# TWO NOTES ON LOCALLY MACAULAY RINGS

BY
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1. **Introduction.** In this paper all rings are assumed to be commutative rings with a unit. The undefined terminology used in this paper (height, altitude, etc.) will be the same as that in [1]. Throughout this paper a number of known properties of locally Macaulay rings are stated, and then are used in the remainder of the paper without explicit mention.

In §2 it is proven that if $R$ is a locally Macaulay ring and if $(a_1, \cdots, a_n)$ is a prime sequence in $R$, the kernel of the natural homomorphism from $P = R[X_1, \cdots, X_{n-1}]$ onto $R' = R[a_2/a_1, \cdots, a_n/a_1]$ is $(a_1X_1 - a_2, a_1X_2 - a_3, \cdots, a_1X_{n-1} - a_n)P$ (Lemma 2.3). As a consequence, $R'$ is a locally Macaulay ring and $(a_1, a_2/a_1, \cdots, a_n/a_1)$ is a prime sequence in $R'$ (Theorem 2.4). Further, if $R[X_1]$ is a Macaulay ring, then $R'$ is a Macaulay ring (Theorem 2.8). An example is given to show that the converses are not in general true.

In §3 it is proven that, with the same $R$ and $a_i$, the Rees ring $R^* = R[ta_1, \cdots, ta_n, 1/t]$ ($t$ an indeterminant) of $R$ with respect to $A = (a_1, \cdots, a_n)R$ is a locally Macaulay ring (a Macaulay ring if $R[X_1]$ is) and $(1/t, ta_1, \cdots, ta_n)$ is a prime sequence in $R^*$ (Theorems 3.1 and 3.3). A form of the converses of Theorems 3.1 and 3.3 is true (Theorem 3.8). Also, for every $e \geqq 1$, $k \geqq e$, and $i = 1, \cdots, n$, $(a_1^e, a_2^e, \cdots, a_i^e)A^{k-e} = (a_1^e, a_2^e, \cdots, a_i^e) \cap A^k$ (Corollary 3.6). Further, for all $k \geqq 1$, every prime divisor of $A^k$ has height $n$, and $A^k : a_1R = A^{k-1}$ (Corollary 3.7). It is also proven that if the Rees ring $R^*$ of a Noetherian ring $R$ with respect to an ideal $A = (a_1, \cdots, a_n)R$ is a locally Macaulay ring (a Macaulay ring), then $R' = R[a_1/a, \cdots, a_n/a]$ is a locally Macaulay ring (a Macaulay ring) for every non-zero-divisor $a \in A$ (Corollary 3.9).

2. **Transformations of locally Macaulay rings by a prime sequence.**

LEMMA 2.1. *Let $R$ be a ring, let $a, b$ be elements in $R$ such that $a$ is not a zero divisor, and let $X$ be an indeterminant. If $aR : bR = aR$, then the kernel $K$ of the natural homomorphism from $R[X]$ onto $R[b/a]$ is generated by $aX - b$.*

**Proof.** Clearly $aX - b \in K$. Let $f(X) = r_nX^n + \cdots + r_0 \in K$. Then $r_nb^n + r_{n-1}ab^{n-1} + \cdots + r_0a^n = 0$, so $r_n \in aR : b^nR = aR$, say $r_n = da$. Since $a$ is not a zero divisor, $g(X) = (db + r_{n-1})X^{n-1} + r_{n-2}X^{n-2} + \cdots + r_0 \in K$, and $f(X) = (aX - b)dX^{n-1} + g(X)$. Hence, by induction on $n$, $f(X) \in (aX - b)R[X]$, so $K$ is generated by $aX - b$, q.e.d.

