]$ converges a.s., which means that $\lim_{n\to\infty}X(s_n)$ exists a.s. The rest of the proof of the existence of the left-hand limit, as well as the right-hand and end point limits, follows exactly that for centered processes with independent increments [5, p. 409].

LEMMA 2.6. *There are at most a countable number of fixed points of discontinuity of the process $X(t)$.*

The proof follows from Lemma 2.5 and a general theorem on separable processes which have stochastic left-hand and right-hand limits at each point [5, p. 356].

Now it will be shown that almost all sample functions $X(t)$ behave like continuous parameter martingale sample functions. The role of $\{t_n\}$ as a separability sequence is used for the first time.

THEOREM 2.1. *Almost all sample functions $X(t)$ have finite left-hand and right-hand limits at each point of (a,b), and corresponding one-sided limits at the endpoints a,b. The discontinuities are jumps, except perhaps at the fixed points of discontinuity.*

Proof. Let $C_1, \cdots, C_n$ be fixed positive constants, and let signs $\pm$ be attached to them, where $\pm$ are each selected independently and with probability 1/2. The set of partial sums $\pm C_1 \pm C_2 \pm \cdots \pm C_k$, $k = 1, \cdots, n$ is a martingale; by the "upcrossing inequality" [5, p. 316], the expected number of upcrossings of a fixed interval $[r_1, r_2]$ is no greater than

$$(2.5)\qquad (r_2 - r_1)^{-1}\left(E\left|\sum_{k=1}^{n} \pm C_k\right| + r_1\right) \leqq (r_2 - r_1)^{-1}\left(\left(\sum_{k=1}^{n} C_k^2\right)^{1/2} + r_1\right);$$

here we have used the independence of the signs and Jensen's inequality to get the inequality above. By Markov's inequality, the probability that there are at least k upcrossings is bounded above by the right-hand side of (2.5) divided by k.

Now let $M_{n,k}$ denote the event that the number of upcrossings of $[r_1, r_2]$ by the finite sequence $X(t_{n,0}), \cdots, X(t_{n,n})$ is at least k. If we follow the proof for separable semi-martingales [5, p. 362], we see that all that we have to establish is that $P(\bigcap_k \bigcup_n M_{n,k}) = 0$; for the rest of the proof of that theorem also proves ours. It will suffice for us to prove that $P(\bigcap_k \bigcup_n M_{n,k} \mid \mathscr{B}) = 0$, a.s.

The monotonicity of $M_{n,k}$ (monotone nonincreasing in k, nondecreasing in n) and the martingale convergence theorem for conditional probabilities enables us to compute $P(\bigcap_k \bigcup_n M_{n,k} \mid \mathscr{B})$ as

$$\lim_{k\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} P(M_{n,k} \mid \mathscr{B}^{(m)}).$$

This is dominated by

$$\limsup_{k\to\infty} \quad P(M_{m,k} \mid \mathscr{B}^{(m)}).$$

The increments $X_{m,k}$, $k = 1, \cdots, m$ are conditionally independent given $\mathscr{B}^{(m)}$ with randomly selected signs; this can be seen from Lemma 2.2 and its proof, where we get the same conditional distribution when conditioning by $\mathscr{B}^{(m)}$ as when conditioning by $\mathscr{B}_m$. By the upcrossing inequality for partial sums of constants with randomly selected signs, we get from (2.5)

$$P(M_{m,k} \mid \mathscr{B}^{(m)}) \leqq k^{-1}(r_2 - r_1)^{-1} \left[\left(\sum_{=1}^{m} X_{m,i}^2 \right)^{1/2} + r_1 \right].$$

Taking the limit over m and then over k, and applying Lemma 2.4, we conclude that $P(\bigcap_k \bigcup_n M_{n,k} \mid \mathscr{B}) = 0$ a.s.

From now on we shall assume that the process $X(t)$ has no *fixed* points of discontinuity, but only *moving* points of discontinuity. By Theorem 2.1, the latter are points where the sample function has a jump.

LEMMA 2.7. *The increments of $X(t)$ over nonoverlapping intervals are conditionally independent given $\mathscr{B}$.*

Proof. By Lemma 2.3 and Corollary 2.1, the conditional characteristic function of increments over nonoverlapping intervals whose endpoints are elements of the sequence $\{t_n\}$ is equal to the product of the conditional characteristic functions; therefore, the assertion of the lemma holds for such intervals. Since the process has no fixed points of discontinuity, the increments over arbitrary nonoverlapping intervals are a.s. limits of increments over the first kind of intervals; therefore, the conditional characteristic functions of the increments over the general intervals are a.s. limits of those for the special intervals. The proof is complete.

Let us denote by $\mathscr{D}$ the class of real valued functions defined on $[a,b]$ and having the properties of our sample functions, i.e., f is in $\mathscr{D}$ if and only if the right-hand and left-hand limits exist at each point in $[a,b]$ (with a suitable interpretation at the endpoints) and whose discontinuities, if any, are jumps. It can be established, by means of the Heine-Borel theorem, that, for any $\varepsilon > 0$, there are at most a *finite* number of jumps whose magnitude exceeds ε; therefore, the jumps of the function, $J_1, J_2, \cdots$ can be put in a sequence ordered in such a way that $|J_1| \geqq |J_2| \geqq \cdots$. Suppose that f is continuous at each point in the sequence $\{t_n\}$, given above; for each n, put $f_{n,k} = f(t_{n,k}) - f(t_{n,k-1})$, $k = 1, \cdots, n$. Suppose that f has ν jumps in $[a,b]$ of magnitude exceeding ε, where ν is some nonnegative integer. For every n, there are at most ν intervals among the n intervals $(t_{n,k-1}, t_{n,k})$ which contain a jump exceeding ε in magnitude. The number of such intervals converges to ν as $n \to \infty$, and *the increments $f_{n,k}$ corresponding to these intervals converge to the respective jumps $J_1, \cdots, J_\nu$; furthermore, the* $\limsup (n \to \infty)$ *of the absolute maximum increment $|f_{n,k}|$ over intervals $(t_{n,k-1}, t_{n,k})$ not containing jumps exceeding ε in magnitude is less than or equal to ε.* This statement, which can be established with the aid of the Heine-Borel theorem, will be referred to as the "truncation principle." It has been tacitly used by Cogburn and Tucker in [4, p. 283].

Returning to the stochastic process $X(t)$, since there are no fixed points of discontinuity, we know that almost all the sample functions are continuous at each point in the sequence $\{t_n\}$. The truncation principle now applies to almost all sample functions.

We shall denote by $J_1, J_2, \cdots$ the (ordered) sequence of jumps of $X(t)$. The following canonical representation of $X(t)$ is implied by the next theorem; it is as follows. An arbitrary random sequence of points $\{\tau_n\}$ is chosen in $[a,b]$. An arbitrary random sequence of positive numbers $\{\lambda_n\}$ satisfying $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ a.s. is selected; so is an arbitrary random continuous nondecreasing nonnegative function $V(t)$, $a \leqq t \leqq b$. The joint distribution of these three processes is arbitrary, and the "stochastic information" about them is summarized in a σ-field $\mathscr{B}$. Now let $\theta_1, \theta_2, \cdots$ be a sequence of independent random variables, independent of $\mathscr{B}$, and each assuming the values ± 1 with probabilities 1/2, 1/2. Let $U(t), t \geqq 0$, be a standard Brownian motion process (separable), which is independent of $\mathscr{B}$ and of the process $\{\theta_n\}$. Let $X'(t)$ be a (random) function on $[a,b]$ defined as follows: it has a jump of magnitude λ_n and of sign θ_n at the point $\tau_n, n = 1, 2, \cdots$, and varies only by jumps. Let $X''(t)$ be defined as $U(V(t))$, $a \leqq t \leqq b$; $X''(t)$ is a Brownian motion process with a continuous random time parameter. The process $X(t) = X'(t) + X''(t)$ has sign-invariant increments; conversely, every process with sign-invariant increments has such a representation. This is analogous to Lévy's representation of the process with independent increments as the convolution of Poisson processes and Brownian motion [6], [5, p. 422]. One immediate consequence of our representation is that a sign-invariant process with almost all continuous sample functions is a Brownian motion process with a continuous random time parameter.

