A semiprime ring may be defined as a ring in which every nonzero right ideal $A$ is potent, that is, $A^n \neq 0$ for all $n > 0$. Evidently one can weaken the condition of semiprimeness by assuming only that some class of right ideals is potent. A natural choice for such a class is the class of all nonzero closed right ideals. A right ideal of a ring $R$ is called closed if it has no essential extension in the lattice $L_r$ of right ideals of $R$. We call ring $R$ (right) potent iff every nonzero closed right ideal of $R$ is potent.

The present paper is concerned with potent rings $R$ for which the (right) singular ideal is zero and the lattice $L_r^*$ of closed right ideals of $R$ is atomic. Necessary and sufficient conditions are given (3.7) for a triangular block matrix ring over a field $F$ to be potent. Such a potent ring is shown to have a full triangular block matrix ring as a classical quotient ring under certain conditions (3.6).

If $R$ is a finite-dimensional potent irreducible ring, then the ideals of $R$ in $L_r^*$ form a chain $R = T_0 > T_1 > \cdots > T_k = 0$. This fact allows us to imbed a potent triangular block matrix ring $S$ in $R$ and, in turn, to imbed $R$ in a full triangular block matrix ring $M$. If $\dim T_i - \dim T_{i+1} > 1$ in $L_r^*$, $i = 1, \cdots, k - 1$, then it is shown that $M$ is a classical quotient ring of $R$ (4.4). This generalizes Goldie's results on prime rings.

1. Atomic potent rings. If $R$ is a ring, then $L_r$ (or $L_r(R)$) denotes the lattice of right ideals and $L_2$ the lattice of 2-sided ideals of $R$. The notation $A'$ is used for the right annihilator of an element or subset $A$ of $R$.

If $L$ is a lattice with 0 and 1 and $A, B \in L$, then $B$ is called an essential extension of $A$ iff $A \subseteq B$ and $A \cap C \neq 0$ whenever $B \cap C \neq 0$, $C \in L$. We call $A \in L$ closed iff $A$ is the only essential extension of $A$ and large iff $I$ is an essential extension of $A$. A minimal element of $L - \{0\}$ is called an atom of $L$; dually, a maximal element of $L - \{1\}$ is called a coatom of $L$. We call lattice $L$ atomic iff each nonzero element of $L$ contains an atom.

The set $R_r^\Delta = \{a \in R \mid a'$ large in $L_r\}$ is an ideal of ring $R$ called the right singular ideal. If $R_r^\Delta = 0$, then each $A \in L_r$ has a unique maximal essential extension $A^*$, and the set $L_r^*$ of closed right ideals of $R$ is a complete complemented modular lattice. If $J_r^*$ denotes the lattice of all annihilating right ideals of $R$, then it is easily
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seen that $J_1^* \subseteq L_1^*$. The lattice $J_1^*$ is not usually a sublattice of $L_1^*$, although intersections are set-theoretic in both lattices. For convenience, we let $L_2^* = L_1^* \cap L_2$ and $J_2^* = J_1^* \cap L_2$. Corresponding left properties of a ring $R$ are indicated by replacing each "r" by an "l".

A ring $R$ is called right atomic iff $R_1^* = 0$ and $L_1^*$ is atomic. The union in $L_r$ of all atoms of $L_r^*$ is denoted by $R_r^*$. A right atomic ring $R$ is called (right) stable in [2] iff $(R_1^*)^r = 0$. If every nonzero closed right ideal of a right atomic ring $R$ is potent then $R$ is called a (right) potent ring, or a P-ring. It is clear that a right atomic ring $R$ is potent iff $A^2 \neq 0$ for every atom $A \in L_r^*$. Hence, a P-ring is also a stable ring.

If $R$ is a right atomic ring, then atoms $A$ and $B$ of $L_r^*$ are called perspective, $A \sim B$, iff they have a common complement in $L_r^*$. It may be shown that if $A \neq B$, then $A \sim B$ iff either $A \cup B$ contains a third atom $C$ or $a' = b'$ for some nonzero $a \in A$ and $b \in B$ [3, p. 540]. The union in $L_r^*$ of all atoms perspective to an atom $A$ is an atom in the center $C_r^*$ of $L_r^*$. It is known that $C_r^*$ is a Boolean algebra and that the elements of $C_r^*$ are ideals of $R$ [3, p. 541]. The ring $R$ is called (right) irreducible iff $C_r^* = \{0, R\}$. We shall call a right atomic, irreducible ring an I-ring. Clearly a right atomic ring $R$ is an I-ring iff $A \sim B$ for all atoms $A, B \in L_r^*$. An I-ring which is also a P-ring will be called a PI-ring.

1.1. Lemma. If $R$ is a P-ring and $A, B \in L_r^*$ then $A^* \subseteq B^*$ iff $A^* \cap B = 0$.

Proof. If $A^* \subseteq B^*$ then $A^* \cap B \subseteq (A^* \cap B)^r \subseteq (A^* \cap B)^2 = 0$, and therefore $A^* \cap B = 0$. Conversely, if $A^* \cap B = 0$ then $A^* \subseteq (AB)^r \subseteq B^*$.

It might be worth observing that the atomicity of $R$ is not needed in 1.1.

1.2. Lemma. If $R$ is a P-ring and $A \sim B$, where $A$ and $B$ are atoms of $L_r^*$, then either $AB \neq 0$ or $BA \neq 0$.

