TWO NOTES ON LOCALLY MACAULAY RINGS

BY

LOUIS J. RATLIFF, JR.

1. Introduction. In this paper all rings are assumed to be commutative rings with a unit. The undefined terminology used in this paper (height, altitude, etc.) will be the same as that in [1]. Throughout this paper a number of known properties of locally Macaulay rings are stated, and then are used in the remainder of the paper without explicit mention.

In §2 it is proven that if $R$ is a locally Macaulay ring and if $(a_1, \ldots, a_n)$ is a prime sequence in $R$, the kernel of the natural homomorphism from $P = R[X_1, \ldots, X_{n-1}]$ onto $R' = R[a_2/a_1, \ldots, a_n/a_1]$ is $(a_1 X_1 - a_2, a_1 X_2 - a_3, \ldots, a_1 X_{n-1} - a_n)P$ (Lemma 2.3). As a consequence, $R'$ is a locally Macaulay ring and $(a_1, a_2/a_1, \ldots, a_n/a_1)$ is a prime sequence in $R'$ (Theorem 2.4). Further, if $R[X_1]$ is a Macaulay ring, then $R'$ is a Macaulay ring (Theorem 2.8). An example is given to show that the converses are not in general true.

In §3 it is proven that, with the same $R$ and $a_i$, the Rees ring $R^* = R[t a_1, \ldots, t a_n, 1/t]$ (t an indeterminant) of $R$ with respect to $A = (a_1, \ldots, a_n)R$ is a locally Macaulay ring (a Macaulay ring if $R[X_1]$ is) and $(1/t, t a_1, \ldots, t a_n)$ is a prime sequence in $R^*$ (Theorems 3.1 and 3.3). A form of the converses of Theorems 3.1 and 3.3 is true (Theorem 3.8). Also, for every $e \geq 1$, $k \geq e$, and $i = 1, \ldots, n$, $(a_1^e, a_2^e, \ldots, a_i^e) A^{k-e} \cap A$ (Corollary 3.6). Further, for all $k \geq 1$, every prime divisor of $A^k$ has height $n$, and $A^k : a_i R = A^{k-1}$ (Corollary 3.7). It is also proven that if the Rees ring $R^*$ of a Noetherian ring $R$ with respect to an ideal $A = (a_1, \ldots, a_n)R$ is a locally Macaulay ring (a Macaulay ring), then $R' = R[a_1/a, \ldots, a_n/a]$ is a locally Macaulay ring (a Macaulay ring) for every non-zero-divisor $a \in A$ (Corollary 3.9).

2. Transformations of locally Macaulay rings by a prime sequence.

**Lemma 2.1.** Let $R$ be a ring, let $a, b$ be elements in $R$ such that $a$ is not a zero divisor, and let $X$ be an indeterminant. If $aR : bR = aR$, then the kernel $K$ of the natural homomorphism from $R[X]$ onto $R[b/a]$ is generated by $aX - b$.

**Proof.** Clearly $aX - b \in K$. Let $f(X) = r_n X^n + \cdots + r_0 \in K$. Then $r_n b^n + r_{n-1} a b^{n-1} + \cdots + r_0 a^n = 0$, so $r_n \in a R : b^n R = aR$, say $r_n = da$. Since $a$ is not a zero divisor, $g(X) = (da + r_{n-1}) X^{n-1} + r_{n-2} X^{n-2} + \cdots + r_0 \in K$, and $f(X) = (aX - b) d X^{n-1} + g(X)$. Hence, by induction on $n$, $f(X) \in (aX - b) R[X]$, so $K$ is generated by $aX - b$, q.e.d.