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A local (Noetherian) ring $R$ is a *Macaulay local ring* in case there exists a system of parameters $(a_1, \cdots, a_n)$ in $R$ such that $a_i$ is not in any prime divisor of $(a_1, \cdots, a_{i-1})R$ $(i = 1, \cdots, n)$. In particular $a_1$ is not a zero divisor. A Noetherian ring $R$ is a *locally Macaulay ring* in case $R_M$ is a Macaulay local ring for every maximal ideal $M$ in $R$. $R$ is a *Macaulay ring* in case $R$ is a locally Macaulay ring such that height $M$ = altitude $R$ for every maximal ideal $M$ in $R$. It is known that if $R$ is a Macaulay local ring of altitude $n$ and if $(a_1, \cdots, a_k)$ is a subset of a system of parameters in $R$, then $R/(a_1, \cdots, a_k)R$ is a Macaulay local ring of altitude $n - k$ [3, p. 397]. Also, $R$ is a locally Macaulay ring if and only if the following theorem (the *unmixedness theorem*) holds: If an ideal $A$ in $R$ is generated by $k$ elements and if height $A = k$ $(k \geqq 0)$, then every prime divisor of $A$ has height $k$ [1, p. 85]. These two facts immediately imply that if $R$ is a locally Macaulay ring (a Macaulay ring) and if $A$ is an ideal in $R$ which is generated by $k$ elements and has height $k$, then $R/A$ is a locally Macualay ring (a Macaulay ring). Finally, it is known that if $X_1, \cdots, X_n$ are algebraically independent over a Noetherian ring $R$, then $R[X_1, \cdots, X_n]$ is a locally Macaulay ring if and only if $R$ is [1, p. 86].

These facts are used in the proof of

COROLLARY 2.2. *Let $R$ be a locally Macaulay ring, and let $a, b$ be elements in $R$ such that $a$ is not a zero divisor and $aR : bR = aR$. Then $R[b/a]$ is a locally Macaulay ring.*

**Proof.** $R[X]$ is a locally Macaulay ring, and the kernel of the natural homomorphism from $R[X]$ onto $R[b/a]$ is generated by $aX - b$ (Lemma 2.1). Since $aX - b$ is not a zero divisor in $R[X]$, $R[b/a]$ is a locally Macaulay ring, q.e.d.

Theorem 2.4 below generalizes the above corollary. To obtain the generalization the following definitions and lemma will be used.

An integral domain $R$ satisfies the *altitude formula* in case the following condition holds: If $R'$ is an integral domain which is finitely generated over $R$, and if $p'$ is a prime ideal in $R'$, then height $p' + \operatorname{trd}(R'/p')/(R/p' \cap R) =$ height $p' \cap R + \operatorname{trd} R'/R$. It is known that if an integral domain $R$ is a homomorphic image of a locally Macaulay ring, then $R$ satisfies the altitude formula [1, p. 130].

If $R$ is a locally Macaulay ring, and if $p \subset q$ are prime ideals in $R$, then $R_q$ is a Macaulay local ring [1, p. 86], so height $p$ + height $q/p$ = height $q$ [3, p. 399]. This fact will be used in the future without explicit mention.

A sequence $(a_1, \cdots, a_n)$ of nonunits in a Noetherian ring $R$ is a *prime sequence* in case $a_1$ is not a zero divisor, $(a_1, \cdots, a_i)R : a_{i+1}R = (a_1, \cdots, a_i)R$ $(i = 1, \cdots, n-1)$, and $(a_1, \cdots, a_n)R \neq R$. It is known that if $R$ is a semi-local ring, and if $(a_1, \cdots, a_n)$ is a prime sequence of elements in the Jacobson radical of $R$, then $(a_{\pi 1}, \cdots, a_{\pi n})$ is a prime sequence for every permutation $\pi$ of $\{1, \cdots, n\}$ [3, pp. 394–395].

LEMMA 2.3. *Let $R$ be a locally Macaulay ring, let $(a_1, \cdots, a_n)$ be a prime sequence in $R$, and let $X_1, \cdots, X_{n-1}$ be algebraically independent over $R$. Then* *the kernel $K$ of the natural homomorphism $\phi$ from $P = R[X_1, \cdots, X_{n-1}]$ onto $R' = R[a_2/a_1, \cdots, a_n/a_1]$ is generated by $(a_1X_1 - a_2, a_1X_2 - a_3, \cdots, a_1X_{n-1} - a_n)$.*

**Proof**(1). The proof is by induction on $n$. The case $n=1$ is trivial, and Lemma 2.1 proves the case $n = 2$. Let $n > 2$ and assume the conclusion holds for the case $n - 1$. Now $\phi = fg$, where $f$ and $g$ are the natural homomorphisms from $S = R[a_2/a_1, X_2, \cdots, X_{n-1}]$ onto $R'$ and from $P$ onto $S$ respectively. Since the kernel of $g$ is $(a_1X_1 - a_2)P$ (Lemma 2.1), and since $R^* = R[a_2/a_1]$ is a locally Macaulay ring (Corollary 2.2), it is sufficient (by induction) to prove that $(a_1, a_3, a_4, \cdots, a_n)$ is a prime sequence in $R^*$. Since $R$ and $R^*$ have the same total quotient ring, $a_1$ is not a zero divisor in $R^*$, hence height $a_1R^* = 1$. Let $A_i^* = (a_1X_1 - a_2, a_1, a_3, \cdots, a_i)P$ $(i \geqq 3)$. Then $A_i^* = (a_1, a_2, a_3, \cdots, a_i)P$, hence height $A_i^* = i$. Consequently, by the unmixedness theorem $(a_1X_1 - a_2, a_1, a_3, \cdots, a_n)$ is a prime sequence in $P$, hence $(a_1, a_3, \cdots, a_n)$ is a prime sequence in $R^*$, q.e.d.