Proof. The lemma is obvious if $A = B$, so let us assume that $A \neq B$. Suppose that $AB = BA = 0$. Then there exists an atom $C \subseteq A \cup B$ such that $C \cap A = C \cap B = 0$. Since $A \cap A' = B \cap B' = 0$, evidently $A(a + b) \neq 0$ and $B(a + b) \neq 0$ for all nonzero $a \in A$ and $b \in B$. Hence, $AC \neq 0$ and $BC \neq 0$ in view of the fact that $C \cap (A + B) \neq 0$. Therefore, $A^r \subseteq C^r$ and $B^r \subseteq C^r$ by 1.1. We cannot have either $CA \neq 0$ or $CB \neq 0$, for then either $C' = A^r$ or $C' = B^r$ and either $A \subseteq B^r \subseteq A^r$ or $B \subseteq A^r \subseteq B^r$ contrary to assumption. Hence, $CA = CB = 0$ and $C(A \cup B) = 0$, contrary to the fact that $C^2 \neq 0$. We conclude that either $AB \neq 0$ or $BA \neq 0$ as desired.

Perhaps we should point out that if $A$ and $B$ are atoms of $L_r^*$ such that $AB \neq 0$ then necessarily $A \sim B$. For if $ab \neq 0$ for some $a \in A$ and $b \in B$, then $(ab)^r = b'$ [3,6.9] and $A \sim B$ by our remarks above.

1.3. Lemma. If $R$ is a PI-ring and $A, B \in L_r^*$ then either $A^* \cap B = 0$ or $A \cap B^* = 0$. 

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Proof. If \( A' \cap B \neq 0 \) and \( A \cap B' \neq 0 \), then there exist atoms \( C, D \in L_r^* \) such that \( C \subseteq A' \cap B \) and \( D \subseteq A \cap B' \). However, then \( CD = DC = 0 \) contrary to 1.2.

1.4. Theorem. If \( R \) is a PI-ring and \( S = \{A' \mid A \in L_r^* \} \), then \( S \) is a chain in \( J_r^* \). Also, \( S = \{0, R\} \) if \( R \) is a prime ring.

Proof. The first part follows from 1.1 and 1.3. If \( R \) is not prime, then \( BC = 0 \) for some nonzero \( B, C \in L_2 \). Since \( B' \in L_r^*, B' \neq R \), evidently \( A \cap C = 0 \) for some atom \( A \in L_r^* \). Clearly \( AC = 0 \) and therefore \( A' \neq 0 \). This proves 1.4.

2. Finite-dimensional rings. A ring \( R \) is said to have finite right rank iff there exists an integer \( n \) such that every independent subset of \( L_r \) has at most \( n \) elements. If \( R_r^* = 0 \), then \( R \) has finite right rank iff the lattice \( L_r^* \) is finite dimensional. The dimension of \( L_r^* \) is called the (right) rank, or dimension, of \( R \) and is denoted by \( \dim R \). A prefix of "F" used in designating a ring indicates that it is assumed to be finite dimensional. The case \( \dim R = 1 \) is uninteresting (if \( RA = 0 \), it means that \( RR^{-1} \) is a field), so we shall always tacitly assume that \( \dim R > 1 \).

2.1. Lemma. If \( R \) is an FPI-ring and \( T \in L_r^* \setminus \{R\} \), then \( T = A' \) for some atom \( A \in L_r^* \).

Proof. Clearly \( BT \subseteq B \cap T = 0 \) for some atom \( B \in L_r^* \). Let atom \( A \in L_r^* \) be chosen so that \( A' \) is a minimal element of \( \{B' \mid B \in L_r^*, B \) an atom, \( B' \supseteq T\} \). If \( T \neq A' \), there exists an atom \( C \in L_r^* \) such that \( C \cap T = 0 \) and \( C \subseteq A' \). Since \( AC = 0 \), necessarily \( CA \neq 0 \) by 1.2 and \( C' \subseteq A', C' \neq A' \), by 1.1. Since \( C' \supseteq T \), this is contrary to the choice of \( A \). Hence, \( T = A' \) as desired.

An interesting consequence of 2.1 is that \( A' = 0 \) for some atom \( A \in L_r^* \). Actually, this is true for any finite-dimensional stable ring by \([2, 2.13]\).

2.2. Theorem. If \( R \) is an FPI-ring, then \( L_r^* \setminus \{R\} \) is a finite chain \( R = T_0 > T_1 > \cdots > T_k = 0 \). If \( A \in L_r^* \setminus \{0\} \) and \( A \subseteq T_j \) then \( A' \subseteq T_{j+1} \). Conversely, if \( A \) is an atom of \( L_r^* \) and \( A' \subseteq T_{j+1} \) then \( A \subseteq T_j \).

Proof. The first part follows directly from 2.1 and 1.4. The other parts are obvious if \( j = 0 \), so let us assume that \( j > 0 \). By 2.1, there exists an atom \( B \in L_r^* \) such that \( B' = T_j \). If \( A \in L_r^* \setminus \{0\} \) and \( A \subseteq T_j \), then \( A' \cap B = 0 \) by 1.3 and \( A' \subseteq B' \), \( A' \neq B' \), by 1.1. Hence, \( A' \subseteq T_{j+1} \). Conversely, if \( A \) is an atom and \( A' \subseteq T_{j+1} \), then \( A' \subseteq B' \), \( A' \neq B' \), and \( A \cap B' \neq 0 \) by 1.1. Hence, \( A \subseteq T_j \). This proves 2.2.