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A local (Noetherian) ring $R$ is a *Macaulay local ring* in case there exists a system of parameters $(a_1, \ldots, a_n)$ in $R$ such that $a_i$ is not in any prime divisor of $(a_1, \ldots, a_{i-1})R$ ($i = 1, \ldots, n$). In particular $a_1$ is not a zero divisor. A Noetherian ring $R$ is a *locally Macaulay ring* in case there exists a system of parameters $(a_1, \ldots, a_n)$ in $R$ such that $a_i$ is not in any prime divisor of $(a_1, \ldots, a_{i-1})R$ ($i = 1, \ldots, n$). In particular $a_1$ is not a zero divisor. A Noetherian ring $R$ is a *locally Macaulay ring* in case $R^M$ is a Macaulay local ring for every maximal ideal $M$ in $R$. $R$ is a *Macaulay ring* in case $R$ is a locally Macaulay ring such that height $M = \text{altitude } R$ for every maximal ideal $M$ in $R$. It is known that if $R$ is a Macaulay local ring of altitude $n$ and if $(a_1, \ldots, a_k)$ is a subset of a system of parameters in $R$, then $R/(a_1, \ldots, a_k)R$ is a Macaulay local ring of altitude $n - k$ [3, p. 397]. Also, $R$ is a locally Macaulay ring if and only if the following theorem (the *unmixedness theorem*) holds: If an ideal $A$ in $R$ is generated by $k$ elements and if height $A = k$ ($k \geq 0$), then every prime divisor of $A$ has height $k$ [1, p. 85]. These two facts immediately imply that if $R$ is a locally Macaulay ring (a Macaulay ring) and if $A$ is an ideal in $R$ which is generated by $k$ elements and has height $k$, then $R/A$ is a locally Macaulay ring (a Macaulay ring). Finally, it is known that if $X_1, \ldots, X_n$ are algebraically independent over a Noetherian ring $R$, then $R[X_1, \ldots, X_n]$ is a locally Macaulay ring if and only if $R$ is [1, p. 86].

These facts are used in the proof of

**Corollary 2.2.** Let $R$ be a locally Macaulay ring, and let $a, b$ be elements in $R$ such that $a$ is not a zero divisor and $aR : bR = aR$. Then $R[b/a]$ is a locally Macaulay ring.

**Proof.** $R[X]$ is a locally Macaulay ring, and the kernel of the natural homomorphism from $R[X]$ onto $R[b/a]$ is generated by $aX - b$ (Lemma 2.1). Since $aX - b$ is not a zero divisor in $R[X]$, $R[b/a]$ is a locally Macaulay ring, q.e.d.

Theorem 2.4 below generalizes the above corollary. To obtain the generalization the following definitions and lemma will be used.

An integral domain $R$ satisfies the *altitude formula* in case the following condition holds: If $R'$ is an integral domain which is finitely generated over $R$, and if $p'$ is a prime ideal in $R'$, then height $p' + \text{trd}(R'/p')/(R'/p' \cap R) = \text{height } p' \cap R + \text{trd } R'/R$. It is known that if an integral domain $R$ is a homomorphic image of a locally Macaulay ring, then $R$ satisfies the altitude formula [1, p. 130].

If $R$ is a locally Macaulay ring, and if $p < q$ are prime ideals in $R$, then $R_q$ is a Macaulay local ring [1, p. 86], so height $p + \text{height } q/p = \text{height } q$ [3, p. 399]. This fact will be used in the future without explicit mention.

A sequence $(a_1, \ldots, a_n)$ of nonunits in a Noetherian ring $R$ is a *prime sequence* in case $a_1$ is not a zero divisor, $(a_1, \ldots, a_n)R : a_{i+1}R = (a_1, \ldots, a_i)R$ ($i = 1, \ldots, n - 1$), and $(a_1, \ldots, a_n)R \neq R$. It is known that if $R$ is a semi-local ring, and if $(a_1, \ldots, a_n)$ is a prime sequence of elements in the Jacobson radical of $R$, then $(a_{n_1}, \ldots, a_{n_n})$ is a prime sequence for every permutation $\pi$ of $\{1, \ldots, n\}$ [3, pp. 394–395].

**Lemma 2.3.** Let $R$ be a locally Macaulay ring, let $(a_1, \ldots, a_n)$ be a prime sequence in $R$, and let $X_1, \ldots, X_{n-1}$ be algebraically independent over $R$. Then
the kernel $K$ of the natural homomorphism $\phi$ from $P = R[X_1, \ldots, X_{n-1}]$ onto $R' = R[a_2/a_1, \ldots, a_n/a_1]$ is generated by $(a_1X_1 - a_2, a_1X_2 - a_3, \ldots, a_1X_{n-1} - a_n)$.