THEOREM 2.4. *With the same notation as Lemma 2.3, $R_i' = R[a_2/a_1, \cdots, a_i/a_1]$ $(2 \leqq i \leqq n)$ is a locally Macaulay ring, and $(a_1, a_{i+1}, \cdots, a_{i+j}, b_1, \cdots, b_k)$ is a prime sequence in $R_i'$, where $\{b_1, \cdots, b_k\}$ is a subset of $\{a_2/a_1, \cdots, a_i/a_1\}$, and $0 \leqq j \leqq n - i$. (For $j = 0$ the sequence is $(a_1, b_1, \cdots, b_k)$.)*

**Proof.** That $R_i'$ is a locally Macaulay ring follows immediately from Lemma 2.3 and the remarks preceding the proof of Corollary 2.2. Let $A^* = (a_1, a_{i+1}, \cdots, a_{i+j}, b_1, \cdots, b_k)R_i'$. Since $(a_1X_1 - a_2, \cdots, a_1X_{i-1} - a_i, a_1)R[X_1, \cdots, X_{i-1}]$ is generated by $(a_1, \cdots, a_i)$, $A^*$ is a proper ideal. Hence by the unmixedness theorem, since $j$ and $k$ are arbitrary, it is sufficient to prove height $A^* = j + k + 1$. Let $p'$ be a minimal prime divisor of $A^*$, let $q'$ be a (minimal) prime divisor of zero in $R_i'$ such that $q' \subset p'$ and let $p = p' \cap R$, $q = q' \cap R$. By the altitude formula (for $R'/q'$ over $R/q$), height $p'/q' + \operatorname{trd} R'/p'/R/p =$ height $p/q$ (since $a_1 \notin q$). Also, height $p'/q' \leqq j + k + 1$, $\operatorname{trd} R'/p'/R/p \leqq i - 1 - k$, and height $p/q =$ height $p \geqq i + j$. Hence, height $p' =$ height $p'/q' = j + k + 1$. Therefore height $A^* = j + k + 1$, q.e.d.

REMARK 2.5. The last step in the proof of Theorem 2.4 shows the following results. For every (minimal) prime divisor $p'$ of $A^*$ and for every prime divisor $q'$ of zero contained in $p'$, $p'/q'$ is a minimal prime divisor of $(A^* + q')/q'$. Since height $p'/q' = j + k + 1$, none of the elements $a_1, \cdots, a_{i+j}, b_1, \cdots, b_k$ are in $q'$. Also the elements $a_2/a_1, \cdots, a_i/a_1$ which are not in $p'$ are such that their $p'$ residues are algebraically independent over $R/(p' \cap R)$.

REMARK 2.6. In Theorem 2.4, if every permutation of $(a_1, \cdots, a_n)$ is a prime sequence in $R$ (for example, if $R$ is a semi-local locally Macaulay ring and $a_1, \cdots, a_n$ are in the Jacobson radical of $R$), then every permutation of $(a_1, a_{i+1}, \cdots, a_n, a_2/a_1, \cdots, a_i/a_1)$ is a prime sequence in $R_i'$.

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(1) The author is indebted to the referee for the following proof which is considerably simpler than the author's original proof, and which leads to a more direct proof of Theorem 2.4.