2.3. Corollary. If \( 0 \leq j < k \), then \( T_j \) is the union of all atoms \( A \in L_r^* \) such that \( A' \subseteq T_{j+1} \).

If \( R \) is an FPI-ring of dimension \( n \), then the lattice \( J_r^* \) (\( J_r^* \)) is shown in \([4]\) to be a complemented lower (upper) semimodular lattice in which every maximal chain has length \( n \). If \( J_r^* \) consists of \( R = T_0 > T_1 > \cdots > T_k = 0 \) as in 2.2, then
$J^*_2$ consists of $0 = T^I_0 < T^I_1 < \cdots < T^I_k = R$. For each atom $A \in L^*_n$, there exists an integer $j$, $0 \leq j < k$, such that $A \subset T_j$ and $A' = T_{j+1}$ by 2.1 and 2.2. Let us select $a \in A$ such that $a^2 \neq 0$, and define $B = a^I$, an atom of $J^*_2$ containing $a$. Clearly $B \subset T_{j+1}$ and $B \cap T^I_j = 0$. Hence, $T^I_j B = 0$ and $B' \subset T^I_j$. Since $B' \in J^*_2$ and $B' \supset T^I_{j+1}$ (for $B' \supset T^I_{j+1}$ implies $B \subset T_{j+1}$ and $a \in T_{j+1}$, contrary to the fact that $A \cap T_{j+1} = 0$), evidently $B' = T^I_j$. Clearly $B$ is potent, and we have proved the following result.

2.4. Lemma. If $R$ is an FPI-ring and $J^*_2 = \{T^I_0, \ldots, T^I_k\}$ as above, then for each integer $j$, $0 \leq j < k$, there exists a potent atom $B \in J^*_i$ such that $B \subset T^I_{j+1}$ and $B' = T^I_j$.

Assume that we have selected an independent set $\{B_1, \ldots, B_p\}$ of potent atoms of $J^*_i$ (i.e., $B_{i+1} \cap (B_1 \cup \cdots \cup B_i) = 0$, $i = 1, \ldots, p - 1$) such that

$C \subset T^I_{j+1}$ and $C \cap T^I_j = 0$ where $C = B_1 \cup \cdots \cup B_p$.

If $C \cup T^I_j \neq T^I_{j+1}$, then $C' \cap T^I_j \neq T^I_{j+1}$ and there exists an atom $A \in L^*_n$ such that $A \subset C' \cap T^I_j$ and $A \cap T^I_{j+1} = 0$. Let $a \in A$, $a^2 \neq 0$, and $B = a^I$, an atom of $J^*_i$. By the proof of 2.4, $B$ is a potent atom such that $B \subset T^I_{j+1}$ and $B \cap T^I_j = 0$. If $B \subset C \cup T^I_j$, then $B' \supset C' \cap T^I_j$ and $a^2 = 0$, contrary to assumption. Hence, $B \cap (C \cup T^I_j) = 0$. By a lattice-theoretic argument (see [4, §4]), $(B \cup C) \cap T^I_j = 0$ also. The result below now follows by induction.

2.5. Lemma. Let $R$ be an FPI-ring and $J^*_2 = \{T^I_0, \ldots, T^I_k\}$ as above. Then for each integer $j$, $0 \leq j < k$, there exists an independent set $\{B_1, \ldots, B_q\}$ of potent atoms of $J^*_i$ such that

$(B_1 \cup \cdots \cup B_q) \cup T^I_j = T^I_{j+1}$, \hspace{1cm} $(B_1 \cup \cdots \cup B_q) \cap T^I_j = 0$.

If $R$ is an FPI-ring of dimension $n$ and $J^*_2 = \{T_0, \ldots, T_k\}$ as above, then $\dim T^I_i$ in $L^*_n$ equals $n - \dim T^I_i$ in $J^*_i$ by [4]. For convenience, let

$d_i = \dim T^I_i \hspace{1cm} i = 0, \ldots, k$.

Thus, $0 = d_0 < d_1 < \cdots < d_k = n$. We shall call

$(d_1 - d_0, d_2 - d_1, \ldots, d_k - d_{k-1})$

the set of block numbers of $R$. By 1.4, $R$ is prime iff $(n)$ is its set of block numbers.

In view of 2.5, there exists an independent set $\{B_1, \ldots, B_n\}$ of potent atoms of $J^*_i$ such that if

$C_j = B_{d_{j+1}} \cup \cdots \cup B_{d_{j+1}}$, \hspace{1cm} j = 0, \ldots, k - 1,$

then

$C_j \cup T^I_j = T^I_{j+1}$, \hspace{1cm} $C_j \cap T^I_j = 0$, \hspace{1cm} j = 0, \ldots, k - 1.$

If we define
\[ A_j = \bigcup_{i=1; i \neq j}^n B_i^* j = 1, \ldots, n, \]

then \( \{A_1, \ldots, A_n\} \) is an atomic basis of \( L_*^n \) contained in \( J_*^n \), as shown in [4]. It is immediate that

\[ A_j^* = \bigcup_{i=1; i \neq j}^n B_i, j = 1, \ldots, n. \]

If \( i \) and \( j \) are selected so that \( d_j < i \leq d_{j+1} \), then \( B_i \subset T_{j}^{j+1} \) and \( B_i \cap T_j^i = 0 \), so that \( B_i \nsubseteq T_{j+1}^{j+1} \) and \( B_i \subset T_j \). Since \( A_i^* \nsubseteq B_i \), clearly \( A_i \cap T_{j+1}^j = 0 \). On the other hand, \( A_i^* \nsubseteq T_j^i \) and therefore \( A_i \subset T_j \). Thus by 2.2, \( A_i^* = T_{j+1}^j \). Since \( B_p \subset T_{j+1}^j \) iff \( p > d_{j+1} \), evidently \( A_i B_p \neq 0 \) iff \( p \leq d_{j+1} \). We assemble these results below.