**Proof**. The proof is by induction on $n$. The case $n = 1$ is trivial, and Lemma 2.1 proves the case $n = 2$. Let $n > 2$ and assume the conclusion holds for the case $n - 1$. Now $\phi = fg$, where $f$ and $g$ are the natural homomorphisms from $S = R[a_2/a_1, X_2, \ldots, X_{n-1}]$ onto $R'$ and from $P$ onto $S$ respectively. Since the kernel of $g$ is $(a_1X_1 - a_2)P$ (Lemma 2.1), and since $R^* = R[a_2/a_1]$ is a locally Macaulay ring (Corollary 2.2), it is sufficient (by induction) to prove that $(a_1, a_3, a_4, \ldots, a_n)$ is a prime sequence in $R^*$. Since $R$ and $R^*$ have the same total quotient ring, $a_1$ is not a zero divisor in $R^*$, hence height $a_1R^* = 1$. Let $A^* = (a_1X_1 - a_2, a_1X_2 - a_3, \ldots, a_1X_{n-1} - a_n)P$ $(i \geq 3)$. Then $A^* = (a_1, a_2, a_3, \ldots, a_i)P$, hence height $A^* = i$. Consequently, by the unmixedness theorem $(a_1X_1 - a_2, a_1X_2 - a_3, \ldots, a_n)$ is a prime sequence in $P$, hence $(a_1, a_3, \ldots, a_n)$ is a prime sequence in $R^*$, q.e.d.

**Theorem 2.4.** With the same notation as Lemma 2.3, $R'_i = R[a_2/a_1, \ldots, a_i/a_1]$ $(2 \leq i \leq n)$ is a locally Macaulay ring, and $(a_1, a_{i+1}, \ldots, a_i + b_1, b_2, \ldots, b_k)$ is a prime sequence in $R'_i$, where $\{b_1, \ldots, b_k\}$ is a subset of $\{a_2/a_1, \ldots, a_i/a_1\}$, and $0 \leq j \leq n - i$. (For $j = 0$ the sequence is $(a_1, b_1, \ldots, b_k)$.)

**Proof.** That $R'_i$ is a locally Macaulay ring follows immediately from Lemma 2.3 and the remarks preceding the proof of Corollary 2.2. Let $A^* = (a_1, a_{i+1}, \ldots, a_i + b_1, b_2, \ldots, b_k)R'_i$. Since $(a_1X_1 - a_2, \ldots, a_iX_{i-1} - a_iA_{i-1})R[X_1, \ldots, X_{i-1}]$ is generated by $(a_1, \ldots, a_i)$, $A^*$ is a proper ideal. Hence by the unmixedness theorem, since $j$ and $k$ are arbitrary, it is sufficient to prove height $A^* = j + k + 1$. Let $p'$ be a minimal prime divisor of $A^*$, let $q'$ be a (minimal) prime divisor of zero in $R'_i$ such that $q' \subset p'$ and let $p = p' \cap R$, $q = q' \cap R$. By the altitude formula (for $R'/q'$ over $R/q$), height $p'/q' + \text{trd} R'/p'/R/p = \text{height} p/q$ (since $a_i \notin q$). Also, height $p'/q' \leq j + k + 1$, trd $R'/p'/R/p \leq i - 1 - k$, and height $p/q = \text{height} p \geq i + j$. Hence, height $p' = \text{height} p'/q' = j + k + 1$. Therefore height $A^* = j + k + 1$, q.e.d.

**Remark 2.5.** The last step in the proof of Theorem 2.4 shows the following results. For every (minimal) prime divisor $p'$ of $A^*$ and for every prime divisor $q'$ of zero contained in $p'$, $p'/q'$ is a minimal prime divisor of $(A^* + q')/q'$. Since height $p'/q' = j + k + 1$, none of the elements $a_1, \ldots, a_i + b_1, \ldots, b_k$ are in $q'$. Also the elements $a_2/a_1, \ldots, a_i/a_1$ which are not in $p'$ are such that their $p'$ residues are algebraically independent over $R/(p' \cap R)$.

**Remark 2.6.** In Theorem 2.4, if every permutation of $(a_1, \ldots, a_n)$ is a prime sequence in $R$ (for example, if $R$ is a semi-local locally Macaulay ring and $a_1, \ldots, a_n$ are in the Jacobson radical of $R$), then every permutation of $(a_1, a_{i+1}, \ldots, a_n, a_2/a_1, \ldots, a_i/a_1)$ is a prime sequence in $R'_i$.