**Proof.** Let $(c_1, \cdots, c_n)$ be a permutation of $(a_1, a_{i+1}, \cdots, a_n, a_2/a_1, \cdots, a_i/a_1)$. Since no $a_i$ is a zero divisor in $R$, $c_1$ is not a zero divisor in $R_i'$. Also $(c_1, \cdots, c_n)R_i' \neq R_i'$. Therefore, by the unmixedness theorem, it remains to prove height $(c_1, \cdots, c_h) R_i' = h$ $(h = 2, \cdots, n-1)$. Let $p'$ be a minimal prime divisor of $(c_1, \cdots, c_h)R_i'$, let $q'$ be a prime divisor of zero in $R_i'$ which is contained in $p'$, and let $p = p' \cap R$, $q = q' \cap R$. If $a_1 \notin p'$, then $\operatorname{trd} R'/p'/R/p = 0$. Hence by the altitude formula (for $R'/q'$ over $R/q$), height $p'/q' =$ height $p/q$. Now height $p' \leqq h$ and height $p \geqq h$ (by the assumption on $(a_1, \cdots, a_n)$), so height $p' =$ height $p = h$. If $a_1 \in p'$, let $k$ of the elements $c_1, \cdots, c_h$ be in $\{a_2/a_1, \cdots, a_i/a_1\}$. Then height $p \geqq i + (h - 1 - k)$ (by the assumption on $(a_1, \cdots, a_n)$), and $\operatorname{trd} R'/p'/R/p \leqq i - 1 - k$. By the altitude formula for $R'/q'$ over $R/q$, height $p' =$ height $p'/q' = h$, q.e.d.

Remark 2.6 is of some interest because of the following

LEMMA 2.7. *Let $R$ be a locally Macaulay ring, and let $(a_1, \cdots, a_n)$ be a prime sequence in $R$ such that every permutation of $(a_1, \cdots, a_n)$ is a prime sequence in $R$. Let $A = (a_1, \cdots, a_n)R$. Then, for all $k \geqq 1$,* (1) *every prime divisor of $A^k$ has height $n$, and* (2) $A^k \colon a_i R = A^{k-1}$ $(i = 1, \cdots, n)$.

**Proof.** This can be proved in the same way as Lemmas 5 and 6 in [3, pp. 401–402]. Without assuming that every permutation of $(a_1, \cdots, a_n)$ is a prime sequence in $R$, Corollary 3.7 below proves (1) is still true and (2)′ $A^k \colon a_1 R = A^{k-1}$ (for all $k \geqq 1$), q.e.d.

It is known that if $R$ is a Macaulay ring and if $X_1, \cdots, X_n$ are algebraically independent over $R$, then $R[X_1, \cdots, X_n]$ is a Macaulay ring if and only if there does not exist an ideal $p$ in $R$ such that $R/p$ is a semi-local integral domain of altitude one [1, p. 87]. Hence if $R[X_1]$ is a Macaulay ring, then $R[X_1, \cdots, X_n]$ is a Macaulay ring. This fact is used in the proof of the next theorem.

THEOREM 2.8. *If $R$ and $R[X]$ are Macaulay rings ($X$ transcendental over $R$), and if $(a_1, \cdots, a_n)$ is a prime sequence in $R$, then $R' = R[a_2/a_1, \cdots, a_n/a_1]$ is a Macaulay ring.*

**Proof.** The kernel $K$ of the natural homomorphisms from $P = R[X_1, \cdots, X_{n-1}]$ onto $R'$ has height $n - 1$. Since $P$ is a Macaulay ring, if $M$ is a maximal ideal in $P$ which contains $K$, then altitude $R + n - 1 =$ altitude $P =$ height $M =$ height $M/K +$ height $K$. Hence, if $M'$ is a maximal ideal in $R'$, then height $M' =$ altitude $P -$ height $K =$ altitude $R$. Since $R'$ is a locally Macaulay ring by Theorem 2.4, $R'$ is a Macaulay ring, q.e.d.

REMARK 2.9. If $R$ is a locally Macaulay ring (a Macaulay ring such that $R[X]$ is a Macaulay ring), and if $(a_1, \cdots, a_n)$ is a prime sequence in $R$, then $R[a_1/a, \cdots, a_n/a]$ is a locally Macaulay ring (a Macaulay ring) for every non-zero-divisor $a \in (a_1, \cdots, a_n)R$. This follows from Theorems 3.1 and 3.3 and Corollary 3.9 below.