2.6. Theorem. Let \( R \) be an FPI-ring of dimension \( n \) with block numbers \( (b_1, \ldots, b_k) \). Then there exist potent atomic bases \( \{B_1, \ldots, B_n\} \) for \( J_*^n \) and \( \{A_1, \ldots, A_n\} \) for \( L_*^n \) such that:

1. \( A_i = (\bigcup_{j \neq i} B_j)^c \) and \( B_i = (\bigcup_{j \neq i} A_j)^c \), \( i = 1, \ldots, n \).
2. \( J_*^n = \{A_i^* \mid i = 1, \ldots, n\}, J_*^n = \{B_i^* \mid i = 1, \ldots, n\} \).
3. \( A_i^* \geq A_2^* \geq \cdots \geq A_n^* = 0 \) and \( 0 \neq B_1^* \geq B_2^* \geq \cdots \geq B_n^* \).
4. \( A_i^* = B_i^* \) iff \( d_0 + \cdots + d_i < i \) and \( j \leq d_0 + \cdots + d_p + 1 \) for some \( p \), where \( d_0 = 0 \).
5. \( A_i B_j \neq 0 \) iff \( i > d_0 + \cdots + d_p \) and \( d_0 + \cdots + d_p < j \leq d_0 + \cdots + d_{p+1} \) for some \( p \).

3. Triangular-block matrix rings. We shall give examples of FPI-rings in this section. To this end, let \( F \) be a (skew) field and \( F_{ij}, i, j = 1, \ldots, n \), be additive subgroups of \( F \) such that

\[ F_{ij} \cap F_{jk} = F_{ik}, i, j, k = 1, \ldots, n, \]

and let

\[ S = \sum_{i,j=1}^n F_{ij} e_{ij}, \]

where the \( e_{ij} \) are the usual \( n \times n \) unit matrices. Clearly \( S \) is a subring of \( (F)_n \), the ring of all \( n \times n \) matrices over \( F \).

The ring \( S \) will be called a T-ring (triangular-block matrix ring) in \( (F)_n \) iff there exist integers \( 0 = d_0 < d_1 < \cdots < d_k = n \) such that

\[ F_{ij} \neq 0 \] iff \( i > d_0 \) and \( d_p < j \leq d_{p+1} \), \( p = 0, \ldots, k - 1 \).

Associated with \( S \) is the full T-ring

\[ M = \sum_{i,j=1}^n F_{ij}' e_{ij}, \text{ where } F_{ij}' = F \text{ whenever } F_{ij} \neq 0, \]

\[ \text{and } F_{ij}' = 0 \text{ otherwise.} \]
It is clear that $M$ is closed under inverses. We shall call $M$ the full cover of $S$.

The $T$-ring $S$ described above can be thought of as a ring of $k \times k$ matrices whose elements are rectangular matrices over $F$. Thus, 

$$(3.4) \quad S = (S_{rs} \mid r, s = 1, \ldots, k)$$

where $S_{rs}$ is a set of $m_r \times m_s$ matrices $(m_i = d_i - d_{i-1})$ of the form

$$(f_{ij} \mid i = d_r + 1, \ldots, d_r + 1, j = d_s + 1, \ldots, d_s + 1), \quad f_{ij} \in F_{ij}.$$ 

The $k \times k$ matrices of $S$ are triangular, having zeros above the main diagonal. The full cover $M$ of $S$ has the form $(M_{rs})$, where $M_{rs}$ is the set of all $m_r \times m_s$ matrices over $F$ if $r \geq s$, and is zero otherwise. It is easily shown that the matrix ring $S_{ii}$ is prime for each $i$.

A ring $R$ is called a (right) quotient ring of ring $S$, and we write $S \leq R$, iff $S \subset R$ and $aS \cap S \neq 0$ for every nonzero $a \in R$. If ring $R$ has a unit and $S \subset R$, then $R$ is called a (right) classical quotient ring of $S$ iff every regular element $b \in S$ (i.e., $b^r = b^l = 0$) has an inverse in $R$ and $R = \{ab^{-1} \mid a, b \in S, b \text{ regular}\}$. If $R$ is a classical quotient ring of $S$, we write $R = SS^{-1}$. Clearly $S \leq R$ whenever $SS^{-1} = R$. If $M$ is a full $T$-ring, then $MM^{-1} = M$.

3.5. Theorem. If $S$ is a $T$-ring in $(F)_n$ given by 3.2, then $S \leq (F)_n$ iff $F_{11}F_{11}^{-1} = F$.

Proof. If $B = S_{e_{11}}$, then $B^l = 0$ and therefore $B \leq S$. If $S \leq (F)_n$, then $B \leq (F)_n$ and for every nonzero $f \in F$, $(f_{e_{11}})B \cap B \neq 0$. Hence, $fF_{11} \cap F_{11} \neq 0$ and $F \in F_{11}F_{11}^{-1}$. Thus, $F_{11}F_{11}^{-1} = F$.