(1) The author is indebted to the referee for the following proof which is considerably simpler than the author's original proof, and which leads to a more direct proof of Theorem 2.4.
Proof. Let \((c_1, \ldots, c_n)\) be a permutation of \((a_1, a_{i+1}, \ldots, a_n, a_2/a_1, \ldots, a_i/a_1)\). Since no \(a_i\) is a zero divisor in \(R\), \(c_i\) is not a zero divisor in \(R'_i\). Also \((c_1, \ldots, c_n)R'_i \neq R'_i\). Therefore, by the unmixedness theorem, it remains to prove \(\text{height } (c_1, \ldots, c_n) R'_i = h\) \((h = 2, \ldots, n - 1)\). Let \(p'\) be a minimal prime divisor of \((c_1, \ldots, c_n) R'_i\), let \(q'\) be a prime divisor of zero in \(R\) which is contained in \(p'\), and let \(p = p' \cap R, q = q' \cap R\). If \(a_1 \notin p'\), then \(\text{trd } R'/p'/R/p = 0\). Hence by the altitude formula \(\text{height } p'/q' = \text{height } p/q\). Now \(\text{height } p' \leq h\) and \(\text{height } p \geq h\) \((\text{by the assumption on } (a_1, \ldots, a_n))\), so \(\text{height } p' = \text{height } p = h\). If \(a_1 \in p'\), let \(k\) of the elements \(c_1, \ldots, c_n\) be in \(\{a_2/a_1, \ldots, a_i/a_1\}\). Then \(\text{height } p \geq i + (h - 1 - k)\) \((\text{by the assumption on } (a_1, \ldots, a_n))\), and \(\text{trd } R'/p'/R/p \leq i - 1 - k\). By the altitude formula for \(R'/q'\) over \(R/q\), \(\text{height } p' = \text{height } p'/q' = h\), q.e.d.

Remark 2.6 is of some interest because of the following

**Lemma 2.7.** Let \(R\) be a locally Macaulay ring, and let \((a_1, \ldots, a_n)\) be a prime sequence in \(R\) such that every permutation of \((a_1, \ldots, a_n)\) is a prime sequence in \(R\). Let \(A = (a_1, \ldots, a_n)R\). Then, for all \(k \geq 1\), \((1)\) every prime divisor of \(A^k\) has height \(n\), and \((2)\) \(A^k: a_iR = A^{k-1}\) \((i = 1, \ldots, n)\).

**Proof.** This can be proved in the same way as Lemmas 5 and 6 in \([3, \text{pp. 401–402}]\). Without assuming that every permutation of \((a_1, \ldots, a_n)\) is a prime sequence in \(R\), Corollary 3.7 below proves \((1)\) is still true and \((2)'\) \(A^k: a_iR = A^{k-1}\) \((\text{for all } k \geq 1)\), q.e.d.

It is known that if \(R\) is a Macaulay ring and if \(X_1, \ldots, X_n\) are algebraically independent over \(R\), then \(R[X_1, \ldots, X_n]\) is a Macaulay ring if and only if there does not exist an ideal \(p\) in \(R\) such that \(R/p\) is a semi-local integral domain of altitude one \([1, \text{p. 87}]\). Hence if \(R[X_1]\) is a Macaulay ring, then \(R[X_1, \ldots, X_n]\) is a Macaulay ring. This fact is used in the proof of the next theorem.

**Theorem 2.8.** If \(R\) and \(R[X]\) are Macaulay rings \((X\mbox{ transcendental over } R)\), and if \((a_1, \ldots, a_n)\) is a prime sequence in \(R\), then \(R' = R[a_2/a_1, \ldots, a_n/a_1]\) is a Macaulay ring.

**Proof.** The kernel \(K\) of the natural homomorphisms from \(P = R[X_1, \ldots, X_n]\) onto \(R'\) has height \(n - 1\). Since \(P\) is a Macaulay ring, if \(M\) is a maximal ideal in \(P\) which contains \(K\), then altitude \(R + n - 1 = \text{altitude } P = \text{height } M = \text{height } M/K + \text{height } K\). Hence, if \(M'\) is a maximal ideal in \(R'\), then \(\text{height } M' = \text{altitude } P - \text{height } K = \text{altitude } R\). Since \(R'\) is a locally Macaulay ring by Theorem 2.4, \(R'\) is a Macaulay ring, q.e.d.

**Remark 2.9.** If \(R\) is a locally Macaulay ring \((\mbox{a Macaulay ring such that } R[X]\mbox{ is a Macaulay ring)}\), and if \((a_1, \ldots, a_n)\) is a prime sequence in \(R\), then \(R[a_1/a, \ldots, a_n/a]\) is a locally Macaulay ring \((\mbox{a Macaulay ring)}\) for every non-zero-divisor \(a \in (a_1, \ldots, a_n)R\). This follows from Theorems 3.1 and 3.3 and Corollary 3.9 below.