It will now be shown that the converses of Theorems 2.4 and 2.8 are not in general true. Let $S = k[X, Y]$, where $k$ is a field and $X$ and $Y$ are algebraically independent over $k$. Let $P = (X-1, Y)S$, $R_1 = S_P$, and $N_1 = PR_1$. Let $Q = (X, Y)S$, $R_2 = S_Q$, and $N_2 = QR_2$. Let $R' = R_1 \cap R_2$, $M_1 = N_1 \cap R'$, and $M_2 = N_2 \cap R'$. Further let $R = k + (M_1 \cap M_2)$, and let $M = (M_1 \cap M_2)R$. Then $R'$ is the intersection of two regular local rings, hence $R'$ is normal. The following statements are easily verified: (1) $M_1$ and $M_2$ are the maximal ideals in $R'$, and $R'_{M_i} = R_i$ is Noetherian ($i = 1,2$). Therefore $R'$ is Noetherian [1, p. 203], so $R'$ is a normal semi-local Macaulay domain. (2) Since $R'/M_i = k$ $(i = 1,2)$, $R$ is a local domain and $R'$ is its derived normal ring [1, p. 204]. (3) $XY, Y \in R$, $X \notin R$, $R' = R[XY/Y]$, and $(Y, X = XY/Y)$ is a prime sequence in $R'$. (4) If $p$ is a height one prime ideal in $R$, then $R_p$ is a regular local ring. Since $R \neq R'$, $M$ is an imbedded prime divisor of every nonzero element in $M$ [1, p. 41], hence $R$ is not a Macaulay domain.

3. **The Rees ring of a locally Macaulay ring.** Let $R$ be a Noetherian ring, let $A = (a_1, \cdots, a_n)R$ be an ideal in $R$, let $t$ be an indeterminant, and set $u = t^{-1}$. The graded Noetherian ring $R^* = R[ta_1, \cdots, ta_n, u]$ is called the *Rees ring* of $R$ with respect to $A$.

THEOREM 3.1. *Let $R$ be a locally Macaulay ring, and let $a_1, \cdots, a_n$ be a prime sequence in $R$. Then the Rees ring $R_i^*$ of $R$ with respect to $(a_1, \cdots, a_i)R$ $(1 \leqq i \leqq n)$ is a locally Macaulay ring, and $(u, a_{i+1}, \cdots, a_{i+j}, b_1, \cdots, b_k)$ is a prime sequence in $R_i^*$, where $\{b_1, \cdots, b_k\}$ is a subset of $\{ta_1, \cdots, ta_i\}$ and $0 \leqq j \leqq n - i$. (For $j = 0$ the sequence is $(u, b_1, \cdots, b_k)$.)*

**Proof.** Since $u$ is transcendental over $R$, $R[u]$ is a locally Macaulay ring, hence $(u, a_1, \cdots, a_n)$ is a prime sequence in $R[u]$. Since $ta_j = a_j/u$, $R_i^*$ is a locally Macaulay ring and $(u, a_{i+1}, \cdots, a_{i+j}, b_1, \cdots, b_k)$ is a prime sequence in $R_i^*$ by Theorem 2.4, q.e.d.

REMARK 3.2. In Theorem 3.1, if every permutation of $(a_1, \cdots, a_n)$ is a prime sequence in $R$, then every permutation of $(u, a_1, \cdots, a_n)$ is a prime sequence in $R[u]$ (since $R[u]$ is a locally Macaulay ring and $u$ is transcendental over $R$), hence by Remark 2.6 every permutation of $(u, a_{i+1}, \cdots, a_n, ta_1, \cdots, ta_i)$ is a prime sequence in $R_i^*$.

THEOREM 3.3. *If $R$ and $R[X]$ are Macaulay rings ($X$ transcendental over $R$), and if $a_1, \cdots, a_n$ is a prime sequence in $R$, then the Rees ring $R^*$ of $R$ with respect to $(a_1, \cdots, a_n)$ $R$ is a Macaulay ring.*

**Proof.** Considering the natural homomorphism from $R[u, X_1, \cdots, X_n]$ onto $R^*$ and the ideal $(u, a_1, \cdots, a_n)$ of $R^*$, the proof is the same as the proof of Theorem 2.8, q.e.d.

LEMMA 3.4. *Let $R^*$ be the Rees ring of a locally Macaulay ring $R$ with respect to a prime sequence $(a_1, \cdots, a_n)$ in $R$. Then $(ta_1, \cdots, ta_i, u)$ is a prime sequence in $R^*$ $(i = 1, \cdots, n)$.*

**Proof.** Since $R^*$ is a locally Macaulay ring and height $(u, ta_1, \cdots, ta_i)R^* = i+1$ (Theorem 3.1), it is sufficient to prove height $(ta_1, \cdots, ta_i)R^* = i$. Let $p$ be a minimal prime divisor of $A_i^* = (ta_1, \cdots, ta_i)R^*$. Then height $p \leqq i$, hence $u \notin p$. Let $T = R[u, t]$, so $T$ is a quotient ring of $R^*$. Since $pT$ is a minimal prime divisor of $A_i^*T = (a_1, \cdots, a_i)T$, and since height $(a_1, \cdots, a_i)R[u] = i$, height $A_i^*T = i$. Therefore height $p = i$, so height $A_i^* = i$, q.e.d.