Conversely, if $F_{11}F_{11}^{-1} = F$ then $F_{ii}F_{jj}^{-1} = F$ for all $i$ and $j$ by [4, Lemma 1.1]. Let $a = \sum_{i} a_{ij}e_{ij} \in (F)_n$, where $a_{ij} \in F$ and some $a_{rs} \neq 0$. For each $i$, there exists some nonzero $f_i \in F_{ii}$ such that $a_{ii}f_i \in F_{ii}$. Since \( \bigcap_{i=1}^{n} F_{ii} \neq 0 \) by [4, Lemma 1.1], there exists $g_i \in F_{ii}$ for some $i = 1, \ldots, n$. Clearly $a(f_{e_{ii}}) \in S$ and $a(f_{e_{ii}}) \neq 0$. Hence, $S \leq (F)_n$.

If $S$ is a $T$-ring in $(F)_n$, then necessarily $S^2 = 0$ and $L^*_n(S) \cong L^*_n((F)_n)$. Since $(F)_n$ is an FI-ring, $S$ is also an FI-ring.

3.6. Theorem. Let $S$ be a $T$-ring in $(F)_n$ given by 3.2 and $M$ be its full cover. Then $SS^{-1} = M$ iff $F_{ii}F_{ii}^{-1} = F$, $i = 1, \ldots, n$.

Proof. If $SS^{-1} = M$ and $S = (S_{rs})$ and $M = (M_{rs})$ are represented as in 3.4, then evidently $S_{ii}S_{ii}^{-1} = M_{ii}$ for each $i$. Hence, $F_{ii}F_{ii}^{-1} = F$ for each $i$ by [4, Theorem 1.2].

Conversely, if $F_{ii}F_{ii}^{-1} = F$ for each $i$ then $F_{ij}F_{ij}^{-1} = F_{ij}F_{ij}^{-1} = F$ for all $i$ and $j$ for which $F_{ij} \neq 0$ by [4, Lemma 1.1]. Let $d = (d_{ij}) \in M$, $d_{ij} \in F$, $d \neq 0$. Then $d_{ij} = a_{ij}b_{ij}^{-1}$ for some $a_{ij} \in F_{ij}$ and $b_{ij} \in F_{jj}$, $i, j = 1, \ldots, n$. Now $\bigcap_{i=1}^{n} b_{ij}F_{ij} \neq 0$ for each $j$ by [4, Lemma 1.1], and hence there exist nonzero $b_j \in F_{ij}$ and $c_{ij} \in F_{jj}$ such
that $b_j = b_{ij}c_{ij}$, $i,j = 1, \ldots, n$. Clearly $d = ab^{-1}$ where $a = \sum a_{ij}c_{ij}e_{ij}$ and $b = \sum b_{ij}e_{ij}$. This proves 3.6.

3.7. Theorem. Let $S$ be a T-ring in $(F)_n$ given by 3.2 such that $S \subseteq (F)_n$. Then $S$ is potent iff

$$F_{jj}F_{kk}^{-1} = F, \quad j < k, \ j, k = 2, \ldots, n.$$  

Proof. Assume that $S$ is potent. Let $j$ and $k$ be integers such that $2 \leq j < k \leq n$ and let $d \in F$, $d \neq 0$. Then $d = a_{ij}a_{kj}^{-1}$ for some $a \in F_{ik}$ by 3.5. If

$$a = a_{jj}e_{jj} + a_{kk}e_{kk},$$

then $a \in S$ and $A = (aR)^*$ is an atom of $L^*_s$ (since $a^r$ is a coatom). By assumption, $b^2 \neq 0$ for some $b \in A$. If $b = \sum b_{rs}e_{rs}$, $b_{rs} \in F_{rs}$, then $b_{rs} = 0$ if $r \neq j$ or $k$ and $b_{js} \neq 0$ iff $b_{ks} \neq 0$. For if $b_{rs} \neq 0$ with $r \neq j$ or $k$, or if $b_{js} \neq 0$ and $b_{ks} = 0$, then $(be_{rs}) \cap aR \neq 0$ contrary to the atomicity of $A$. Hence, either $b_{jj} \neq 0$ or $b_{kk} \neq 0$.

If $b_{jj} \neq 0$, then $(be_{jj}) \cap aR \neq 0$ and $b_{jj}f = a_{jj}g$, $b_{jj}f = a_{jj}g$ for some nonzero $f, g \in F$. Hence, $a_{ij}a_{kk}^{-1} = b_{jj}b_{jj}^{-1} \in F_{jj}F_{jj}^{-1}$. If $b_{kk} \neq 0$, then $(be_{kk}) \cap aR \neq 0$ and $a_{ij}a_{kk}^{-1} = b_{jk}b_{kk}^{-1} \in F_{jk}F_{kk}^{-1}$ by the same reasoning. However, if both $F_{jk}$ and $F_{kj}$ are nonzero, then $F_{kk}F_{kk}^{-1} \subseteq F_{jk}F_{kj}^{-1}$. Therefore, $d \in F_{jj}F_{kj}^{-1}$. We conclude that $F_{jj}F_{kj}^{-1} = F$.