It will now be shown that the converses of Theorems 2.4 and 2.8 are not in
general true. Let $S = k[X, Y]$, where $k$ is a field and $X$ and $Y$ are algebraically independent over $k$. Let $P = (X - 1, Y)S$, $R_1 = SP$, and $N_1 = PR_1$. Let $Q = (X, Y)S$, $R_2 = SQ$, and $N_2 = QR_2$. Let $R' = R_1 \cap R_2$, $M_1 = N_1 \cap R'$, and $M_2 = N_2 \cap R'$. Further let $R = k + (M_1 \cap M_2)$, and let $M = (M_1 \cap M_2)R$. Then $R'$ is the intersection of two regular local rings, hence $R'$ is normal. The following statements are easily verified: (1) $M_1$ and $M_2$ are the maximal ideals in $R'$, and $R_{u_i} = R_i$ is Noetherian $(i = 1, 2)$. Therefore $R'$ is Noetherian [1, p. 203], so $R'$ is a normal semi-local Macaulay domain. (2) Since $R'/M_i = k$ $(i = 1, 2)$, $R$ is a local domain and $R'$ is its derived normal ring [1, p. 204]. (3) $XY, Y \in R$, $X \notin R$, $R' = R[XY/Y]$, and $(Y, X = XY/Y)$ is a prime sequence in $R'$. (4) If $p$ is a height one prime ideal in $R$, then $R_p$ is a regular local ring. Since $R \neq R'$, $M$ is an imbedded prime divisor of every nonzero element in $M$ [1, p. 41], hence $R$ is not a Macaulay domain.

3. The Rees ring of a locally Macaulay ring. Let $R$ be a Noetherian ring, let $A = (a_1, \ldots, a_n)R$ be an ideal in $R$, let $t$ be an indeterminant, and set $u = t^{-1}$. The graded Noetherian ring $R^* = R[t a_1, \ldots, t a_n, u]$ is called the Rees ring of $R$ with respect to $A$.

**Theorem 3.1.** Let $R$ be a locally Macaulay ring, and let $a_1, \ldots, a_n$ be a prime sequence in $R$. Then the Rees ring $R^*_i$ of $R$ with respect to $(a_1, \ldots, a_i)R$ $(1 \leq i \leq n)$ is a locally Macaulay ring, and $(u, a_{i+1}, \ldots, a_{i+j}, b_1, \ldots, b_k)$ is a prime sequence in $R^*_i$, where $\{b_1, \ldots, b_k\}$ is a subset of $\{ta_1, \ldots, ta_i\}$ and $0 \leq j \leq n - i$. (For $j = 0$ the sequence is $(u, b_1, \ldots, b_k)$.)

**Proof.** Since $u$ is transcendental over $R$, $R[u]$ is a locally Macaulay ring, hence $(u, a_1, \ldots, a_n)$ is a prime sequence in $R[u]$. Since $ta_j = a_j/u$, $R^*_i$ is a locally Macaulay ring and $(u, a_{i+1}, \ldots, a_{i+j}, b_1, \ldots, b_k)$ is a prime sequence in $R^*_i$ by Theorem 2.4, q.e.d.

**Remark 3.2.** In Theorem 3.1, if every permutation of $(a_1, \ldots, a_n)$ is a prime sequence in $R$, then every permutation of $(u, a_{i+1}, \ldots, a_n, ta_1, \ldots, ta_i)$ is a prime sequence in $R^*_i$.

**Theorem 3.3.** If $R$ and $R[X]$ are Macaulay rings ($X$ transcendental over $R$), and if $a_1, \ldots, a_n$ is a prime sequence in $R$, then the Rees ring $R^*$ of $R$ with respect to $(a_1, \ldots, a_n)$ is a Macaulay ring.

**Proof.** Considering the natural homomorphism from $R[u, X_1, \ldots, X_n]$ onto $R^*$ and the ideal $(u, a_1, \ldots, a_n)$ of $R^*$, the proof is the same as the proof of Theorem 2.8, q.e.d.

**Lemma 3.4.** Let $R^*$ be the Rees ring of a locally Macaulay ring $R$ with respect to a prime sequence $(a_1, \ldots, a_n)$ in $R$. Then $(ta_1, \ldots, ta_i, u)$ is a prime sequence in $R^*$ $(i = 1, \ldots, n)$.
Proof. Since $R^*$ is a locally Macaulay ring and height $(u, ta_1, \cdots, ta_n)R^* = i + 1$ (Theorem 3.1), it is sufficient to prove height $(ta_1, \cdots, ta_n)R^* = i$. Let $p$ be a minimal prime divisor of $A_*^i = (ta_1, \cdots, ta_n)R^*$. Then height $p \leq i$, hence $u \notin p$. Let $T = R[u, i]$, so $T$ is a quotient ring of $R^*$. Since $pT$ is a minimal prime divisor of $A_*^i T = (a_1, \cdots, a_i)T$, and since height $(a_1, \cdots, a_i)R[u] = i$, height $A_*^i T = i$. Therefore height $p = i$, so height $A_*^i = i$, q.e.d.