For $\varepsilon > 0$, we define the truncated increments $X_{n,k}^{(\varepsilon)}$ of the process with sign-invariant increments as $X_{n,k}^{(\varepsilon)} = X_{n,k}$ if $|X_{n,k}| \leqq \varepsilon$, and 0 if $|X_{n,k}| > \varepsilon$.

THEOREM 2.2. *The truncated increments* $X_{n,k}^{(\varepsilon)}$ *satisfy*

(2.6)
$$\begin{aligned} V_{[a,b]} &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \sum_{k=1}^{n} \left| X_{n,k}^{(\varepsilon)} \right|^2 \\ &= \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \sum_{k=1}^{n} \left| X_{n,k}^{(\varepsilon)} \right|^2 < \infty, \ a.s. \end{aligned}$$

For each u, *we have*

(2.7)
$$\begin{aligned} &E[\cos u(X(b)-X(a)) \,|\, \mathscr{B}] \\ &\qquad = \prod_{k=1}^{\infty} \cos uJ_k \cdot \exp(-\tfrac{1}{2}u^2 V_{[a,b]}), \ a.s. \end{aligned}$$

Proof. According to Lemma 2.3, (2.7) is equivalent to

$$\lim_{n\to\infty} \prod_{k=1}^{n} \cos uX_{n,k} = \prod_{k=1}^{\infty} \cos uJ_k \cdot \exp(-\tfrac{1}{2}u^2 V_{[a,b]}), \text{ a.s.} \tag{2.8}$$

The expression $\prod_k \{\cos uJ_k : |J_k| > \varepsilon\}$ is to stand for the product over factors for which $|J_k| > \varepsilon$. Equation (2.8) is trivial for $u = 0$. We choose $u \neq 0$, and then $\varepsilon > 0$ so small that $|\varepsilon u|$ is also very small. The product $\prod_{k=1}^{n} \cos uX_{n,k}$ may be written as

$$\prod_{k=1}^{n} \cos uX_{n,k} = \prod_{k=1}^{n} \{\cos uX_{n,k} : |X_{n,k}| > \varepsilon\} \cdot \prod_{k=1}^{n} \cos uX_{n,k}^{(\varepsilon)}. \tag{2.9}$$

By the truncation principle for functions in the class $\mathscr{D}$, the first product on the right-hand side of (2.9) converges a.s. to $\prod_k \{\cos uJ_k : |J_k| > \varepsilon\}$. For small ε, the factors $\cos uX_{n,k}^{(\varepsilon)}$ are positive and close to 1 so that their logarithms are defined; furthermore,

(2.10)
$$\begin{aligned} &\lim_{\varepsilon\to 0} \left\{ \sum_{k=1}^{n} \log\cos uX_{n,k}^{(\varepsilon)} \Big/ -\frac{1}{2}u^2 \sum_{k=1}^{n} \left| X_{n,k}^{(\varepsilon)} \right|^2 \right\} \\ &\qquad = 1 \ \textit{uniformly in } n, \text{ a.s.} \end{aligned}$$

In the proof of Lemma 2.4, we showed the existence of a number $u \neq 0$ in an arbitrarily small neighborhood of 0 such that the left-hand side of (2.8) is not 0 with probability arbitrarily near 1. By that same argument we can show that $\cos uJ_k > 0$ for all k with probability close to 1 for u sufficiently near 0. (We throw out the event of small probability that $\sup |J_k|$ exceeds some large fixed number.) Assume now that u satisfies both of these requirements, and let $n \to \infty$ and then $\varepsilon \to 0$ in (2.9). With probability near to 1 the left-hand side approaches a nonzero limit. With probability close to 1 the sequence of partial products of factors $\{\cos uJ_k : |J_k| > \varepsilon\}$ has only positive factors; hence, the sequence is decreasing and approaches a limit as $\varepsilon \to 0$; and, by (2.10) and Lemma 2.4, the second product in (2.9) is bounded away from 0. It follows that with probability close to 1 the first product on the right-hand side of (2.9) converges to a *positive* limit; hence, by (2.10), the second product does also, and so (2.6) holds with probability close to 1. Since the event in (2.6) is independent of the particular value of u, and since the probability of the event is arbitrarily close to 1, its probability *is* 1, and (2.6) holds. Having obtained (2.6), we verify (2.8) for any value of u by reapplying the same argument to (2.9); this time it does not matter whether the left side of (2.8) is 0 or not.

This proof and Corollary 2.1 imply that the conditional characteristic function of an increment $X(t'') - X(t')$, for t' and t'' in the sequence $\{t_n\}$ is of the same form as (2.7), but with the jumps J_k restricted to those in (t', t'') and with $V_{[t',t'']}$ defined via truncated increments only over (t', t''). We would like to extend the validity of the form (2.7) to every increment $X(t) - X(s)$, $a \leqq s < t \leqq b$. The process has no fixed points of discontinuity so that it may be assumed that, for fixed s, t, almost no sample function has jumps at s, t or $t_1, t_2, \cdots$. The conditional characteristic function of $X(t) - X(s)$ is the a.s. limit of the conditional characteristic function of $X(t'') - X(t')$, $t'' \to t$, $t' \to s$, $t', t'' \in \{t_n\}$; therefore, by (2.7), it is equal to the product of $\cos uJ_k$ over all jumps J_k in (s, t) multiplied by the limit of $\exp(-\frac{1}{2}u^2 V_{[t',t'']})$ for $t'' \to t$, $t' \to s$. What we have to show is that this limit is independent of the particular choices of t' and t''.

Let us define a (random) point function $V(t)$ for each t in the sequence $\{t_n\}$ as $V(t) = V_{[a,t]}$; by definition, $V(t)$ is monotonically nondecreasing on its domain. Now define $V(t)$ for every $t, a \leqq t \leqq b$, by defining $V(t)$ at every value outside the sequence $\{t_n\}$ as the limit from the right taken along t_n-values; this uniquely determines $V(t)$, $a \leqq t \leqq b$. we shall show, in the next two lemmas, that almost all sample functions $V(t)$ are continuous on $[a, b]$.

LEMMA 2.8. *For each $\tau, a \leqq \tau \leqq b$, $V(t)$ is continuous a.s. at τ.*

Proof. Let τ be a point in (a, b); the proof for the endpoints is similar. Choose two subsequences of $\{t_n\}, \{t_n'\}$ and $\{t_n''\}$, such that $t_n' \uparrow \tau$, $t_n'' \downarrow \tau$. The aim here is to show that $V(t_n'') - V(t_n')$, which is $V_{[t_n', t_n'']}$, converges a.s. to 0. Since almost no sample function has a discontinuity at τ, $X(t_n'') - X(t_n') \to 0$ a.s. and so $E[\cos u(X(t_n'') - X(t_n')) \mid \mathscr{B}] \to 1$ a.s. for each u. But now, looking at the form (2.7) of the conditional characteristic function, we see that $V_{[t_n', t_n'']} \to 0$ a.s.

LEMMA 2.9. *Almost every sample function $V(t)$ is continuous on $[a, b]$.*

Proof. By Lemma 2.8, almost every sample function $V(t)$ is continuous at the points of the sequence $\{t_n\}$. For $\varepsilon > 0$, define the sequence of monotone approximating functions $V_{n,\varepsilon}(t)$ as

$$\begin{aligned} V_{n,\varepsilon}(t) &= 0 \quad \text{for } t = a \\ &= \sum_{j=1}^{k} |X_{n,j}^{(\varepsilon)}|^2 \quad \text{for } t = t_{n,k}, \qquad k = 1,\cdots,n, \\ &= V_{n,\varepsilon}(t_{n,k}) \quad \text{for } t_{n,k-1} < t < t_{n,k}, \qquad k = 1,\cdots,n, \end{aligned}$$

for $n = 1,2,\cdots$. The jumps of $V_{n,\varepsilon}(t)$ are, in magnitude, bounded by ε. By Theorem 2.2, $V_{n,\varepsilon}(t)$ converges to $V(t)$ at all $\{t_n\}$-points a.s. as $n \to \infty$ and then as $\varepsilon \to 0$. $V(t)$ is a.s. a bounded and monotonic function, and its only discontinuities are jumps; therefore, since it is approximable on a dense continuity set by $V_{n,\varepsilon}(t)$, it is continuous.

