ESSENTIAL CLUSTER SETS(1)

BY
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1. In 1928, W.H. Young [6] showed that an arbitrary real function of a real variable has a remarkable symmetry property. Let $f: R \to R$ be arbitrary. A number y is a right limit of f at $x \in R$ if there is a sequence $x_n > x$, $n = 1, \cdots$ such that $\lim x_n = x$ and $\lim f(x_n) = y$. A left limit is defined similarly. The theorem of Young asserts that for every f the set of right limits is the same as the set of left limits for every $x \in R - A$, where A is a countable set. Young obtained an analogous result for functions of two variables. Here the role of right and left limits is assumed by limits within sectors with the point in question as vertex. Each sector yields a limit set. The result is that all the sectorial limit sets are the same at every $p \in R^2 - A$, where A is of planar measure zero and first category, but not necessarily countable.

In 1938, H. Blumberg [2] obtained a variety of results along the lines of those obtained by Young. He pointed out, for example, that in the planar case, if the angles of the sectors considered exceed 180°, then the countable character of the exceptional set A is restored.

Returning to the original theorem of Young, Zahorski asked if the result still holds if sets of density zero are neglected. L. Belowska [1] has shown that this is false by constructing a function where in this case the exceptional set is uncountable. However, M. Kulbacka [4] has shown that the exceptional set is always of the first category and measure zero. A simple proof of these results was given by C. Goffman [3].

It is natural to ask whether there are theorems similar to the ones of Young and Blumberg in higher dimensions, but with sets of density zero neglected. This paper gives an affirmative answer to this question by showing that a theorem analogous to the two dimensional result of Young holds if one disregards sets of density zero. In this connection, various examples are given relating to the structure of the exceptional set.

2. Let $f: R^2 \to R$ be arbitrary and let $p \in R^2$. If σ is a sector with vertex at the origin, then we denote by σ_p the image of σ under the translation taking the origin into p.

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By the essential cluster set of f at p, we mean the set of all real numbers w for which the set $f^{-1}((w-\varepsilon,\ w+\varepsilon))$ has positive, upper exterior density at p, for each $\varepsilon > 0$. The set of all real numbers w for which the set

$$\sigma_p \cap f^{-1}((w-\varepsilon, w+\varepsilon))$$

has positive, upper, exterior density at p for each $\varepsilon > 0$ is the essential sectorial cluster set of f at p in σ_p. These sets we designate by $C_e(f,p)$ and $C_e(f,p,\sigma_p)$, respectively. We are interested in the set

$$c_e(f,p) = \cap\,[C_e(f,p,\sigma_p)\colon \sigma_p].$$

Clearly, $c_e(f,p) \subset C_e(f,p)$ for each p.

We shall show that these two sets are actually equal except at a set of points which is both of the first category and measure zero, but may be uncountable.

THEOREM. *The set* $E = \{p\colon C_e(f,p) \neq c_e(f,p)\}$ *is of the first category and measure zero.*

The proof will follow from:

LEMMA. *If* σ *is a sector at the origin, then the set*

$$A = \{p\colon C_e(f,p) \neq C_e(f,p,\sigma_p)\}$$

is of the first category and measure zero.

Proof. Since, for each $p, C_e(f,p,\sigma_p) \subset C_e(f,p)$, we need only show that the set

$$\{p\colon C_e(f,p) \not\subset C_e(f,p,\sigma_p)\}$$

is of the first category and measure zero. We observe that if $p \in A$ then there are $\varepsilon > 0$ and $w \in R$ such that

$$\bar{D}_p[f^{-1}((w-\varepsilon,\ w+\varepsilon))] > 0$$

and

$$D_p[\sigma_p \cap f^{-1}((w-\varepsilon,\ w+\varepsilon))] = 0,$$

where the symbols $\bar{D}_p$ and D_p denote upper exterior and exterior densities, respectively. Thus, if $p \in A$, there are rational numbers r and $s, r < s$, such that