Remark 3.5. Let $(a_1, \cdots, a_n)$ be a prime sequence in a locally Macaulay ring $R$. Then the radical of $(a_1, \cdots, a_n)R$ is the radical of $(a_1^{e_1}, a_2^{e_2}, \cdots, a_n^{e_n})R$ ($e_i \geq 1, i = 1, \cdots, n$). Hence, by the unmixedness theorem, $(a_1^{e_1}, a_2^{e_2}, \cdots, a_n^{e_n})$ is a prime sequence in $R$. Therefore $R[a_2, a_3, \cdots, a_n/a_1^{f_1}]$ and $R[t a_1, \cdots, t a_n, u]$ are locally Macaulay rings.

Let $R$ be a Noetherian ring and let $R^*$ be the Rees ring of $R$ with respect to an ideal $A$ in $R$. Let $T = R[t, u]$, so $T$ is a quotient ring of $R^*$. For any ideal $B$ in $R$ let $B' = BT \cap R^*$. For any homogeneous ideal $B^*$ in $R^*$ let $[B^*]_k$ be the set of elements $r \in R$ such that $rt^k \in B^*$. It is immediately seen that $[B^*]_k$ is an ideal in $R$ and $A^k \supseteq [B^*]_k \supseteq [B^*]_{k+1} \supseteq A^*[B^*]_k$ for all integers $k$ (with the convention that $A^k = R$ if $k \leq 0$). Also, since $R^*$ is Noetherian, if $k$ is greater than or equal to the maximum degree of the generators of $B^*$, then $[B^*]_{k+1} = A^*[B^*]_k$, and if $k$ is less than or equal to the degree of the generators of $B^*$, then $[B^*]_{k-1} = [B^*]_k [2]$. Let $B = (b_1, \cdots, b_k)R \subseteq A^*$. Clearly $B^* = BT \cap R^* \supseteq (b_1 t^*, b_2 t^*, \cdots, b_k t^*)R^* = B^*$, and for $k \leq e$, $[B^*]_k = B \cap A^k = B \supseteq [B^*]_k = [B^*]_k \supseteq B$. Hence for $k > e$, $[B^*]_k = B \cap A^k \supseteq [B^*]_k = BA^{k-e}$. Since $B^* = B^*T = BT$, $B^* = B'$ if and only if $u$ is not in any prime divisor of $B^*$. Hence if $(b_1 t^*, b_2 t^*, \cdots, b_k t^*, u)$ is a prime sequence in $R^*$, then $B^* = B'$. In particular, by Lemma 3.4 and Remark 3.5 we have proved the following

Corollary 3.6. Let $R$ be a locally Macaulay ring, let $(a_1, \cdots, a_n)$ be a prime sequence, and let $A = (a_1, \cdots, a_n)R$. Then, for every $e \geq 1$, $k \geq e$, and $i = 1, \cdots, n$, $(a_1^{e_1}, a_2^{e_2}, \cdots, a_n^{e_n})A^{k-e} = (a_1^{e_1}, a_2^{e_2}, \cdots, a_n^{e_n})R \cap A^k$.

Corollary 3.7. Let $(a_1, \cdots, a_n)$ be a prime sequence in a locally Macaulay ring $R$. Set $A = (a_1, \cdots, a_n)R$. Then, for all $k \geq 1$, (1) every prime divisor of $A^k$ has height $n$, and (2) $A^k : a_1 R = A^{k-1}$.