This completes the program in the paragraph following the proof of Theorem 2.2. The (random) interval function $V_{[s,t]}$, first defined for endpoints in the set $\{t_n\}$, is uniquely extended to arbitrary endpoints in $[a,b]$. This uniquely defines the conditional characteristic function of any increment $X(t) - X(s)$, given $\mathscr{B}$. The joint characteristic function of a finite set of increments over disjoint intervals is, by the conditional independence (Lemma 2.7), equal to the expected value of the product of the conditional characteristic functions. The stochastic process, whose conditional characteristic functions are given by the Gaussian form $\exp(-\frac{1}{2}u^2 V_{[s,t]})$, is representable in the canonical form $U(V(t))$, where $U(t)$, $t \geqq 0$, is a separable, independent Brownian motion process.

We conclude with a result first obtained by Cogburn and Tucker [4] for a process with independent increments. As a matter of fact, they proved it first for a process with *symmetric* independent increments, and then, by symmetrization, for the general process with independent increments.

THEOREM 2.3.

$$\lim_{n\to\infty} \sum_{k=1}^{n} |X_{n,k}|^2 = \sum J_k^2 + V_{[a,b]} < \infty, \ a.s.$$

The same is true for increments $X_{n,k}$ whose squares are summed over (t',t''), for t',t'' in $\{t_n\}$, with the jumps J_k taken over (t',t'').

Proof. By the truncation principle, the sum of $|X_{n,k}|^2$ taken over $|X_{n,k}| > \varepsilon$, converges a.s. to the sum of J_k^2, taken over $|J_k| > \varepsilon$; this sum converges to $\sum J_k^2$ as $\varepsilon \to 0$. On the other hand, by (2.6), the sum of $|X_{n,k}|^2$, taken over $|X_{n,k}| \leqq \varepsilon$, is close to $V_{[a,b]}$ for large n and small ε. The proof of the second assertion of the theorem follows as the remark made after the proof of Theorem 2.2. The finiteness of the limit follows from Lemma 2.4.

3. **The limit of the sum of the γth powers of the increments.** We continue to consider the separable stochastic process $X(t)$, $a \leqq t \leqq b$, with sign-invariant increments and no fixed points of discontinuity.

THEOREM 3.1. *For any* γ, $0<\gamma<2$,

$$\lim_{n\to\infty}\sum_{k=1}^{n}|X_{n,k}|^{\gamma} \tag{3.1}$$

exists a.s. (and may assume an infinite value).

Proof. For any $\varepsilon>0$, the sum in (3.1) may be written as a sum extended over terms for which $|X_{n,k}|>\varepsilon$ plus the sum of the truncated increments, $|X^{(\varepsilon)}_{n,1}|^{\gamma}+\cdots+|X^{(\varepsilon)}_{n,n}|^{\gamma}$. The truncation principle implies that the first sum converges a.s. to $\sum_k\{|J_k|^{\gamma}:|J_k|>\varepsilon\}$, where this notation stands for the sum of $|J_k|^{\gamma}$ over all quantities such that $|J_k|>\varepsilon$. We conclude that (3.1) exists a.s. if and only if

$$\lim_{n\to\infty}\sum_{k=1}^{n}|X^{(\varepsilon)}_{n,k}|^{\gamma} \text{ exists a.s. for all } \varepsilon>0. \tag{3.2}$$

The same reasoning leads us to the conclusion that

$$\lim_{n\to\infty}\sum_{k=1}^{n}|\sin X_{n,k}|^{\gamma} \text{ exists a.s.} \tag{3.3}$$

if and only if

$$\lim_{n\to\infty}\sum_{k=1}^{n}|\sin X^{(\varepsilon)}_{n,k}|^{\gamma} \text{ exists a.s. for all } \varepsilon>0; \tag{3.4}$$

furthermore, (3.2) *and* (3.4) *are equivalent.* The theorem will be proved by showing the validity of (3.3). This will be done by showing that the sums in (3.3) form a reversed lower semi-martingale sequence relative to the σ-fields $\mathscr{B}^{(n)}$, $n=1,2,\cdots$.

The difference between the successive sums $|\sin X_{n,1}|^{\gamma}+\cdots+|\sin X_{n,n}|^{\gamma}$ and $|\sin X_{n+1,1}|^{\gamma}+\cdots+|\sin X_{n+1,n+1}|^{\gamma}$ is of the form $|\sin(X+Y)|^{\gamma}-|\sin X|^{\gamma}-|\sin Y|^{\gamma}$, where X and Y are two successive $X_{n+1,k}$ increments. We note that $|\sin X|^{\gamma}+|\sin Y|^{\gamma}=|\sin|X||^{\gamma}+|\sin|Y||^{\gamma}$ and so is $\mathscr{B}^{(n+1)}$-measurable; the aim is to establish the semi-martingale inequality

$$E(|\sin(X+Y)|^{\gamma}|\mathscr{B}^{(n+1)})\leqq|\sin X|^{\gamma}+|\sin Y|^{\gamma}.$$

Jensen's inequality for conditional expectations [5, p. 33] and the elementary trigonometric identity $\sin(x+y)=\sin x\cos y+\sin y\cos x$ yield the inequality

$$\begin{aligned} E(|\sin(X+Y)|^{\gamma}|\mathscr{B}^{(n+1)}) &\leqq E^{\gamma/2}(\sin^2(X+Y)|\mathscr{B}^{(n+1)})\\ &= [\sin^2 X\cos^2 Y+\cos^2 X\sin^2 Y\\ &\quad + 2E(\sin X\sin Y\cos X\cos Y|\mathscr{B}^{(n+1)})]^{\gamma/2}. \end{aligned} \tag{3.5}$$

The conditional expectation in the last member of (3.5) vanishes because X and Y are conditionally sign-invariant given $\mathscr{B}^{(n+1)}$ (this was shown in the latter part of the proof of Lemma 2.2). By the elementary inequality

$|a+b|^\alpha \leqq |a|^\alpha + |b|^\alpha$, $0 \leqq \alpha \leqq 1$, the last member of (3.5) is bounded above by $|\sin X \cos Y|^\gamma + |\cos X \sin Y|^\gamma \leqq |\sin X|^\gamma + |\sin Y|^\gamma$. This confirms the semi-martingale inequality. The appropriate convergence theorem [5, p. 329] now implies (3.3).

We now define the *index* β of the process $X(t)$ as

$$\beta = \inf\{\alpha : \alpha > 0,\ \sum |J_k|^\alpha < \infty \text{ a.s.}\};$$

by Theorem 2.3, $\beta \leqq 2$.

THEOREM 3.2. *The limit* (3.1) *is* $+\infty$ *a.s. on the set where* $V_{[a,b]}$ (*in* (2.7)) *is positive. If* $V_{[a,b]}$ *vanishes a.s. and* $X(t)$ *is of index* β, *then, for* $\gamma, \beta < \gamma < 2$, *the limit* (3.1) *is equal to* $\sum |J_k|^\gamma$ *a.s.*

Proof. Since $\gamma < 2$, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} |X_{n,k}|^\gamma \geqq \lim_{n\to\infty} \sum_{k=1}^{n} |X_{n,k}^{(\varepsilon)}|^\gamma$$
$$\geqq \varepsilon^{\gamma-2} \limsup_{n\to\infty} \sum_{k=1}^{n} |X_{n,k}^{(\varepsilon)}|^2 .$$

Let $\varepsilon \to 0$ and invoke Theorem 2.2 to complete the proof of the first statement of the theorem.