REMARK 3.5. Let $(a_1, \cdots, a_n)$ be a prime sequence in a locally Macaulay ring $R$. Then the radical of $(a_1, \cdots, a_n)R$ is the radical of $(a_1^{e_1}, a_2^{e_2}, \cdots, a_n^{e_n})R$ $(e_i \geqq 1, i = 1, \cdots, n)$. Hence, by the unmixedness theorem, $(a_1^{e_1}, a_2^{e_2}, \cdots, a_n^{e_n})$ is a prime sequence in $R$. Therefore $R[a_2^{e_2}/a_1^{e_1}, \cdots, a_n^{e_n}/a_1^{e_1}]$ and $R[ta_1^{e_1}, \cdots, ta_n^{e_n}, u]$ are locally Macaulay rings.

Let $R$ be a Noetherian ring and let $R^*$ be the Rees ring of $R$ with respect to an ideal $A$ in $R$. Let $T = R[t, u]$, so $T$ is a quotient ring of $R^*$. For any ideal $B$ in $R$ let $B' = BT \cap R^*$. For any homogeneous ideal $B^*$ in $R^*$ let $[B^*]_k$ be the set of elements $r \in R$ such that $rt^k \in B^*$. It is immediately seen that $[B^*]_k$ is an ideal in $R$ and $A^k \supseteq [B^*]_k \supseteq [B^*]_{k+1} \supseteq A[B^*]_k$ for all integers $k$ (with the convention that $A^k = R$ if $k \leqq 0$). Also, since $R^*$ is Noetherian, if $k$ is greater than or equal to the maximum degree of the generators of $B^*$, then $[B^*]_{k+1} = A[B^*]_k$, and if $k$ is less than or equal to the degree of the generators of $B^*$, then $[B^*]_{k-1} = [B^*]_k$ [2]. Let $B = (b_1, \cdots, b_i)R \subseteq A^e$. Clearly $B' = BT \cap R^* \supseteq (b_1t^e, b_2t^e, \cdots, b_it^e)R^* = B^*$, and for $k \leqq e$, $[B']_k = B \cap A^k = B \supseteq [B^*]_k = [B]_e \supseteq B$. Hence for $k > e$, $[B']_k = B \cap A^k \supseteq [B^*]_k = BA^{k-e}$. Since $B'T = B^*T = BT$, $B^* = B'$ if and only if $u$ is not in any prime divisor of $B^*$. Hence if $(b_1t^e, b_2t^e, \cdots, b_it^e, u)$ is a prime sequence in $R^*$, then $B' = B^*$. In particular, by Lemma 3.4 and Remark 3.5 we have proved the following

COROLLARY 3.6. *Let $R$ be a locally Macaulay ring, let $(a_1, \cdots, a_n)$ be a prime sequence, and let $A = (a_1, \cdots, a_n)R$. Then, for every $e \geqq 1$, $k \geqq e$, and $i = 1, \cdots, n$, $(a_1^e, a_2^e, \cdots, a_i^e)A^{k-e} = (a_1^e, a_2^e, \cdots, a_i^e)R \cap A^k$.*

COROLLARY 3.7. *Let $(a_1, \cdots, a_n)$ be a prime sequence in a locally Macaulay ring $R$. Set $A = (a_1, \cdots, a_n)R$. Then, for all $k \geqq 1$,* (1) *every prime divisor of $A^k$ has height $n$, and* (2) *$A^k : a_1R = A^{k-1}$.*

**Proof.** By Corollary 3.6, $a_1A^{k-1} = a_1R \cap A^k$. Since $a_1$ is not a zero divisor in $R$, $A^{k-1} = a_1A^{k-1} : a_1R = (a_1R \cap A^k) : a_1R = A^k : a_1R$, hence (2) holds. For (1), $u^kR^* \cap R = A^k$, where $R^* = R[ta_1, \cdots, ta_n, u]$, and $k \geqq 1$. Since $R^*$ is a locally Macaulay ring, every prime divisor of $uR^*$ has height one, and the prime divisors of $u^kR^*$ are the prime divisors of $uR^*$ (Remark 3.5). Let $p'$ be a prime divisor of $uR^*$, let $q'$ be a minimal prime divisor of zero in $R^*$ which is contained in $p'$, and let $p = p' \cap R$, $q = q' \cap R$. Applying Remark 2.5 (with $A^* = uR^*$) and the altitude formula for $R^*/q'$ over $R/q$, height $p = n$ (since $\mathrm{trd} R^*/q'/R/q = 1$), so $p$ is a prime divisor of $A^k$. Since $u^kR^* \cap R = A^k$, (1) holds, q.e.d.