Proof. By Corollary 3.6, $a_1 A^{k-1} = a_1 R \cap A^k$. Since $a_1$ is not a zero divisor in $R$, $A^{k-1} = a_1 A^{k-1}$; $a_1 R = (a_1 R \cap A^k)$; $a_1 R = A^k : a_1 R$, hence (2) holds. For (1), $u^k R^* \cap R = A^k$, where $R^* = R[t a_1, \cdots, t a_n, u]$, and $k \geq 1$. Since $R^*$ is a locally Macaulay ring, every prime divisor of $uR^*$ has height one, and the prime divisors of $u^k R^*$ are the prime divisors of $uR^*$ (Remark 3.5). Let $p'$ be a prime divisor of $uR^*$, let $q'$ be a minimal prime divisor of zero in $R^*$ which is contained in $p'$, and let $p = p' \cap R$, $q = q' \cap R$. Applying Remark 2.5 (with $A^* = uR^*$) and the altitude formula for $R^*[q'/q]$ over $R/q$, height $p = n$ (since $\text{trd} R^*[q'/q] = 1$), so $p$ is a prime divisor of $A^k$. Since $u^k R^* \cap R = A^k$, (1) holds, q.e.d.
If \((a_1, \ldots, a_n)\) is a prime sequence in a locally Macaulay ring \(R\), then \((ta_1, \ldots, ta_n, u)\) is a prime sequence in the locally Macaulay ring \(R[t, a, u]\) (Theorem 3.1 and Lemma 3.4). Theorem 3.8 contains the converse of this.

**Theorem 3.8.** Let \(R\) be a Noetherian ring and let \(A\) be an ideal in \(R\). If the Rees ring \(R^*\) of \(R\) with respect to \(A\) is a locally Macaulay ring (a Macaulay ring), then \(R\) is a locally Macaulay ring (\(R\) and \(R[X]\) are Macaulay rings). If also there are elements \(b_1, \ldots, b_n\) in \(A\) such that \((b_1t^{e_1}, \ldots, b_nt^{e_n}, u)\) is a prime sequence in \(R^*\), then \((b_1, \ldots, b_n)\) is a prime sequence in \(R\).

**Proof.** Let \(R^*\) be a locally Macaulay ring. Then, since \(T = R^*[t]\) is a quotient ring of \(R^*\), \(T\) is a locally Macaulay ring. Let \(M\) be a maximal ideal in \(R\). Since \(T\) is a quotient ring of \(R^*\) and of \(R[\{u\}]\), \(T_M = R[\{u\}]_{M \cap \{u\}}\) is a Macaulay local ring. Since \(u\) is transcendental over \(R\), a system of parameters in \(R^*\) is a system of parameters in \(R[\{u\}]_{M \cap \{u\}}\). It is known that if a local ring has one system of parameters which form a prime sequence, then each system of parameters forms a prime sequence [3, p. 399]. Hence \(R\) is a locally Macaulay ring. Therefore, if \((b_1t^{e_1}, \ldots, b_nt^{e_n}, u)\) is a prime sequence in \(R^*\), then, for \(i = 1, \ldots, n\), every prime divisor of \((b_1t^{e_1}, \ldots, b_nt^{e_n})T = (b_1, \ldots, b_n)T\) has height \(i\). Hence height \((b_1, \ldots, b_n)R = i\), and so \((b_1, \ldots, b_n)\) is a prime sequence in \(R\). Let \(R^*\) be a Macaulay ring. By what has already been proved, \(R\) and \(R[X]\) are locally Macaulay rings. To prove that \(R\) is a Macaulay ring, let \(M\) be a maximal ideal in \(R\). Then \(N^* = (M, u - 1)T \cap R^*\) is a maximal ideal in \(R^*\). Therefore, altitude \(R + 1 = \text{altitude } R^* = \text{height } N^* = \text{height } N^*T = \text{height } M + 1\), hence \(R\) is a Macaulay ring. Finally, let \(N\) be a maximal ideal in \(R[\{u\}]\). If there is a maximal ideal \(N^*\) in \(R^*\) such that \(N^* \cap \{u\} = N\), then altitude \(R[\{u\}] = \text{altitude } R^* = \text{height } N^* = (\text{since } R^*/N^*\text{ is a field}) \text{ height } N^* + \text{trd } R^*/N^*/R[\{u\}] = (\text{altitude formula}) \text{ height } N \leq \text{ altitude } R[\{u\}].\) If there does not exist such \(N^*\), then \(NT = T\), hence \(u \in N\). Therefore \(R/N \cap R = R[\{u\}]/N\) is a field, so altitude \(R[\{u\}] = \text{altitude } R + 1 = \text{height } N \cap R + 1 = \text{height } N^*\). Hence \(R[X] \cong R[\{u\}]\) is a Macaulay ring, q.e.d.

**Corollary 3.9.** Let \(R\) be a Noetherian ring. If there exists an ideal \(A = (a_1, \ldots, a_n)R\) in \(R\) such that the Rees ring \(R^*\) of \(R\) with respect to \(A\) is a locally Macaulay ring (a Macaulay ring), then for every non-zero-divisor \(a \in A\), \(R' = R[\{a\}/a, \ldots, a_n/a]\) is a locally Macaulay ring (a Macaulay ring).