By separating $|X_{n,1}|^\gamma + \cdots + |X_{n,n}|^\gamma$ into a sum over terms for which $|X_{n,k}| > \varepsilon$ and a sum over terms for which $|X_{n,k}| \leqq \varepsilon$, and by using the truncation principle argument leading to (3.2), one can show that the conclusion to he second statement of our theorem is equivalent to

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \sum_{k=1}^{n} |X_{n,k}^{(\varepsilon)}|^\gamma = 0, \text{ a.s.}$$

The *existence* of this limit is a consequence of the equivalence of (3.1) and (3.2); we have only to *evaluate* the limit. For this purpose it is sufficient to show that the conditional expectation of the limit, given $\mathscr{B}$, is 0; furthermore, by Fatou's lemma, it is enough to show that

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} E\left(\sum_{k=1}^{n} |X_{n,k}^{(\varepsilon)}|^\gamma \,|\, \mathscr{B}\right) = 0, \text{ a.s.} \tag{3.6}$$

By Jensen's inequality, we have

$$E(|X_{n,k}^{(\varepsilon)}|^\gamma \,|\, \mathscr{B}) \leqq E^{\gamma/2}(|X_{n,k}^{(\varepsilon)}|^2 \,|\, \mathscr{B});$$

from this, and from the relation $1 - \cos u \sim u^2/2$, $u \to 0$, we can see that (3.6) is implied by

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \sum_{k=1}^{n} E^{\gamma/2}(1 - \cos X_{n,k}^{(\varepsilon)} \,|\, \mathscr{B}) = 0, \text{ a.s.} \tag{3.7}$$

Let $\xi_k^{(n)}(\varepsilon)$ be the indicator function of the event that the interval $(t_{n,k-1}, t_{n,k})$ has in it no jump of $X(t)$ greater in magnitude than ε; $\xi_k^{(n)}(\varepsilon)$ is $\mathscr{B}$-measurable because $\mathscr{B}$ contains all "information" about the location and magnitude of the jumps. The truncation principle implies that (3.7) follows from

$$\lim_{\varepsilon \to 0} \liminf_{n \to \infty} \sum_{k=1}^{n} \xi_k^{(n)}(\varepsilon) E^{\gamma/2}(1 - \cos X_{n,k} \mid \mathscr{B}) = 0, \text{ a.s.} \tag{3.8}$$

In fact, there are at most a fixed finite number of subintervals $(t_{n,k-1}, t_{n,k})$ such that $\xi_k^{(n)}(\varepsilon) = 0$, and on these, $\cos X_{n,k}^{(\varepsilon)} \to 1$ a.s.; on the other hand, we always have $\cos X_{n,k} \leqq \cos X_{n,k}^{(\varepsilon)}$.

From the representation (2.7) and the present assumption that $V_{[a,b]} = 0$, we have

$$\begin{aligned} \xi_k^{(n)}(\varepsilon) E(1 - \cos X_{n,k} \mid \mathscr{B}) &= \xi_k^{(n)}(\varepsilon)\left(1 - \prod_{(n,k)} \cos J\right) \\ &= \xi_k^{(n)}(\varepsilon)\left(1 - \prod_{(n,k)} \{\cos J : |J| \leqq \varepsilon\}\right), \end{aligned} \tag{3.9}$$

where $\prod_{(n,k)}$ is the product over all jumps in $(t_{n,k-1}, t_{n,k})$. Let $\sum_{(n,k)}$ stand for summation over all jumps in $(t_{n,k-1}, t_{n,k})$, and let α be a real number such that $2\beta/\gamma < \alpha < 2$. The four elementary inequalities

$$\begin{aligned} 1 - \cos u &\leqq 2|u|^{\alpha} && \textit{for all } u, \\ 1 - u &\geqq e^{-2u} && \textit{for all small } u, \\ 1 - e^{-u} &\leqq u && \textit{for } u \geqq 0, \\ (\Sigma |J|)^{c} &\leqq \Sigma |J|^{c} && \textit{for } 0 < c < 1 \end{aligned}$$

yield the successive inequalities

$$\begin{aligned} &\sum_{k=1}^{n} \left(1 - \prod_{(n,k)} \{\cos J : |J| \leqq \varepsilon\}\right)^{\gamma/2} \\ &\leqq \sum_{k=1}^{n} \left(1 - \prod_{(n,k)} [1 - 2\{|J|^{\alpha} : |J| \leqq \varepsilon\}]\right)^{\gamma/2} \\ &\leqq \sum_{k=1}^{n} \left(1 - \exp\left[-4 \sum_{(n,k)} \{|J|^{\alpha} : |J| \leqq \varepsilon\}\right]\right)^{\gamma/2} \\ &\leqq \sum_{k=1}^{n} 2^{\gamma}\left(\sum_{(n,k)} \{|J|^{\alpha} : |J| \leqq \varepsilon\}\right)^{\gamma/2} \\ &\leqq 2^{\gamma} \sum_{k=1}^{n} \sum_{(n,k)} \{|J|^{\alpha\gamma/2} : |J| \leqq \varepsilon\}. \end{aligned}$$

The last double sum is over all jumps in (a,b) of magnitude $\leqq \varepsilon$, and since $\alpha\gamma/2 > \beta$, it tends to 0 a.s. as $\varepsilon \to 0$ by definition of β. From this and (3.9) we deduce the relation (3.8) and complete the proof.

4. **Application to diffusion.** In this section and the next one some of the results for processes with sign-invariant increments are extended to other general kinds of stochastic processes. In this section we show that Theorem 2.3 can be extended to the general Ito process, described in [5, pp. 273–291]. This result was first found under strong differentiability conditions on the diffusion coefficients [2]. (There the main result was incorrectly stated. I am indebted to Dr. V. Baumann of Cologne for his kindness in pointing out that error and to Professor G. Baxter for supplying the correction in [S. M. Berman, *Oscillation of sample functions in diffusion processes*, Math. Reviews No. 642 **28** (1964), p. 132].) Now we shall remove the differentiability assumptions but can prove only a weaker form of Theorem 2.3, namely, a.s. convergence of the sum of the squares of the increments over a sufficiently fine subsequence of partitions. The difficulty of proving convergence over the original sequence is that the latter does not seem to form a martingale sequence.

Let $X(t)$, $a \leqq t \leqq b$, be an Ito process: a separable, real valued Markov process with continuous sample functions, and satisfying the stochastic integral equation

$$X(t) - X(a) = \int_a^t m[s, X(s)]\, ds + \int_a^t \sigma[s, X(s)]\, dY(s), \tag{4.1}$$

where $Y(s)$ is a Brownian motion process such that for each t_0 in (a,b), the increments of $Y(s)$ over subintervals of $[t_0, b]$ are distributed independently of the $X(t)$ process in $[a, t_0)$. It is assumed that m and σ are Baire functions of their arguments and that there exists a constant K such that m and σ satisfy the following growth and regularity conditions

$$\begin{aligned} |m[t,\xi]| &\leqq K(1+\xi^2)^{1/2}, \\ 0 &\leqq \sigma[t,\xi] \leqq K(1+\xi^2)^{1/2}, \\ |m[t,\xi_2] - m[t,\xi_1]| &\leqq K|\xi_2 - \xi_1|, \\ |\sigma[t,\xi_2] - \sigma[t,\xi_1]| &\leqq K|\xi_2 - \xi_1|. \end{aligned} \tag{4.2}$$

We continue to use the notation of §2: $t_{n,k}$ are the subdivision points of $[a,b]$ and $X_{n,k}$ are the corresponding increments, $k = 1, \cdots, n$.