Conversely, let us assume that 3.8 holds. Every atom $A$ of $L^*_s$ contains a nonzero element $a$ of the form $a = a_{kk}e_{kk} + \cdots + a_{ie}e_{ie}$, $a_i \in F_{ik}$, $a_k \neq 0$. If $k = 1$, then $a^2 \neq 0$ and $A$ is potent. If $k > 1$, we claim that there exists some $b = b_{ie}e_{ie} + \cdots + b_{ne}e_{ne} \in A$, $b_i \in F_{ik}$, with $b_i \neq 0$ iff $a_i \neq 0$. Since $b^2 \neq 0$, this will prove that $A$ is potent and hence will prove the theorem.

Such a $b \in A$ exists iff $af = b_i g$, $i = k, \ldots, n$, for some nonzero $f, g \in F$; i.e., iff

$$a^{-1}b_k = a^{-1}b_i \quad \text{for each } i \text{ for which } a_i \neq 0.$$  

Assume, for simplicity of notation, that $a_i \neq 0$ if $k \leq i \leq p$ and that $a_i = 0$ if $i > p$. If we have found nonzero $b_i \in F_{ik}$, $i = 1, \ldots, m - 1$, for which (1) holds, with $m \leq p$, then let us select nonzero $b_m \in F_{mk}$ and $c \in F_{kk}$ such that $a_i^{-1}b_ic = a_m^{-1}b_m$ (which we can do, since $F_{kk}F_{kk}^{-1} = F$). Then $a_i^{-1}b_lc = a_i^{-1}b_mc = a_m^{-1}b_m$, $i = 1, \ldots, m - 1$, and (1) follows by induction. This proves 3.7.

If a T-ring is potent, then its block numbers are the obvious ones according to the result below.

3.9. Theorem. Let $S$ be a T-ring in $(F)_n$ whose blocks are defined by the numbers $0 = d_0 < d_1 < \cdots < d_k$ as in 3.2. If $S \subseteq (F)_n$ and $S$ is potent, then

$L^*_s = \{T_0, \ldots, T_k\}$ where $T_0 = R$, $T_k = 0$, and

$$T_i = \sum_{r > m} \sum_{j=1}^n F_{rj}e_{rj}$$

where $m = d_i$; $i = 1, \ldots, k - 1$. 
Proof. If \( A = e_{jj}S \), where \( j = d_i \) for some \( i, \ 0 < i \leq k \), then clearly \( A' = T \).
Conversely, if \( B \) is an atom of \( L^*_n \), let \( r \) be the maximum integer for which \( be_{rr} \neq 0 \) for some \( b \in B \). We claim that \( r = d_i \) for some \( i \). Otherwise, \( d_{i-1} < r < d_i \) for some \( i \) and we can find a nonzero \( ce_{uu} \in S \), where \( u = d_i \), and nonzero \( f \in F_{11} \)
\( g \in F_{ii} \) such that \( b(f_{ee}) = (ce_{uu})(ge_{nn}) \) just as we did in the proof of 3.7. Clearly \( ce_{uu} \in B \), contrary to the choice of \( r \). Since \( B \) contains nonzero elements of the form \( be_{rr} \) for \( r = 1, \cdots, d_i \) and \( Be_{ss} = 0 \) if \( s > d_i \), evidently \( B' = T_i \). This proves 3.9.

The block numbers of the potent ring \( S \) of 3.9 clearly are \((d_1 - d_0, \cdots, d_k - d_{k-1})\).
Thus, 3.9 gives us a way of constructing FPI-rings having any prescribed block numbers. In particular, any full T-ring in \((F)_n\) is an FPI-ring. A T-ring over the ring of integers is also an FPI-ring.

As a slightly different example, let \( F \) be a field which has a nonzero subring \( K \)
such that \( KK^{-1} \neq F \). Then the \( 2 \times 2 \) matrix rings
\[
S = Fe_{11} + Fe_{21} + Ke_{22}, \quad M = Fe_{11} + Fe_{21} + Fe_{22}
\]
are both potent. However, \( SS^{-1} \neq M \), i.e., \( S \) doesn’t have \( M \) as a classical quotient ring.

We point out that if \( S \) is a potent T-ring in \((F)_n\) such that \( S \leq (F)_n \) and \( F_{kj} \) and \( F_{k'j} \) are nonzero for some \( j \) and \( k \), say with \( k > j \), then
\[
(1) \quad F_{jj}F_{jj}^{-1} = F_{kk}F_{kk}^{-1} = F.
\]
To prove (1), we have that for any nonzero \( a, b, c, d \in F \) there exist \( f \in F_{jj} \) and \( g \in F_{kk} \) such that \( a^{-1}db = fg^{-1} \), or \( d = (afc)(bgc)^{-1} \). By letting \( a, c \in F_{jj} \) and \( b \in F_{kk} \), we see that \( d \in F_{jj}F_{jj}^{-1} \); and by letting \( a \in F_{kk} \), \( b \in F_{kk} \), and \( c \in F_{kk} \), we can see that \( d \in F_{kk}F_{kk}^{-1} \). Since \( c \) is any nonzero element of \( F \), (1) is proved.

The situation described in the preceding paragraph will occur in T-ring \( S \) of 3.2 iff \( d_{i+1} - d_i > 1 \) for some \( i \). If \( S \leq (F)_n \), then \( F_{11}^{-1}F_{11} = F \) irrespective of whether or not \( d_1 - d_0 > 1 \). If, in addition, \( S \) is potent and \( d_{i+1} - d_i > 1 \) for all \( i > 0 \), then \( F_{kk}F_{kk}^{-1} = F \) for all \( k \) by (1) above. This proves the following corollary of 3.6 and 3.7.

