THE PRINCIPAL SEMI-ALGEBRA IN A BANACH ALGEBRA(1)

BY

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INTRODUCTION

0.1. **Orientation.** A subset of a real linear associative algebra is called a *semi-algebra* iff it is a wedge [8, p. 20] closed under multiplication. The theory of such a structure has been developed principally by F. F. Bonsall, who originally considered semi-algebras of non-negative-valued functions, but later extended his research to include locally compact and convolution semi-algebras [1], [2], [3]. Independently, S. Bourne [6] defined semi-algebras and proved a representation theorem for a special class of them.

The present paper carries the study in a new direction. The question of determining maximal closed subalgebras of certain Banach algebras has received a great deal of attention (see, for example, [10]), but the corresponding problem for semi-algebras seems not to have been considered at all. That some interesting results might be obtained was suggested by the observation that, with real Euclidean 3-space considered naturally as a Banach algebra (operations defined pointwise), the solid $\{(x,y,z)\colon x,y,z \geqq 0,\ z \leqq x^{1/2}y^{1/2}\}$ is a maximal closed subsemi-algebra of the positive octant. I am grateful to Professor L. Nachbin, who, by conjecturing a possible generalization for continuous real-valued functions, led me into a deeper consideration of the problem, and to Professor F. F. Bonsall for his encouragement and supervision.

The setting for Chapter 1 is a real Banach algebra with multiplicative identity 1. The role of the positive octant in the above example is played by the principal semi-algebra, defined to be the least closed semi-algebra whose interior contains 1. A representation of this semi-algebra is given, along with some results and examples concerning its maximal closed subsemi-algebras. Chapter 2 deals with the particular Banach algebra of all continuous real-valued functions defined on a compact Hausdorff space, whose principal semi-algebra is the set of all its non-negative-valued functions. In this case, not only can *all* maximal

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closed subsemi-algebras of the principal semi-algebra be found, but a simple characterization of all their intersections can be given.

The paper which is probably most closely related to the results of Chapter 2 is one by W. B. Jurkat and G. G. Lorentz [11], which gives necessary and sufficient conditions involving simple geometric means that a pair of functions generate the uniformly closed semi-algebra of non-negative continuous functions on a compact Hausdorff space. It seems that it may be possible to obtain this result as part of the present theory. I note also that geometric means of functions have arisen in work by D. G. Bourgin [5] and J. F. C. Kingman [13].

The remainder of the introduction contains a list of results for later reference.

0.2. **The geometric mean of a non-negative continuous function.** Here, and throughout the paper, E denotes a compact Hausdorff space, $C(E)$ the Banach algebra (with algebraic operations defined pointwise and the uniform norm) of all continuous real-valued functions defined on E, $C^+(E)$ the set of all non-negative functions in $C(E)$ and $C^{\#}(E)$ the set of all those functions in $C(E)$ which are everywhere strictly positive. Consider $C(E)$ as being ordered with positive cone $C^+(E)$. See N. Bourbaki [4] for background in integration theory.

Let μ be a Radon probability measure on E (i.e. a positive measure of total mass 1). The *geometric mean with weight* μ, denoted by GM_μ, is defined on $C^+(E)$ by

$$GM_\mu f = \begin{cases} \exp\displaystyle\int \log f \, d\mu & \text{if } \log f \text{ is } \mu\text{-integrable} \\ 0 & \text{if } \log f \text{ is not } \mu\text{-integrable.} \end{cases}$$

(Note that $\log f$ may be an extended real-valued function.) For $f \in C^{\#}(E)$, $\log f$ is μ-integrable and the first alternative of the definition is valid. Observe that, if the support of μ is a finite subset of E, then $GM_\mu f$ is expressible as a finite product.

GM_μ is a non-negative monotone function on $C^+(E)$ with the following properties (where $f,\ g \in C^+(E),\ \lambda \geqq 0$):

(a) $GM_\mu \lambda = \lambda$;

(b) $\lim_{\lambda\to 0^+} GM_\mu(f+\lambda g) = GM_\mu f$;

(c) $0 \leqq GM_\mu f \leqq \mu(f) \leqq \|f\|$;

(d) $GM_\mu fg = (GM_\mu f)(GM_\mu g)$; $GM_\mu \lambda f = \lambda GM_\mu f$; $GM_\mu(f+g) \geqq GM_\mu f + GM_\mu g$; $GM_\mu f^\lambda = (GM_\mu f)^\lambda$.