THEOREM 4.1. *Under the above stated assumptions, the process $X(t)$ has the property*

$$\lim_{n' \to \infty} \sum_{k=1}^{n'} |X_{n',k}|^2 = \int_a^b \sigma^2[s, X(s)]\, ds, \text{ a.s.} \tag{4.3}$$

where $\{n'\}$ *is a subsequence of the positive integers with the property that*

$$\sum_{n'} \max_{1 \leqq k \leqq n'} (t_{n',k} - t_{n',k-1})^{1/2} < \infty .$$

Proof. We shall not use the property of the sequence $\{n'\}$ until later so that we shall write n instead of n' for the present.

We first show that the existence of the limit (4.3) is an event whose probability is the same for all functions m satisfying (4.2); and, the value of the limit, when it exists, is the same for all such m. Using the integral representation (4.1), we may write the sum on the left-hand side of (4.3) as the sum of three terms:

$$\sum_{k=1}^{n} \left(\int_{t_{n,k-1}}^{t_{n,k}} m[s, X(s)]\, ds \right)^2 + 2 \sum_{k=1}^{n} \left(\int_{t_{n,k-1}}^{t_{n,k}} m[s, X(s)]\, ds \right) \left(\int_{t_{n,k-1}}^{t_{n,k}} \sigma[s, X(s)]\, dY(s) \right) + \sum_{k=1}^{n} \left(\int_{t_{n,k-1}}^{t_{n,k}} \sigma[s, X(s)]\, dY(s) \right)^2 = A_n + 2B_n + C_n .$$

The only terms involving m are A_n and B_n; we shall show that both of these converge a.s. to 0. Successive use of the Schwarz inequality, the first condition in (4.2), and continuity (hence integrability) of $X(t)$ yields

$$\begin{aligned} A_n &\leqq \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1}) \int_{t_{n,k-1}}^{t_{n,k}} m^2[s, X(s)]\, ds \\ &\leqq \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1}) \int_{t_{n,k-1}}^{t_{n,k}} K^2(1 + X^2(s))\, ds \\ &\leqq K^2 \int_a^b (1 + X^2(s))\, ds \cdot \max_{1 \leqq k \leqq n} (t_{n,k} - t_{n,k-1}) \to 0, \text{ a.s.} \end{aligned}$$

The application of (4.2), the uniform continuity of almost all $Y(t)$ sample functions, and the boundedness of $X(t)$ leads to

$$|B_n| \leqq K \max_{a \leqq s \leqq b} (1 + X^2(s))^{1/2} \cdot \max_{1 \leqq k \leqq n} |Y(t_{n,k}) - Y(t_{n,k-1})| \cdot \int_a^b K(1 + X^2(s))^{1/2}\, ds \to 0, \text{ a.s.}$$

This proves the assertion made at the beginning of the proof; since the function $m \equiv 0$ satisfies (4.2), we may proceed with our computations, assuming $m \equiv 0$ in (4.1). In this case (4.1) has the special form

$$X(t) - X(a) = \int_a^t \sigma[s, X(s)]\, dY(s), \qquad a \leqq t \leqq b. \tag{4.4}$$

The sum of squares of the increments of the process in (4.4) is also decomposable into three terms:

$$\sum_{k=1}^{n}\left\{\int_{t_{n,k-1}}^{t_{n,k}}(\sigma[s,X(s)]-\sigma[t_{n,k-1},X(t_{n,k-1})])\,dY(s)\right\}^2$$

$$+\,2\sum_{k=1}^{n}\int_{t_{n,k-1}}^{t_{n,k}}(\sigma[s,X(s)]-\sigma[t_{n,k-1},X(t_{n,k-1})])\,dY(s)$$
$$\cdot\,\sigma[t_{n,k-1},X(t_{n,k-1})]\cdot(Y(t_{n,k})-Y(t_{n,k-1}))$$

$$+\sum_{k=1}^{n}\sigma^2[t_{n,k-1},X(t_{n,k-1})](Y(t_{n,k})-Y(t_{n,k-1}))^2 = A'_n+2B'_n+C'_n.$$

By a fundamental expectation property of the stochastic integral [5, p. 427], the expected value of A'_n is

$$\sum_{k=1}^{n}\int_{t_{n,k-1}}^{t_{n,k}}E(\sigma[s,X(s)]-\sigma[t_{n,k-1}X(t_{n,k-1})])^2\,ds.$$

The last condition in (4.2) implies that this is no greater than

$$K^2\sum_{k=1}^{n}\int_{t_{n,k-1}}^{t_{n,k}}E(X(s)-X(t_{n,k-1}))^2\,ds. \tag{4.5}$$

By (4.4), the same fundamental expectation property of the stochastic integral, and the second line of (4.2), we obtain, for $t_{n,k-1}<s\leqq t_{n,k}$, (cf. [5, p. 283])

$$\begin{aligned}E(X(s)-X(t_{n,k-1}))^2 &= E\left[\int_{t_{n,k-1}}^{s}\sigma[u,X(u)]\,dY(u)\right]^2\\ &= \int_{t_{n,k-1}}^{s}E(\sigma^2[u,X(u)])\,du\\ &\leqq K^2(s-t_{n,k-1})\left[1+\max_{t_{n,k-1}\leqq u\leqq t_{n,k}}EX^2(u)\right].\end{aligned}$$

Integration over s yields

$$\int_{t_{n,k-1}}^{t_{n,k}}E(X(s)-X(t_{n,k-1}))^2\,ds$$
$$\leqq \frac{1}{2}K^2(t_{n,k}-t_{n,k-1})^2\left[1+\max_{t_{n,k-1}\leqq u\leqq t_{n,k}}EX^2(u)\right].$$

Since $1+EX^2(s)$ is integrable over $[a,b]$ [5, p. 277], we obtain

$$EA'_n\leqq \text{constant}\max_k\,(t_{n,k}-t_{n,k-1}).$$

We now use the property of the sequence $\{n'\}$; by Markov's inequality, we have, for $\varepsilon>0$,

$$\sum_{n'} P\{A'_{n'} > \varepsilon\} \leqq \varepsilon^{-1} \sum_{n'} EA'_{n'} < \infty;$$

hence, by the Borel-Cantelli lemma, $A'_{n'} \to 0$ a.s.

Applying the Schwarz inequality and the expectation property of the stochastic integral, we get

$$EB'_n \leqq \sum_{k=1}^{n} \left(\int_{t_{n,k-1}}^{t_{n,k}} E(\sigma[s, X(s)] - \sigma[t_{n,k-1}, X(t_{n,k-1})])\, dY(s) \right)^{1/2} \cdot \sigma[t_{n,k-1}, X(t_{n,k-1})](t_{n,k} - t_{n,k-1}).$$

In accordance with the foregoing calculations, the latter sum is no greater than a constant multiple of

$$\sum_{k=1}^{n} (t_{n,k} - t_{n,k-1})^{3/2} \left[1 + \max_{t_{n,k-1} \leqq a \leqq t_{n,k}} EX^2(u) \right] \leqq \text{constant} \max_{1 \leqq k \leqq n} (t_{n,k} - t_{n,k-1})^{1/2}$$

Again we employ the property of $\{n'\}$ and the Borel-Cantelli lemma and conclude that $B'_{n'} \to 0$ a.s.

We now cite Lévy's original version of (4.3) for the special case of the standard Brownian motion process: *the squares of* $Y(t_{n,k}) - Y(t_{n,k-1})$, *summed over any fixed subinterval of* $[a, b]$, *converges a.s. to the length of the subinterval* [8], [5, p. 395]. (4.2) implies that $\sigma^2[s, X(s)]$ is (uniformly) continuous in s for almost all sample functions. Fix an arbitrary integer $m > 0$; then, for $n > m$, C'_n is bounded above by

$$\sum_{k=1}^{m} \max_{t_{m,k-1} \leqq s \leqq t_{m,k}} \sigma^2[s, X(s)] \cdot \sum_{t_{n,j} \in [t_{m,k-1}, t_{m,k})} [Y(t_{n,j+1}) - Y(t_{n,j})]^2,$$

and below by a corresponding expression with "min" instead of "max." Applying Lévy's cited theorem to the $Y(t)$-process, we conclude that the upper bound converges a.s. to

$$\sum_{k=1}^{m} \max_{t_{m,k-1} \leqq s \leqq t_{m,k}} \sigma^2[s, X(s)](t_{m,k} - t_{m,k-1})$$

and the lower bound to

$$\sum_{k=1}^{m} \min_{t_{m,k-1} \leqq s \leqq t_{m,k}} \sigma^2[s, X(s)](t_{m,k} - t_{m,k-1}).$$

The bounds converge ($m \to \infty$) to the common limit given by the (Riemann) integral in (4.3), because $\sigma^2[s, X(s)]$ is continuous.