3.10. Corollary. Let \( S \) be a T-ring in \((F)_n\) defined by 3.2, \( S \leq (F)_n \), and \( M \) be the full cover of \( S \). If \( S \) is potent and \( d_{i+1} - d_i > 1 \) for all \( i > 0 \), then \( SS^{-1} = M \).

4. FPI-rings as matrix rings. It is well known that every \( n \)-dimensional
I-ring \( R \) has a full ring \( Q \) of linear transformations of an \( n \)-dimensional vector space over a field as a quotient ring and that \( L^*_n(Q) \cong L^*_n(R) \) under the correspondence \( A \to A \cap R, \ A \in L^*_n(Q) \). (See [1] for references.) Let \( R \) be an FPI-ring, and the \( A_i \) and \( B_j \) be as given in 2.6. Corresponding to the basis \( \{A_1, \cdots, A_n\} \)
of \( L^*_n(R) \) is an atomic basis \( \{A'_1, \cdots, A'_n\} \) of \( L^*_n(Q) \). By [5, Proposition 5, p. 52], there exists a set \( \{e_{ij} | i, j = 1, \cdots, n\} \) of matrix units in \( Q \) such that \( A'_i = e_{ii}Q \) and hence \( A_i = (e_{ii}Q) \cap R, \ i = 1, \cdots, n \). Clearly \( B_i = (\bigcup_{j \neq i} A_j)^\ell = [(\Sigma_{j \neq i} e_{jj}Q) \cap R]^\ell = (Qe_{ii}) \cap R, \ i = 1, \cdots, n \).
Relative to the given set of matrix units in \( Q \), there exists a field \( F \) such that [6; Proposition 6, p. 52]

\[
Q = \sum_{i,j=1}^{n} F e_{ij} \cong (F)^n.
\]

Then

\[
A_i \cap B_j = F e_{ij}, \quad i, j = 1, \ldots, n,
\]

for some additive subgroups \( F_{ij} \) of \( F \) satisfying 3.1. If we let

\[
S = \sum_{i,j=1}^{n} F e_{ij},
\]

then \( S \) is a subring of \( R \). By 2.6 (5), \( F_{ij} \neq 0 \) iff \( i > d_p \) and \( d_p < j \leq d_p + 1 \) for some \( p \). Thus, \( S \) is a T-ring in \( (F)^n \) with the same block numbers as \( R \).

Since \( B_1 = 0 \), we know that \( B_1 \leq R \). Also, \( \{ A_1 \cap B_1, \ldots, A_n \cap B_1 \} \) is an atomic basis of \( L^*_n(B_1) \). Hence, \( F_{11}e_{11} + \cdots + F_{nn}e_{nn} \leq B_1 \leq R \), and therefore (since \( F_{11}e_{11} + \cdots + F_{nn}e_{nn} \subset S \)) \( S \leq R \leq Q \).

Associated with the T-ring \( S \) is the full T-ring \( M \) over \( F \) with the same block numbers as \( S \).

4.2. Lemma. If \( R \) is an FPI-ring and rings \( Q, S, \) and \( M \) are defined as above then \( S \subset R \subset M \).

Proof. If \( b \in R \), then \( b \in Q \) and \( b = \sum b_{ij} e_{ij} \) for some \( b_{ij} \in F \). If \( b_{rs} \neq 0 \) then \( (e_{rf})b(e_{sg}) \in R \) for any nonzero \( f \in F_{rs} \) and \( g \in F_{as} \); i.e., \( c = f b_{rs} g e_{rs} \in R \). Since \( c \in A_r \cap B_s \), evidently \( f b_{rs} g \in F_{rs} \) and \( F_{rs} \neq 0 \). Thus, \( b \in M \). This proves 4.2.

4.3. Theorem. If \( R \) is an FPI-ring and \( S \subset R \subset M \) as in 4.2, then \( S \) is an FPI-ring having the same dimensions and same block numbers as \( R \).

Proof. Since \( S \leq Q \), \( \dim S = \dim R \) and \( F_{j1} F_{m1}^{-1} = F \) for all \( j \) and \( m \). Let \( a_{j1} + a_{m} e_{m1} = a \in S \), where \( 1 < j < m \), \( a_{j1} \in F_{j1} \), and \( a_{j} \neq 0 \) and \( a_{m} \neq 0 \). Also let \( d = a_{j} a_{j}^{-1} \) and \( e = e_{jj} + d e_{mj} \). Clearly \( e^2 = e \) and \( eQ \) is an atom of \( L^*_n(Q) \). Hence, \( A = eQ \cap R \) is an atom of \( L^*_n(R) \). As such, it is potent. Let \( b \in A \), \( b^2 \neq 0 \).

Now \( b = \sum (e_{ji} + d e_{mi}) c_i \) for some \( c_i \in F \). Since \( b^2 \neq 0 \), either \( c_j \neq 0 \) or \( c_m \neq 0 \). Assume that \( c_j \neq 0 \). Then \( (e_{jj} f)(b(e_{jj} g)) \in F_{jj} e_{jj} \) and \( (e_{mm} h)(b(e_{jj} g)) \in F_{mj} e_{mj} \) for all nonzero \( f, g \in F_{jj} \) and \( h \in F_{mm} \). Hence, \( f c_j g \in F_{jj} \), \( h d c_j g \in F_{mj} \), and \( (h d f^{-1})(f c_j g) \in F_{mj}, h d f^{-1} \in F_{mj} F_{jj}^{-1} \). Since \( d \) ranges over \( F \), so does \( h d f^{-1} \) and therefore \( F_{mj} F_{jj}^{-1} = F \) (or, taking inverses, \( F_{jj} F_{mj}^{-1} = F \)).