$$\bar{D}_p[f^{-1}((r,s))] > 0$$

and

$$D_p[f^{-1}((r,s)) \cap \sigma_p] = 0.$$

Let for r and s rational, $r < s$,

$$A_{r,s} = \{p\colon \bar{D}_p[f^{-1}((r,s))] > 0$$

and

$$D_p[\sigma_p \cap f^{-1}((r,s))] = 0\}.$$

A routine calculation shows that

$$A = \bigcup_{r,s} A_{r,s}.$$

In order to prove the lemma, it now suffices to show that if $S \subset R^2$ is any set, the set T of points p for which

$$\bar{D}_p(S) > 0 \text{ and } D_p(\sigma_p \cap S) = 0$$

is of the first category and measure zero.

It will be necessary to consider the set $\sigma_p \cap \bar{\cup}\,(p,\delta)$ where $\bar{\cup}\,(p,\delta)$ is the closed disc with center p and radius $\delta > 0$. This set we shall designate by $\sigma_{p,\delta}$. We now let, for every pair of natural numbers μ and ν.

$$E_{\mu,\nu} = \left\{ p: \bar{D}_p(S) > \frac{1}{\nu} \text{ and } \frac{m_e(S \cap \sigma_{p,\delta})}{m_e(\sigma_{p,\delta})} < \frac{1}{2\nu} \text{ if } 0 < m_e(\sigma_{p,\delta}) < \frac{1}{\mu} \right\}.$$

Clearly, $T \subset \bigcup_{\mu,\nu} E_{\mu,\nu}$.

We show that the sets $E_{\mu,\nu}$ are nowhere dense.

Suppose an $E_{\mu,\nu}$ is dense in some open disc I with $m(I) < 1/\mu$. Then for each $p \in I$ and $\delta > 0$ there is a sequence $\{p_n\} \subset E_{\mu,\nu}$ such that $\lim p_n = p$ and

$$\frac{m_e(S \cap \sigma_{p_n,\delta})}{m_e(\sigma_{p_n,\delta})} < \frac{1}{2\nu}, \quad n = 1, 2, \cdots$$

whenever $\sigma_{p_n,\delta} \subset I$ and $\sigma_{p,\delta} \subset I$. Furthermore, $F(p) = m_e(S \cap \sigma_{p,\delta}) / m_e(\sigma_{p,\delta})$ is a continuous function of p on the set of points p for which $\sigma_{p,\delta} \subset I$. It follows that $m_e(S \cap \sigma_{p,\delta}) / m_e(\sigma_{p,\delta}) \leqq 1/2\nu$ whenever $\sigma_{p,\delta} \subset I$. Let $J \subset I$ be any open disc. The collection of sets

$$\{\sigma_{p,\eta} : \sigma_{p,\eta} \subset J, \eta > 0\},$$

covers J in the sense of Vitali [5]. Hence, by the Vitali Covering Theorem there exist σ_{p_i,η_i}; $i = 1, 2, \cdots n$ all contained in J, with

$$\sigma_{p_i,\eta_i} \cap \sigma_{p_j,\eta_j} = \varnothing, \quad i \neq j,$$

and

$$J = Q \cup \left(\bigcup_{i=1}^{n} \sigma_{p_i,\eta_i} \right),$$

where

$$m_e(Q) < \frac{m_e(J)}{2\nu}.$$

It follows that

$$\begin{aligned} m_e(S \cap J) &= m_e(S \cap Q) + \sum_{i=1}^{n} m_e(S \cap \sigma_{p_i,\eta_i}) \\ &\leqq m_e(Q) + \sum_{i=1}^{n} m_e(\sigma_{p_i,\delta_i}) \cdot \frac{1}{2v} \\ &< \frac{m_e(J)}{2v} + \frac{m_e(J)}{2v} = \frac{m_e(J)}{v}. \end{aligned}$$

Since, for every open disc $J \subset I$, we have $m_e(S \cap J)/m_e(J) < 1/v$, it follows that $\bar{D}_p(S) \leqq 1/v$ for all $p \in I$.