5. **Application to processes with independent increments.** In this section, $X(t)$, $a \leqq t \leqq b$, is a separable stochastic process with independent increments

which is centered and has no fixed points of discontinuity. By the classical theorem of Lévy [6], the logarithm of the characteristic function of an increment $X(t) - X(s)$, $a \leqq s < t \leqq b$, is of the form

$$\psi(u; s, t) = iu[\alpha(t) - \alpha(s)] - \frac{1}{2}u^2[\sigma^2(t) - \sigma^2(s)] \tag{5.1}$$
$$+ \int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) dL(x; s, t),$$

where $\alpha(t)$ and $\sigma^2(t)$ are real continuous functions and $\sigma^2(t)$ is nondecreasing. The properties of the function $L(x; s, t)$ are discussed in the standard texts (e.g. [5, p. 421]); we record the facts that the increments $L(x_2; s, t) - L(x_1; s, t)$ for either $-\infty \leqq x_1 < x_2 < 0$ or $0 < x_1 < x_2 \leqq \infty$, $a \leqq s < t \leqq b$, are nonnegative and define a completely additive set function (measure) over the Borel sets in the infinite strip $-\infty \leqq x \leqq \infty$, $a \leqq t \leqq b$, with the following two properties: the measure of every Borel set outside any open strip over the line $x = 0$ is finite and

$$\int_{[0 < |x| < 1]} x^2 \, dL(x; a, b) < \infty.$$

All integrals with respect to $L(x; s, t)$, both in (5.1) and in what follows, are tacitly assumed to be over the domain $\{x \neq 0\}$.

Extending the definition of Blumenthal and Getoor [3], given by them for processes with *stationary* independent increments, we define the index β of a process $X(t)$ with general independent increments as

$$\beta = \inf \left\{\alpha: \int_{-1}^{1} |x^{\alpha}| \, dL(x; a, b) < \infty\right\};$$

thus $\beta \leqq 2$. The finiteness of $\int_{-1}^{1} |x|^{\alpha} dL(x; a, b)$ implies the a.s. finiteness of $\sum_k |J_k|^{\alpha}$, where $J_1, J_2, \cdots$ are the jumps of the process. For, on one hand, $\sum_k \{|J_k|^{\alpha}: |J_k| > 1\}$ is finite a.s. because there are at most a finite number of jumps J such that $|J| > 1$. On the other hand, we have

$$E\left[\sum_k \{|J_k|^{\alpha}: |J_k| \leqq 1\}\right] = \int_{-1}^{1} |x|^{\alpha} dL(x; a, b),$$

because $dL(x; a, b)$ is the expected number of jumps J such that $x < J \leqq x + dx$ [6], [5, p. 423].

Some of the computations in Lemmas 5.1 and 5.2 below are related to [3]. Put $\psi_{n,k}(u) = \psi(u; t_{n,k-1}, t_{n,k})$ and $L_{n,k}(x) = L(x; t_{n,k-1}, t_{n,k})$, $k = 1, \cdots, n$.

LEMMA 5.1. *Let $X(t)$ have increment characteristic functions whose logarithms are of the form* (5.1) *with $\alpha(t) \equiv \sigma^2(t) \equiv 0$, and which is of index β. For a given, fixed number $\varepsilon > 0$, assume that the measure induced by L on the strip*

$-\infty \leqq x \leqq \infty$, $a \leqq t \leqq b$, assigns measure 0 to that portion outside the rectangle $-\varepsilon < x < \varepsilon$, $a \leqq t \leqq b$. Then, for any γ, $\max(1,\beta) < \gamma \leqq 2$,

$$\sup_n \sum_{k=1}^n |\psi_{n,k}(u)| \leqq 4|u|^\gamma \int_{-\varepsilon}^{\varepsilon} |x|^\gamma dL(x;a,b) \tag{5.2}$$
$$+ |u\varepsilon| \int_{-\varepsilon}^{\varepsilon} |x|^2 \, dL(x;a,b)$$

holds for each u.

Proof. Put $R_n(u) = |\operatorname{Re}\psi_{n,1}(u)| + \cdots + |\operatorname{Re}\psi_{n,n}(u)|$, $I_n(u) = |\operatorname{Im}\psi_{n,1}(u)| + \cdots + |\operatorname{Im}\psi_{n,}(u)|$. By the elementary inequality $1 - \cos t \leqq 2|t|^\gamma$, we have

$$R_n(u) \leqq 2|u|^\gamma \sum_{k=1}^n \int_{-\varepsilon}^{\varepsilon} |x|^\gamma dL_{n,k}(x)$$
$$= 2|u|^\gamma \int_{-\varepsilon}^{\varepsilon} |x|^\gamma dL(x;a,b).$$

By the inequality $|\sin t - t| \leqq 2|t|^\gamma$, and by the triangle inequality, we have $|\sin t - t(1+t^2)^{-1}| \leqq 2|t|^\gamma + t^3/(1+t^2)$; therefore, application of the last inequality gives

$$I_n(u) \leqq 2|u|^\gamma \int_{-\varepsilon}^{\varepsilon} |x|^\gamma dL(x;a,b)$$
$$+ |u\varepsilon| \int_{-\varepsilon}^{\varepsilon} |x|^2 y dL(x;a,b), \quad \textit{for all } u.$$

(5.2) follows from the ineqiualties for $R_n(u)$ and $I_n(u)$ applied to the inequality $|z| \leqq |\operatorname{Re} z| + |\operatorname{Im} z|$.

Let $f_\gamma(y)$ be the density function of the symmetric stable law of index γ, $0 < \gamma < 2$, which is uniquely represented by its (real) characteristic function

$$\int_{-\infty}^{\infty} e^{iuy} f_\gamma(y) dy = \int_{-\infty}^{\infty} \cos uy f_\gamma(y) \, dy = e^{-|u|^\gamma}.$$

By a well-known property of the stable law [7, p. 201], the integral

$$\int_{-\infty}^{\infty} f_\gamma(y)|y|^\alpha \, dy$$

is finite for $\alpha < \gamma$. Now let Y be a random variable with the characteristic function $\phi(u) = E\{\exp(iuY)\}$. Blumenthal and Getoor [3] have given a representation of $E\{\exp(|Y|^\gamma)\}$ in terms of the stable density: interchanging the order of integration and expectation, they get

$$E(e^{-|Y|^\gamma}) = \int_{-\infty}^{\infty} E(e^{iuY}) f_\gamma(u) \, du = \int_{-\infty}^{\infty} \phi(u) f_\gamma(u) \, du.$$

We shall use this formula in proving the next lemma.

LEMMA 5.2. *Under the hypotheses of Lemma* 5.1, *we have*

$$\limsup_{n\to\infty} \sum_{k=1}^{n} E[1-\exp(-|X_{n,k}|^{\gamma})]$$

$$(5.3) \qquad \leqq 8\int_{-\infty}^{\infty} |u|^{\alpha} f_{\gamma}(u)du \cdot \int_{-\varepsilon}^{\varepsilon} |x|^{\alpha} dL(x;a,b)$$

$$+ 2\varepsilon\int_{-\infty}^{\infty} |u| f_{\gamma}(u)\,du \cdot \int_{-\varepsilon}^{\varepsilon} x^2 dL(x;a,b),$$

where α *is any number satisfying* $\max(1,\beta)<\alpha<\gamma$, *and* $f_{\gamma}(u)$ *is the density function of the symmetric stable law of index* γ.