**Proof.** Since \((a - u)R[t, u] = (at - 1)R[t, u]\) is the kernel of the mapping from \(R[t, u]\) onto \(R[1/a, a]\) (Lemma 2.1), and since \(R[t, u]\) is a quotient ring of \(R^*\), to prove the two statements about \(R'\) it is sufficient to prove that \(u\) is not in any prime divisor of \((ta - 1)R^*\). If \(u\) is in some (minimal) prime divisor \(p\) of \((ta - 1)R^*\), then \(p\) is a prime divisor of \(uR^*\). But \(uR^*\) is a graded ideal, hence \(p\) is a graded ideal. This implies the contradiction \(1 \in p\). Therefore \(u\) is not in any prime divisor of \((ta - 1)R^*\), q.e.d.
Theorem 3.8 is of some interest, since the Rees ring $R^*$ of a locally Macaulay ring $R$ with respect to an ideal $A$ which cannot be generated by a prime sequence may be a locally Macaulay ring. For example, let $R$ be a semi-local Macaulay ring of altitude $n \geq 2$, and let $(a_1, \ldots, a_n)$ be a prime sequence in the Jacobson radical of $R$. Let $A = (a_1, \ldots, a_n)R$ and fix an integer $e \geq 2$. Then $A^e$ cannot be generated by $n$ elements, but the Rees ring of $R$ with respect to $A^e$ is a locally Macaulay ring. For convenience of notation this will be proved for the case $n = 2$ (the general case being exactly the same). Let $a = a_1$ and $b = a_2$, and let $N$ be a maximal ideal in $R^* = R[t^e, \ldots, t^e b^{-f}, \ldots, t^e b^e, u]$. If $u \notin N$, then $R^*_N$ contains $T = R[t, u]$. Since $T$ is a locally Macaulay ring, $R^*_N$ is a Macaulay local ring. If $(t^e, \ldots, t^e b^{-f}, \ldots, t^e b^e)R^*$ is not contained in $N$, say $t^e b^{-f} \notin N$. Then $t^e b^{-f-1} t^e b^{-f-1} = a/b \in R^*_N$ (if $f < e$), and/or $b/a \in R^*_N$ (if $f > 0$). Since $(a, b)$ and $(b, a)$ are prime sequences in $R$, $R_a = R[a/b]$, $R_0 = R[b/a]$, and $R_f = R[a/b, b/a]$ are locally Macaulay rings, and at least one of these rings (call it $R_f$) is contained in $R^*_N$. Hence $S = R'[t^e b^{-f}]$ is a locally Macaulay ring contained in $R^*_N$, and $S$ contains $R[t^e, \ldots, t^e b^{-f}, \ldots, t^e b^e]$. Since $t^e b^{-f} \notin N = NR^*_N \cap S$, $u = a^e b^{-f} t^e b^{-f} \in S_R$. Hence $R^*_N = S_R$ is a locally Macaulay ring. Clearly the only maximal ideals in $R^*$ which contain $(t^e, t^e b, u)$ are prime sequences in $R^*$ (since the $N_i$ contain this sequence). Since $(a^e, b^e)$ is a prime sequence in the locally Macaulay ring $R^*[t]$, to prove $(t^e, t^e b, u)$ is a prime sequence, it is sufficient to prove that $u$ is not in any prime divisor of either of the ideals $t^e R^*$ or $(t^e, t^e b) R^*$. This is equivalent to proving $t^e R^* = a^e T \cap R^*$ and $(t^e, t^e b) R^* = (a^e, b^e) T \cap R^*$, where $T = R[t, u]$. With the notation used in the proof of Corollary 3.6, these latter equalities are equivalent to $[a^e R^*]_k = [a^e T \cap R^*]_k$ and $[(t^e, t^e b) R^*]_k = [(a^e, b^e) T \cap R^*]_k$ for all $k$. Since the degrees of the generators of the four ideals are all non-negative, and since $[a^e R^*]_0 = [a^e T \cap R^*]_0 = a^e R$ and $[(t^e, t^e b) R^*]_0 = [(a^e, b^e) T \cap R^*]_0 = (a^e, b^e) R$, it must be shown that $a^e (A^e)^{k-1} = a^e R \cap (A^e)^k$ and $(a^e, b^e) (A^e)^k = (a^e, b^e) R \cap (A^e)^k$ for all $k \geq 1$. These equalities hold by Corollary 3.6.

**References**


**University of California, Riverside, California**