This contradicts the assumption that $E_{\mu,v}$ is dense in I. Thus, $E_{\mu,v}$ is nowhere dense and $\bigcup_{\mu,v} E_{\mu,v}$ is of the first category from which it follows that $T \subset \bigcup_{\mu,v} E_{\mu,v}$ is also of the first category.

That T is of measure zero follows from the fact that T is a subset of the set of points at which the exterior metric density of S does not exist and this set has measure zero by the Lebesgue Density Theorem.

Returning to the proof of the theorem, we consider a fixed, countable, dense collection of sectors at the origin. For example, we may choose those sectors whose sides have rational slopes. We order these

$$\sigma^1, \sigma^2, \cdots, \sigma^n, \cdots.$$

We observe that

$$\begin{aligned} E &= \{p: C_e(f,p) \neq c_e(f,p)\} \\ &= \{p: C_e(f,p) \neq \cap [C_e(f,p,\sigma_p^n): n = 1,2,\cdots]\} \\ &\subset \bigcup_{n=1}^{\infty} \{p: C_e(f,p) \neq C_e(f,p,\sigma_p^n)\}. \end{aligned}$$

By the lemma the sets

$$\{p: C_e(f,p) \neq C_e(f,p,\sigma_p^n)\}, \quad n = 1,2,\cdots$$

are of the first category and measure zero, from which it follows that E is of the first category and measure zero.

3. Concerning the original problem posed by Zahorski, L. Belowska [1] and C. Goffman [3] have constructed functions where the exceptional sets are uncountable. We would now like to give an example showing that the exceptional set may be uncountable in every neighborhood. We put this in the following form:

PROPOSITION 1. *There is set $S \subset (0,1)$ and a set E such that $E \cap I$ is uncountable for every open interval $I \subset (0,1)$, and the upper right density of S is positive at every $x \in E$ and the left density of S is zero at every $x \in E$.*

Proof. This construction makes use of the basic ideas used by Goffman in [3] where he constructs sets S and E contained in the interval $(0,1)$, with E uncountable and with the property that the upper right density of S is positive at each $x \in E$ and the left density of S is zero at each $x \in E$.

Let $I_0 = (0,1)$ and let $I_0, I_1, I_2, \cdots, I_m, \cdots$ be an enumeration of the open rational intervals in I_0. We designate by S_1 the union of all the subintervals of I_0 of the form

$$\left(\cdot \underbrace{xx\cdots x}_{n}1,\ \cdot \underbrace{xx\cdots x}_{n}1\ \underbrace{00\cdots 0}_{n}1 \right)$$

where $x = 0$ or 2 and $n = 1, 2, \cdots$. (The ternary system of representing numbers in I_0 is being used.)

Assume we already have the sets

$$S_1, S_2, \cdots, S_m.$$

We construct a set S_{m+1} as follows: Suppose $I_m = (a_m, b_m)$.

CASE 1. If $I_m \cap (\bigcup_{j=1}^m S_j) = \varnothing$, let

$$T_m = (A_m - B_m, A_m + B_m)$$

where $A_m = (a_m + b_m)/2$ and $B_m = (b_m - a_m)/3^m$. Then, let T'_m be the union of all subintervals of T_m of the form

$$\left(A_m - B_m + 2B_m\left(\cdot \underbrace{xx\cdots x}_{n}1\right), A_m - B_m + 2B_m\left(\cdot \underbrace{xx\cdots x}_{n}1\ \underbrace{00\cdots 0}_{n}1 \right)\right)$$

where $x = 0$, or 2 and $n = 1, 2, \cdots$. Define

$$S_{m+1} = (S_m - T_m) \cup T'_m.$$

CASE 2. If $I_m \cap (\bigcup_{j=1}^m (S_j) \neq \varnothing$, then

$$I_m \cap \left(\bigcup_{j=1}^m S_j \right) = \bigcup_k I_m^k$$

a countable union of disjoint open intervals. Let $I_m^k = (a_m^k, b_m^k)$. Let k be arbitrary but fixed and let $T_m = [A_m^k - B_m^k, A_m^k + B_m^k]$ where $A_m^k = (a_m^k + b_m^k)/2$ and $B_m^k = (b_m^k - a_m^k)/3^m$. Let T'_m be the union of all subintervals of T_m of the form