Proof. By the previous formula, the left-hand side of (5.3) is equal to

$$(5.4) \qquad \limsup_{n\to\infty} \int_{-\infty}^{\infty} f_{\gamma}(y) \sum_{k=1}^{n} [1-\exp\psi_{n,k}(y)]\,dy.$$

By the uniform smallness of the increments of the process with independent increments, $\max_k |1-\exp\psi_{n,k}(y)|$ converges to 0 for each Y as $n\to\infty$ [5, p. 132]. From the inequality for complex z,

$$|1-e^z| \leqq 2|z|, \quad |z| \textit{ sufficiently small},$$

we now obtain for each y

$$|1-\exp\psi_{n,k}(y)| \leqq 2|\psi_{n,k}(y)|, \qquad k=1,\cdots,n \quad \textit{for all large } n.$$

From this inequality, (5.2), and Fatou's lemma, (5.4) is no greater than

$$2\int_{-\infty}^{\infty} f_{\gamma}(y) \limsup_{n\to\infty} \sum_{k=1}^{n} |\psi_{n,k}(y)|\,dy.$$

We apply (5.2) with α instead of γ, to the above integrand and integrate to get the right-hand side of (5.3).

LEMMA 5.3. *Under the same hypotheses as Lemma* 5.1,

$$\limsup_{n\to\infty} \sum_{k=1}^{n} E|X_{n,k}|^{\gamma}$$

is no greater than twice the left-hand side of (5.3).

Proof. For $\varepsilon>0$, write $|X_{n,k}|^{\gamma}$ as a sum $|X_{n,k}|^{\gamma} = (|X_{n,k}|^{\gamma} - |X_{n,k}^{(\varepsilon)}|^{\gamma}) + |X_{n,k}^{(\varepsilon)}|^{\gamma}$. From the elementary inequality

$$|x|^{\gamma} \leqq 2[1-\exp(-|x|^{\gamma})] \quad \textit{for all small } x,$$

we get, for small ε,

$$\sum_{k=1}^{n} E|X_{n,k}^{(\varepsilon)}|^{\gamma} \leqq 2\sum_{k=1}^{n} E[1-\exp(-|X_{n,k}^{(\varepsilon)}|^{\gamma})]$$

$$\leqq 2\sum_{k=1}^{n} E[1-\exp(-|X_{n,k}|^{\gamma})].$$

To complete the proof, we show that

$$\lim_{n\to\infty}\sum_{k=1}^{n} E(|X_{n,k}|^{\gamma}-|X_{n,k}^{(\varepsilon)}|^{\gamma}) = 0.$$

Let $\theta_{n,k}(\varepsilon)$ be the indicator function of the event $|X_{n,k}|>\varepsilon$; then

$$|X_{n,k}|^{\gamma}-|X_{n,k}^{(\varepsilon)}|^{\gamma} = |X_{n,k}|^{\gamma}\theta_{n,k}(\varepsilon).$$

By the Hölder inequality,

$$E(|X_{n,k}|^{\gamma}\theta_{n,k}(\varepsilon)) \leqq (EX_{n,k}^{2})^{\gamma/2}(P\{|X_{n,k}|>\varepsilon\})^{1-\gamma/2} \;;$$

by the Hölder inequality for sums, the sum over the terms on the right, from 1 to n, is less than or equal to

$$\left(\sum_{k=1}^{n} E|X_{n,k}|^{2}\right)^{\gamma/2}\left(\sum_{k=1}^{n} P\{|X_{n,k}|>\varepsilon\}\right)^{(2-\gamma)/2}$$

Since $-E|X_{n,k}|^{2}$ is the second derivative of $\psi_{n,k}$ at the origin, the first of the two factors above is

$$\left[\int_{-\varepsilon}^{\varepsilon} x^{2}dL(x;a,b)\right]^{\gamma/2} .$$

According to the necessary and sufficient conditions for the convergence of the distribution of $X_{n,1}+\cdots+X_{n,n}$ to the distribution of $X(b)-X(a)$ [9, p. 311], the second of the two factors must tend to 0 because

$$\lim_{n\to\infty}\sum_{k=1}^{n} P\{|X_{n,k}|>\varepsilon\} = \int_{|x|>\varepsilon} dL(x;a,b) = 0.$$

These three lemmas lead to the following theorem, which supplements the results in [4] for $\gamma=2$.

THEOREM 5.1. *Let $X(t)$, $a\leqq t\leqq b$, have increment characteristic functions whose logarithms are of the form* (5.1), *and which is of index β. If $\sigma^2(t)$ is not identically equal to* 0, *then, for every γ, $0<\gamma<2$,*

$$\lim_{n\to\infty}\sum_{k=1}^{n}|X_{n,k}|^{\gamma} = +\infty \ a.s. \tag{5.5}$$

If $\sigma^2(t)$ is identically equal to 0, *and $\alpha(t)$ is of bounded variation, then, for any γ, $\max(1,\beta)<\gamma<2$,* (5.6) $$\lim_{n\to\infty} \sum_{k=1}^{n} |X_{n,k}|^\gamma = \sum_k |J_k|^\gamma \ a.s.$$

Proof. Let $Y(t)$ be a process distributed identically as and independently of $X(t)$; let $Y_{n,k}$ denote the increment corresponding to $X_{n,k}$. As is well known, the process $X(t) - Y(t)$ has independent and symmetric increments, and, therefore, sign-invariant increments.

Case 1. $\sigma^2(t) \not\equiv 0$. The elementary inequality $|x+y|^\gamma \leqq \max(1, 2^{\gamma-1})(|x|^\gamma + |y|^\gamma)$ yields

$$\sum_{k=1}^{n} |X_{n,k} - Y_{n,k}|^\gamma \leqq \max(1, 2^{\gamma-1}) \left(\sum_{k=1}^{n} |X_{n,k}|^\gamma + \sum_{k=1}^{n} |Y_{n,k}|^\gamma \right).$$

The process $X(t) - Y(t)$ has a Gaussian component and the result of Cogburn and Tucker [4] implies that $V_{[a,b]}$, given by Theorem 2.3, is a.s. positive (in fact constant). The first part of Theorem 3.2 then implies that the left-hand member of the last stated inequality converges a.s. to $+\infty$. This implies (5.5) because the sum of two independent random variables with the same distribution converges a.s. to $+\infty$ if and only if each summand does.

Case 2. $\sigma^2(t) \equiv 0$. Here $\gamma > 1$. For any fixed $\varepsilon > 0$, the process $X(t)$ may be written as the sum of two independent processes with independent increments, $X_1(t)$ and $X_2(t)$, and a deterministic function $g(t)$, where

$$\log E[\exp(iu[X_1(t) - X_1(a)])] = \int_{|x|>\varepsilon} (e^{iux} - 1)\, dL(x; a, t),$$

$$\log E[\exp(iu[X_2(t) - X_2(a)])] = \int_{-\varepsilon}^{\varepsilon} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) dL(x; a, t),$$

$$g(t) = \alpha(t) - \alpha(a) - \int_{|x|>\varepsilon} \frac{x}{1+x^2}\, dL(x; a, t).$$

$X_1(t)$ and $X_2(t)$ are continuous except for a countable number of jumps, and, because of the mutual independence of the two processes, the jump points of one process are all continuity points of the other process a.s. $X_1(t)$ has a finite number of jumps in $[a, b]$, each of magnitude greater than ε, and is constant between successive jumps. $X_2(t)$ has a countable number of jumps, each of magnitude no greater than ε. $g(t)$ is a continuous function because $X(t)$ has no fixed points of discontinuity. The following argument shows that $g(t)$ is also of bounded variation. First of all, $\alpha(t)$ is of bounded variation by hypothesis. Next, since $dL(x; a, t)$ is the expected number of jumps J of the function $X(s)$, $a \leqq s \leqq t$, such that $x < J \leqq x + dx$, the two integrals

$$\int_{\varepsilon}^{\infty} \frac{x}{1+x^2} dL(x; a, t) \quad \text{and} \quad -\int_{-\infty}^{-\varepsilon} \frac{x}{1+x^2}\, dL(x; a, t)$$

are monotonically nondecreasing functions of t, so that their difference is of bounded variation; hence, $g(t)$ is of bounded variation.