The proofs are straightforward. For (b), because of the monotonicity of GM_μ, it suffices to consider the integrals of the functions in the decreasing sequence $\{\log(f+n^{-1}g)\}$ with pointwise limit $\log f$. If the sequence contains any non-μ-integrable member, or if the integrals are not bounded below, then $\log f$ is not μ-integrable. Otherwise, an application of the Beppo Levi Theorem [4, pp. 138, 149] yields the result. Since the second inequality of (c) and the superadditivity property in (d) are not quite obvious, a proof following that in [9, p. 138] is outlined. When $\mu(f) > 0$ and $\log f$ is μ-integrable, it follows from the positivity of μ and the inequality $\log \beta \leqq \beta - 1$ for positive real β, that

$$\begin{aligned} \int \log f\, d\mu &= \int \log(f/\mu(f))\, d\mu + \log \int f\, d\mu \\ &\leqq \int [(f/\mu(f)) - 1]\, d\mu + \log \int f\, d\mu = \log \int f\, d\mu . \end{aligned}$$

For the remaining cases, note that, for positive λ, $GM_\mu(f+\lambda) \leqq \mu(f+\lambda)$ and use (b) and the continuity of μ. To show the superadditivity of GM_μ, observe that when $f + g \in C^{\#}(E)$,

$$GM_\mu(f/(f+g)) + GM_\mu(g/(f+g)) \leqq \mu(f/(f+g)) + \mu(g/(f+g)) = 1;$$

when $f+g$ has a zero, approximate to f by $f+\lambda$.

Finally, when μ_1 and μ_2 are probability measures on E and $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ for $0 < \alpha < 1$, then, for each $f \in C^+(E)$,

$$GM_\mu f = (GM_{\mu_1} f)^{\alpha} \cdot (GM_{\mu_2} f)^{1-\alpha}.$$

0.3. **The theorem of Choquet and Deny.** A subset P of $C(E)$ is an *upper semi-lattice* iff, whenever $f, g \in P$, it is true that $f \cup g \in P$, where

$$(f \cup g)(\xi) \equiv \max\{f(\xi), g(\xi)\} \qquad (\xi \in E).$$

Let μ be a positive Radon measure on E and ξ be a point of E. Then the sets

$$\begin{aligned} U_{\mu,\xi} &\equiv \{f : f \in C(E),\ \mu(f) \geqq f(\xi)\}, \\ U_\mu &\equiv \{f : f \in C(E),\ \mu(f) \geqq 0\} \end{aligned}$$

are both uniformly closed wedges which are upper semi-lattices. The following theorem states that every upper semi-lattice uniformly closed wedge is the intersection of wedges of the types $U_{\mu,\xi}$ and U_μ. Let δ_ξ denote the measure with all of its unit mass concentrated at the point ξ, and W' denote the dual wedge of all continuous real linear functionals taking non-negative values on the wedge W.

THEOREM 0.1 (CHOQUET-DENY). *Let W be a uniformly closed wedge contained in $C(E)$ which is an upper semi-lattice. Suppose that $\mathscr{U}_1$ is the family of all pairs (μ, ξ) with $\xi \in E$ and μ a positive Radon measure such that $\mu(\{\xi\}) = 0$ and $\mu - \delta_\xi \in W'$; suppose that $\mathscr{U}_2$ is the family of all positive Radon measures μ such that $\mu \in W'$. Then*:

$$W = \Big(\bigcap_{(\mu,\xi) \in \mathscr{U}_1} U_{\mu,\xi}\Big) \cap \Big(\bigcap_{\mu \in \mathscr{U}_2} U_\mu\Big).$$

For the proof, see [7]. The theorem is valid even if $\mathscr{U}_1$ is void and $\mathscr{U}_2$ consists of the zero measure alone. (The convention that a void intersection is the whole of the space is adopted.)

CHAPTER 1

1.1. **The principal semi-algebra.** Throughout this chapter, B is a real Banach algebra with multiplicative identity 1. Denote by G the set of regular elements of B. The space Φ_B^c of nontrivial complex homomorphisms on B is compact in the Gelfand topology; the family Φ_B of all nontrivial real homomorphisms is a closed subset of Φ_B^c. A nonvoid subset A of B is a *semi-algebra* iff

$$x, y \in A,\ \lambda \geqq 0 \Rightarrow x + y,\ xy,\ \lambda x \in A.$$

Denote by B^* the dual space consisting of all continuous real linear functionals on B, by A' the subset of those functionals in B^* taking non-negative values on A. Let $\boldsymbol{R}$ be the set of reals, $\boldsymbol{R}^+$ the set of non-negative reals. The proof of the following theorem is similar to the proof of a theorem given by Kadison in [12, p. 8].

THEOREM 1.1. *Let A be any closed subsemi-algebra of B whose interior contains 1. Then $A = \bigcap_{\phi \in \Psi_A} A_\phi$, where $\Psi_A \equiv \{\phi : \phi \in \Phi_B \cap A'\}$ and $A_\phi \equiv \{x : x \in B,\ \phi(x) \geqq 0\}$.*

Proof. Let $M \equiv \{\mu : \mu \in A',\ \mu(1) = 1\}$. Then, because 0 is the only functional in A' which can vanish at the interior point 1, $A' = (\bigcup_{\lambda \geqq 0} \lambda M)$. Note that $\Psi_A \subseteq M$.