If $(a_1,\cdots,a_n)$ is a prime sequence in a locally Macaulay ring $R$, then $(ta_1,\cdots,ta_n,u)$ is a prime sequence in the locally Macaulay ring $R[ta_1,\cdots,ta_n,u]$ (Theorem 3.1 and Lemma 3.4). Theorem 3.8 contains the converse of this.

THEOREM 3.8. *Let $R$ be a Noetherian ring and let $A$ be an ideal in $R$. If the Rees ring $R^*$ of $R$ with respect to $A$ is a locally Macaulay ring (a Macaulay ring), then $R$ is a locally Macaulay ring ($R$ and $R[X]$ are Macaulay rings). If also there are elements $b_1,\cdots,b_n$ in $A$ such that $(b_1t^{e_1},\cdots,b_nt^{e_n},u)$ is a prime sequence in $R^*$, then $(b_1,\cdots,b_n)$ is a prime sequence in $R$.*

**Proof.** Let $R^*$ be a locally Macaulay ring. Then, since $T = R^*[t]$ is a quotient ring of $R^*$, $T$ is a locally Macaulay ring. Let $M$ be a maximal ideal in $R$. Since $T$ is a quotient ring of $R^*$ and of $R[u]$, $T_{MT} = R[u]_{MR[u]}$ is a Macaulay local ring. Since $u$ is transcendental over $R$, a system of parameters in $R_M$ is a system of parameters in $R[u]_{MR[u]}$. It is known that if a local ring has one system of parameters which form a prime sequence, then each system of parameters forms a prime sequence [3, p. 399]. Hence $R$ is a locally Macaulay ring. Therefore, if $(b_1t^{e_1},\cdots,b_nt^{e_n},u)$ is a prime sequence in $R^*$, then, for $i = 1,\cdots,n$, every prime divisor of $(b_1t^{e_1},\cdots,b_it^{e_i})T = (b_1,\cdots,b_i)T$ has height $i$. Hence height $(b_1,\cdots,b_i)R = i$, and so $(b_1,\cdots,b_n)$ is a prime sequence in $R$. Let $R^*$ be a Macaulay ring. By what has already been proved, $R$ and $R[X]$ are locally Macaulay rings. To prove that $R$ is a Macaulay ring, let $M$ be a maximal ideal in $R$. Then $N^* = (M,u-1)T \cap R^*$ is a maximal ideal in $R^*$. Therefore, altitude $R+1 =$ altitude $R^* =$ height $N^* =$ height $N^*T =$ height $M+1$, hence $R$ is a Macaulay ring. Finally, let $N$ be a maximal ideal in $R[u]$. If there is a maximal ideal $N^*$ in $R^*$ such that $N^* \cap R[u] = N$, then altitude $R[u] =$ altitude $R^* =$ height $N^* =$ (since $R^*/N^*$ is a field) height $N^* +$ trd$R^*/N^*/R[u]/N =$ (altitude formula) height $N \leqq$ altitude $R[u]$. If there does not exist such $N^*$, then $NT = T$, hence $u \in N$. Therefore $R/N \cap R = R[u]/N$ is a field, so altitude $R[u] =$ altitude $R + 1 =$ height $N \cap R + 1 =$ height $N^*$. Hence $R[X] \cong R[u]$ is a Macaulay ring, q.e.d.

COROLLARY 3.9. *Let $R$ be a Noetherian ring. If there exists an ideal $A = (a_1,\cdots,a_n)R$ in $R$ such that the Rees ring $R^*$ of $R$ with respect to $A$ is a locally Macaulay ring (a Macaulay ring), then for every non-zero-divisor $a \in A$, $R' = R[a_1/a,\cdots,a_n/a]$ is a locally Macaulay ring (a Macaulay ring).*

**Proof.** Since $(a-u)R[t,u] = (at-1)R[t,u]$ is the kernel of the mapping from $R[t,u]$ onto $R[1/a,a]$ (Lemma 2.1), and since $R[t,u]$ is a quotient ring of $R^*$, to prove the two statements about $R'$ it is sufficient to prove that $u$ is not in any prime divisor of $(ta-1)R^*$. If $u$ is in some (minimal) prime divisor $p$ of $(ta-1)R^*$, then $p$ is a prime divisor of $uR^*$. But $uR^*$ is a graded ideal, hence $p$ is a graded deal. This implies the contradiction $1 \in p$. Therefore $u$ is not in any prime divisor of $(ta-1)R^*$, q.e.d.