Using the random variables $\xi_k^{(n)}(\varepsilon)$, defined in the proof of Theorem 3.2, we write

$$\sum_{k=1}^{n} |X_{n,k}|^{\gamma} = \sum_{k=1}^{n} |X_{n,k}|^{\gamma}\xi_k^{(n)}(\varepsilon) + \sum_{k=1}^{n} |X_{n,k}|^{\gamma}(1 - \xi_k^{(n)}(\varepsilon)).$$

The truncation principle (as that in the proof) implies that the second sum on the right-hand side converges a.s. to $\sum_k \{|J_k|^{\gamma}: |J_k| > \varepsilon\}$. The first sum on the right-hand side is monotonically nonincreasing in ε so that the limit $(\varepsilon \to 0)$ of its lim sup $(n \to \infty)$ exists. To complete the proof of the theorem, we show that the limit is 0 a.s., i.e.

$$\lim_{\varepsilon \to 0} \left[\limsup_{n \to \infty} \sum_{k=1}^{n} |X_{n,k}|^{\gamma}\xi_k^{(n)}(\varepsilon)\right] = 0. \tag{5.7}$$

Since $X_1(t)$ is constant between successive jumps, the increments of $X(t)$ are identical with the increments of $X_2(t) + g(t)$ over the intervals where $X(t)$ has no jumps exceeding ε in magnitude. Put $\bar{X}_{n,k} = X_2(t_{n,k}) - X_2(t_{n,k-1})$, and $g_{n,k} = g(t_{n,k}) - g(t_{n,k-1})$, $k = 1,\cdots,n$; then;

$$\sum_{k=1}^{n} |X_{n,k}|^{\gamma}\xi_k^{(n)}(\varepsilon) = \sum_{k=1}^{n} |\bar{X}_{n,k} + g_{n,k}|^{\gamma}\xi_k^{(n)}(\varepsilon) \leqq \sum_{k=1}^{n} |\bar{X}_{n,k} + g_{n,k}|^{\gamma}, \text{ a.s.} \tag{5.8}$$

According to Minkowski's inequality, the last sum satisfies

$$\left(\sum_{k=1}^{n} |\bar{X}_{n,k} + g_{n,k}|^{\gamma}\right)^{1/\gamma} \leqq \left(\sum_{k=1}^{n} |\bar{X}_{n,k}|^{\gamma}\right)^{1/\gamma} + \left(\sum_{k=1}^{n} |g_{n,k}|^{\gamma}\right)^{1/\gamma}. \tag{5.9}$$

The last term in (5.9) converges to 0 as $n \to \infty$ since $g(t)$ is continuous and of bounded variation, and $\gamma > 1$. (5.8) and (5.9) show that the condition

$$\lim_{\varepsilon \to 0} E\left[\limsup_{n \to \infty} \sum_{k=1}^{n} |\bar{X}_{n,k}|^{\gamma}\right] = 0 \tag{5.10}$$

is sufficient for (5.7).

Let $Y_2(t)$ be a process distributed identically as and independently of $X_2(t)$, with corresponding increments $\bar{Y}_{n,k}$; then, by the inequality

$$|x|^{\gamma} \leqq 2^{\gamma-1}(|x - y|^{\gamma} + |y|^{\gamma}), \quad \gamma > 1,$$

we obtain

$$\sum_{k=1}^{n} |\bar{X}_{n,k}|^{\gamma} \leqq 2^{\gamma-1}\left(\sum_{k=1}^{n} |\bar{X}_{n,k} - \bar{Y}_{n,k}|^{\gamma} + \sum_{k=1}^{n} |\bar{Y}_{n,k}|^{\gamma}\right).$$

To each side of this inequality, we now successively apply the three inequality-preserving operations (cf. [5, p. 338]):

(i) conditional expectation given the process $X_2(t)$. Here we use the fact that $X_2(\cdot)$ and $Y_2(\cdot)$ are independently distributed

(ii) lim sup $(n \to \infty)$;

(iii) unconditional expectation.

After an application of Fatou's lemma, we have

$$E\left(\limsup_{n\to\infty} \sum_{k=1}^{n} |\bar{X}_{n,k}|^\gamma\right) \tag{5.11}$$
$$\leqq 2^{\gamma-1}\cdot\left[E\left(\limsup_{n\to\infty}\sum_{k=1}^{n}|\bar{X}_{n,k}-\bar{Y}_{n,k}|^\gamma\right)+\limsup_{n\to\infty}\sum_{k=1}^{n}E|\bar{Y}_{n,k}|^\gamma\right].$$

The process $Y_2(t)$ satisfies the hypotheses of Lemmas 5.1–5.3. According to these lemmas, the lim sup on the right-hand side of (5.11) is dominated by twice the right-hand side of (5.3). The integrals involving ε in the latter expression converge to 0 as $\varepsilon \to 0$ because the process $X_2(t)$ is of index β. We now estimate the expectation of the lim sup on the right-hand side of (5.11). According to Theorem 3.2, the "lim sup" can be replaced by "lim" under the expectation sign; furthermore, by the same theorem, the limit is $\sum_k |J_k'|^\gamma$, where the sum is over all jumps J_k' of the process $X_2(t) - Y_2(t)$. We have

$$E\left(\sum_k |J_k'|^\gamma\right) = 2\int_{-\varepsilon}^{\varepsilon} |x|^\gamma \, dL(x;a,b); \tag{5.12}$$

in fact, the increment $[X_2(t) - Y_2(t)] - [X_2(s) - Y_2(s)]$ has the characteristic function whose logarithm is

$$2\int_{-\varepsilon}^{\varepsilon} (\cos ux - 1)\, dL(x;s,t),$$

$a \leqq s < t \leqq b$, and $2[dL(x;a,b) + dL(-x;a,b)]$ is the expected number of jumps of the process whose magnitude lies in $[x, x+dx], x > 0$. The right-hand side of (5.12) converges to 0 with ε because the process has index β; (5.10) holds, and the proof is complete.

In Theorem 5.1, we have taken $\gamma > 1$. We shall not make a formal statement of the case $0 < \beta < \gamma \leqq 1$, but shall only sketch the result and the proof. By the inequality

$$|x+y|^\gamma \leqq |x|^\gamma + |y|^\gamma, \qquad 0 < \gamma \leqq 1, \tag{5.13}$$

the sequence $\{\sum_{k=1}^{n} |X_{n,k}|, \; n = 1, 2, \cdots\}$ is a.s. nondecreasing and so must converge to a limit. Under what conditions is this limit actually the same as (5.6)? We decompose $X(t)$ into the sum of the three components $X_1(t)$, $X_2(t)$, and

$g(t)$, as in the previous proof. An examination of the latter shows that (5.6) *holds if the limit, as* $n \to \infty$ *and* $\varepsilon \to 0$, *of the left-hand side of* (5.9) *is a.s.* 0. By (5.13), it suffices to show that the limits of $\sum_{k=1}^{n} |\bar{X}_{n,k}|^{\gamma}$ and $\sum_{k=1}^{n} |g_{n,k}|^{\gamma}$ are both 0. The proof for the former sum follows as in the previous proof; the only difference is that (5.13) is used instead of $|x+y| \leqq 2^{\gamma-1}(|x|^{\gamma}+|y|^{\gamma})$. The vanishing of the limit of $\sum_{k=1}^{n} |g_{n,k}|^{\gamma}$ has to be *assumed*; this can be formulated as an assumption on the smoothness of the functions defining $g(t)$.

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