$$\left(A_m^k - B_m^k + 2B_m^k\left(\cdot \underbrace{xx\cdots x}_{n}1\right), A_m^k - B_m^k + 2B_m^k\left(\cdot \underbrace{xx\cdots x}_{n}1\ \underbrace{00\cdots 0}_{n}1 \right)\right)$$

where $x = 0$ or 2 and $n = 1, 2, \cdots$. Define

$$S_{m+1} = (S_m - T_m) \cap T'_m .$$

We now have the sequence of sets,

$$S_1, S_2, \cdots, S_m, \cdots .$$

(*) NOTE. If $I \subset S_\nu$ is any component interval of S_ν, then

$$m\left[I \cap \left(\bigcup_{j=\nu}^{\infty} S_j \right) \right] > \frac{m(I)}{2}$$

since at the mth stage, if we remove anything, then it is a portion of I whose measure is always less than $m(I)/3^m$.

We define a set S as follows: A point x is in S if $x \in S_m$ for some m and $x \notin T_n$ for all $n \geqq m-1$.

Let $J \subset (0,1)$ be any interval. We shall show that there is an uncountable set $E \subset J$ at which the upper right density of S is positive at each $x \in E$ and the left density of S is zero at each $x \in E$.

In the enumeration of the open rational intervals,

$$I_1, I_2, \cdots, I_m, \cdots$$

let I_m denote the first interval such that $I_m \subset J$.

We have, by construction, either

1. $I_m \cap (\bigcup_{j=1}^{m} S_j) = \varnothing$, or
2. $I_m \cap (\bigcup_{j=1}^{m} S_j) \neq \varnothing$.

CASE 1. If $I_m \cap (\bigcup_{j=1}^{m} S_j) = \varnothing$, then we recall that

$$T_m = (A_m - B_m, A_m + B_m).$$

We consider the Cantor ternary set relative to the interval T_m and designate by E_m those points whose expansions are of the form

$$A_m - B_m + 2B_m(\cdot X022X0222222X0 \cdots)$$

where $X = 0$ or 2 and after pair $X0$ there are as many 2's as there are digits up to and including the pair $X0$.

The set E_m is uncountable. To complete the proof of the proposition, we need only show that at every $x \in E_m$ the upper right density of S at x is positive and the left density of S at x is zero.

If we apply the note (*) and the argument used in [3], then it readily follows that the upper right density of S is positive at each $x \in E_m$.

To show that the left density of S is zero at each $x \in E_m$, we make the following computation:

Let $x \in E_m$ and let $\varepsilon > 0$. There is an $m_0 > m$ such that

$$\sum_{k > m_0} \frac{1}{3^k} < \frac{\varepsilon}{3}.$$

Let I be any interval of length less than

$$\frac{1}{3}\min[m(I_1),\cdots,m(I_{m_0})]$$

with x as right endpoint. By the construction of S only those intervals I_k for which I_k is to the left of x need be considered. In particular, we need only consider those intervals I_k for which $k > m_0$ and $m(I_k) < 3m(I)$. We observe that for $k > m_0$, the contribution of S_k to S is smaller than

$$\frac{1}{3^k}m(I_k) < \frac{1}{3^k}3m(J).$$

Thus,

$$\begin{aligned} m(S\cap I) &< m\left[S\cap\left(\bigcup_{k>m_0} I_k\right)\right] \\ &\leqq \sum_{k>m_0} m(S\cap I_k) < \sum_{k>m_0}\frac{1}{3}m(I_k) \\ &< 3m(I)\sum_{k>m_0}\frac{1}{3k} < m(I)\cdot\varepsilon. \end{aligned}$$

Since $m(S\cap I)/m(I) < \varepsilon$, it follows that the left density of S at x is zero

CASE 2. If $I_m \cap (\bigcup_{j=1}^m S_j) \neq \varnothing$, then

$$I_m \cap \left(\bigcup_{j=1}^{m} S_j\right) = \bigcup_k I_m^k.$$

As in Case 1, we recall that

$$T_m = (A_m^k - B_m^k, A_m^k + B_m^k)$$

and obtain an uncountable set $E_m \subset T_m$ and apply the same argument.