If \( c_j = 0 \) but \( c_m \neq 0 \), then \( (e_{mm} f)(b(e_{mm} g)) \in F_{mm} e_{mm} \) and \( (e_{jj} h)(b(e_{mm} g)) \in F_{jm} e_{jm} \) for all nonzero \( f, g \in F_{mm} \) and \( h \in F_{jj} \). Hence, \( f d c_m g \in F_{mm} \), \( h c_m g \in F_{jm} \), and \( (f d h^{-1})(h c_m g) \in F_{mm}, f d h^{-1} \in F_{mm} F_{jj}^{-1} \). Therefore, \( F_{mm} F_{jm}^{-1} = F \). By previous remarks, we must also have \( F_{jj} F_{mj}^{-1} = F \). Thus, \( S \) is potent by 3.7.
If the block numbers of $S$, and hence also of $R$, are all greater than 1 with the possible exception of the first one, then $F_{ii}F_{ii}^{-1} = F$ for all $i$ and $SS^{-1} = M$ by 3.6. Clearly, then, $RR^{-1} = M$, and we have proved the following result.

4.4. Theorem. Let $R$ be an FPI-ring with block numbers $(m_1,\ldots,m_k)$. If $m_i > 1$ for $i = 2,\ldots,m_k$, then $RR^{-1} = M$, the full T-ring over a field $F$ with block numbers $(m_1,\ldots,m_k)$.

5. Reducible rings. Let us call a ring $R$ reducible iff $R_{\mathfrak{a}} = 0$ and $C_\ast \neq \{0,R\}$. If $H \in C_\ast$, $H \neq 0$ or $R$, then $K = H^t \in C_\ast$ also and $H \cap K = 0$, $H = K^t$[3.6.7]. Hence, $H + K \leq R$. If $R$ is atomic, then each atom of $L_\ast(R)$ is contained in either $H$ or $K$ by our remarks in §1. Therefore, the set of atoms of $H + K$ coincides with the set of atoms of $R$, and $R$ is potent iff $H + K$ is potent.

This leads us to consider the direct sum $R = R_1 \oplus R_2$ of two rings $R_1$ and $R_2$. Clearly, $L_\ast(R_i) \subseteq L_\ast(R)$, $i = 1,2$. Let us assume that $R_i' = 0$ (in $R_i$), $i = 1,2$. Then $C \cap R_i \neq 0$ for each right ideal $C$ of $R$ not contained in $R_j$, $i \neq j$. In particular, if $C$ is a large right ideal of $R$ then $C \cap R_i$ is a large right ideal of $R_i$, $i = 1,2$. Conversely, if $C_i$ is a large right ideal of $R_i$, $i = 1,2$, then $C = C_1 + C_2$ is a large right ideal of $R$. From these remarks, it follows readily that $R^\Delta = R_1^\Delta + R_2^\Delta$. If $R^\Delta = 0$, then it is not difficult to show that

$$L_\ast(R) = \{A_1 + A_2 \mid A_i \in L_\ast(R_i)\}.$$  

Hence, $R$ is potent iff $R_1$ and $R_2$ are potent.

If $R$ is an FP-ring, and $(R_1,\ldots,R_n)$ is the set of atoms of $C_\ast$, then $R_1 + \cdots + R_n$ is a direct sum of ideals of $R$ and $R_1 + \cdots + R_n \leq R$ [3, p. 541]. By our remarks above, each $R_i$ is an FP-ring. Actually, each $R_i$ is an FPI-ring. Let the T-rings $S_i$, $M_i$, and $Q_i$ be selected as in 4.2. $S_i \subseteq R_i \subseteq M_i \subseteq Q_i$, $i = 1,\ldots,n$. Then $Q_1 + \cdots + Q_n$ is the maximal right quotient of $R_1 + \cdots + R_n$, so that

$$S_1 + \cdots + S_n \subseteq R_1 + \cdots + R_n \subseteq R \subseteq Q_1 + \cdots + Q_n.$$  

If $b \in R$, then $b = \sum b_i$, $b_i \in Q_i$, $bR_i \subseteq R_i$, and hence $b_iS_i \subseteq M_i$ for each $i$. Clearly, then, $b_i \in M_i$ for each $i$ and $b \in M_1 + \cdots + M_n$. Thus, $R \subseteq M_1 + \cdots + M_n$.

The theorem below follows from these remarks and the work of the preceeding sections.

5.1. Theorem. If $R$ is an FP-ring then there exist FPI-rings $R_1,\ldots,R_n$ such that $R_1 \oplus \cdots \oplus R_n \leq R$. Furthermore, if $S_i$ and $M_i$ are the associated T-rings of $R_i$, selected as in 4.2, $i = 1,\ldots,n$, then

$$S_1 \oplus \cdots \oplus S_n \leq R \leq M_1 \oplus \cdots \oplus M_n.$$  

If $M = M_1 \oplus \cdots \oplus M_n$, then $RR^{-1} = M$ iff $R_iR_i^{-1} = M_i$, $i = 1,\ldots,n$. 

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