Theorem 3.8 is of some interest, since the Rees ring $R^*$ of a locally Macaulay ring $R$ with respect to an ideal $A$ which cannot be generated by a prime sequence may be a locally Macaulay ring. For example, let $R$ be a semi-local Macaulay ring of altitude $n \geqq 2$, and let $(a_1, \cdots, a_n)$ be a prime sequence in the Jacobson radical of $R$. Let $A = (a_1, \cdots, a_n)R$ and fix an integer $e \geqq 2$. Then $A^e$ cannot be generated by $n$ elements, but *the Rees ring of R with respect to $A^e$ is a locally Macaulay ring*. For convenience of notation this will be proved for the case $n = 2$ (the general case being exactly the same). Let $a = a_1$ and $b = a_2$, and let $N$ be a maximal ideal in $R^* = R[ta^e, \cdots, ta^f b^{e-f}, \cdots, tb^e, u]$. If $u \notin N$, then $R_N^*$ contains $T = R[t, u]$. Since $T$ is a locally Macaulay ring, $R_N^*$ is a Macaulay local ring. If $(ta^e, \cdots, ta^f b^{e-f}, \cdots, tb^e)R^*$ is not contained in $N$, say $ta^f b^{e-f} \notin N$. Then $ta^{f+1} b^{e-f-1}/ta^f b^{e-f} = a/b \in R_N^*$ (if $f < e$), and/or $b/a \in R_N^*$ (if $f > 0$). Since $(a, b)$ and $(b, a)$ are prime sequences in $R$, $R_e = R[a/b]$, $R_0 = R[b/a]$, and $R_f = R[a/b, b/a]$ are locally Macaulay rings, and at least one of these rings (call it $R'$) is contained in $R_N^*$. Hence $S = R'[ta^f b^{e-f}]$ is a locally Macaulay ring contained in $R_N^*$, and $S$ contains $R[ta^e, \cdots, ta^g b^{e-g}, \cdots, tb^e]$. Since $ta^f b^{e-f} \notin N' = NR_N^* \cap S$, $u = a^f b^{e-f}/ta^f b^{e-f} \in S_{N'}$. Hence $R_N^* = S_{N'}$ is a locally Macaulay ring. Clearly the only maximal ideals in $R^*$ which contain $(ta^e, \cdots, ta^f b^{e-f}, \cdots, tb^e, u)R^*$ are the ideals $N_i = (M_i, ta^e, \cdots, ta^f b^{e-f}, \cdots, tb^e, u)R^*$, where $M_i$ is a maximal ideal in $R$. Therefore it remains to prove that the semi-local ring $R^*_{R^* - \cup N_i}$ is a Macaulay ring. For this, it will be shown that $(ta^e, tb^e, u)$ is a prime sequence in $R^*$ (since the $N_i$ contain this sequence). Since $(a^e, b^e)$ is a prime sequence in the locally Macaulay ring $R^*[t]$, to prove $(ta^e, tb^e, u)$ is a prime sequence, it is sufficient to prove that $u$ is not in any prime divisor of either of the ideals $ta^e R^*$ or $(ta^e, tb^e)R^*$. This is equivalent to proving $ta^e R^* = a^e T \cap R^*$ and $(ta^e, tb^e)R^* = (a^e, b^e)T \cap R^*$, where $T = R[t, u]$. With the notation used in the proof of Corollary 3.6, these latter equalities are equivalent to $[ta^e R^*]_k = [a^e T \cap R^*]_k$ and $[(ta^e, tb^e)R^*]_k = [(a^e, b^e)T \cap R^*]_k$ for all $k$. Since the degrees of the generators of the four ideals are all non-negative, and since $[ta^e R^*]_0 = [a^e T \cap R^*]_0 = a^e R$ and $[(ta^e, tb^e)R^*]_0 = [(a^e, b^e)T \cap R^*]_0 = (a^e, b^e)R$, it must be shown that $a^e(A^e)^{k-1} = a^e R \cap (A^e)^k$ and $(a^e, b^e)(A^e)^{k-1} = (a^e, b^e)R \cap (A^e)^k$ for all $k \geqq 1$. These equalities hold by Corollary 3.6.

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