4. By a slight modification of the previous example we easily obtain an analogous result for the two-dimensional case.

PROPOSITION 2. *There exist* $\tilde{S}\subset(0,1)\times(0,1)$, *sectors* σ *at the origin greater than* 180° *and less than* 180°, *and a set* $\tilde{E}$ *with the property that* $\tilde{E}\cap J$ *is uncountable for every open disc* $J\subset(0,1)\times(0,1)$ *and such that* $\bar{D}_p(\tilde{S}) > 0$ *and* $D(\tilde{S}\cap\sigma_p) = 0$ *for every* $p\in\tilde{E}$.

Proof.

CASE 1. For sectors less than 180° we may take for $\tilde{S}$ and $\tilde{E}$ the sets $S\times(0,1)$ and $E\times(0,1)$, respectively, where S and E are the sets constructed in Proposition 1 with the property that the upper right density of S is positive at every $x \in E$ and the left density of S is zero at every $x \in E$. Then, if σ is any sector at the origin, less than 180°, and symmetric about the positive x-axis, an argument similar to the one used in Proposition 1 establishes the desired result.

CASE 2. For sectors greater than 180° we make a construction similar to the one used in Proposition 1. Let $\tilde{J} = (0,1) \times (0,1)$. The role of the open rational intervals used in Proposition 1 will be assumed by open discs in $\tilde{J}$ whose centers have rational coordinates and whose radii are rational. As in Proposition 1 we obtain an inductive sequence of sets

$$\tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_m, \cdots.$$

We shall only construct the first set $\tilde{S}_1$. Since the remainder of the construction is quite similar to the preceeding one, we omit it. Let l denote the line segment passing through the center of $\tilde{J}$ and parallel to the x-axis. We consider all subintervals of l of the form

$$\left(\cdot \underbrace{xx\cdots x1}_{n}, \cdot \underbrace{xx\cdots x}_{n} \underbrace{100\cdots 01}_{n} \right) \times \left\{ \frac{1}{2} \right\}$$

where $x = 0$ or 2 and $n = 1, 2, \cdots$.

Let τ denote a sector at the origin less than 180° and symmetric about the positive x-axis and let σ denote the sector at the origin complementary to τ. Then $\sigma > 180°$.

Define $\tilde{S}_1$ to be the union of all regions of the form

$$\left[\tau_p, \frac{1}{3^{2n}} \quad : p = \left(\cdot xx\cdots x1, \frac{1}{2} \right), x = 0 \text{ or } 2 \text{ and } n = 1, 2, \cdots \right].$$

After obtaining the inductive sequence of sets

$$\tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_m, \cdots,$$

we define a set $\tilde{S}$ as follows:

A point $p \in \tilde{S}$ if $p \in \tilde{S}_m$ for some m and $p \notin \tilde{S}_n$ for all $n \geqq m - 1$.

If $J \subset \tilde{J}$ is any open disc, by an argument similar to the one used in Proposition 1 there is an uncountable set $\tilde{E}_m \subset J$ such that

$$\bar{D}_p(\tilde{S}) > 0 \text{ and } D_p(\tilde{S} \cap \sigma_p) = 0$$

for every $p \in \tilde{E}_m$ and σ as described above.

Concerning the problem raised by Zahorski on neglecting sets of density zero, we observe that the characteristic function of the set S constructed in Proposition 1 is an example of a function which has associated with it an exceptional set E that is uncountable in every neighborhood. It would be interesting to know if an exceptional set can be described in terms of its Hausdorff measure. In particular we ask if the Cantor set can ever be such an exceptional set.

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