Suppose that M is not void. Then M is convex and weak* closed. For some positive δ, A contains the open δ-ball with centre 1, so that M is contained in the closed δ^{-1}-ball with centre 0. Thus M is weak* compact. The Krein-Milman Theorem [8, p. 78] asserts that M possesses extreme points, of which it is the weak* closed convex hull. Let μ be one of these extreme points. If $u \in A$, $\| u \| < \delta$, $0 < \mu(u) < 1$ and $\mu_1(x) \equiv (\mu(u))^{-1}\mu(ux)$, $\mu_2(x) \equiv (\mu(1-u))^{-1}\mu(x-ux)$ $(\forall x \in B)$, then $1 - u \in A$, $\mu_1, \mu_2 \in M$ and $\mu = \mu(u)\mu_1 + \mu(1-u)\mu_2$. Since μ is extreme, $\mu = \mu_1$ so that $\mu(u)\mu(x) = \mu(ux)$ $(\forall x \in B)$. Evidently, this equation will hold for any element u of A. Since A has a nonvoid interior, $B = A - A$, so that μ is in fact a multiplicative functional, hence a member of Ψ_A. If $x \in \bigcap_{\phi \in \Psi_A} A_\phi$, then $\mu(x) \geqq 0$ for each functional μ in A', so that, since A is closed, $x \in A$. (See [8, p. 22].) On the other hand, for any $x \in A$, clearly $x \in \bigcap_{\phi \in \Psi_A} A_\phi$.

If M is void, then Ψ_A is void and $A' = \{0\}$. If A were proper in B, it would be possible by Theorem 6 of [8, p. 22] to find a nontrivial functional in A'. Hence, in this case also, $A = B = \bigcap_{\phi \in \Psi_A} A_\phi$.

COROLLARY. *Any closed subsemi-algebra of B whose interior contains 1 must contain $\bigcap_{\phi \in \Phi_B} A_\phi$. Thus $\bigcap_{\phi \in \Phi_B} A_\phi$ is the least closed semi-algebra containing 1 as interior point.*

Proof. The first statement is clear. To see the second, observe that each A_ϕ contains the open unit ball with centre 1, Definition. The *principal semi-algebra B_+ contained in B* is the least closed semi-algebra containing 1 as an interior point.

The corollary asserts that B_+ certainly exists. One property of B_+ which will be used later is that, for each element u belonging to $G \cap B_+$, the inverse u^{-1} also belongs to B_+.

1.2. Maximal closed subsemi-algebras of B_+.

Definition. A closed subsemi-algebra of B_+ which is not properly contained in any proper closed subsemi-algebra of B_+ is said to be *maximal closed* in B_+.

Theorem 1.2. (i) *If Φ_B is void or contains at least two elements, each maximal closed subsemi-algebra of B_+ contains the identity* 1.

(ii) *If Φ_B consists of the single element ϕ, then $\{x: x \in B_+,\ \phi(x) = 0\}$ is the unique maximal closed subsemi-algebra of B_+ which does not contain* 1.

Before proving this theorem, we need some results about adjoining the identity to a subsemi-algebra A of B. Let

$$A_1 \equiv \{a + \alpha : a \in A,\ \alpha \geqq 0\}.$$

Lemma 1.1. *If A is a closed subsemi-algebra of B, then A_1 is also a closed subsemi-algebra. Furthermore*

$$A_1{}' = \{\mu : \mu \in A',\ \mu(1) \geqq 0\}.$$

Proof. The result that A_1 is a semi-algebra and the equality for A_1' are straightforward. If $1 \in A$, then $A_1 = A$ is closed. Suppose that 1 does not belong to A and that z lies in the closure of A_1. Then there exists a sequence $\{x_n + \alpha_n : x_n \in A,\ \alpha_n \in \boldsymbol{R}^+\}$ such that $\lim_{n\to\infty}(x_n + \alpha_n) = z$. If $\{\alpha_n\}$ has no convergent subsequence, then there is a subset D of the positive integers such that $\lim\{\alpha_n : n \to \infty, n \in D\} = \infty$. Because $\| \alpha_n^{-1} x_n + 1 \| \leqq \alpha_n^{-1} \| x_n + \alpha_n - z \| + \alpha_n^{-1} \| z \|$, it follows that $-1 = \lim\{\alpha_n^{-1} x_n : n \to \infty,\ n \in D\}$, so that $-1 \in A$. But then $1 = (-1)^2 \in A$, giving a contradiction. Hence $\{\alpha_n\}$ has a convergent subsequence with non-negative limit α, say. It can be checked that $\{x_n\}$ has a corresponding subsequence with limit $z - \alpha$, which must lie in A. Thus $z = (z - \alpha) + \alpha \in A_1$, and A_1 is closed.

Lemma 1.2. (i) *If Φ_B is void, or contains at least two elements, then there is no proper closed subsemi-algebra A of B_+ with $A_1 = B_+$.*

(ii) *If Φ_B consists of the single element ϕ, then $A \equiv \{x : x \in B, \phi(x) = 0\}$ is the unique closed subsemi-algebra of B_+ with $A_1 = B_+$.*

Proof. By Theorem 1.1, if Φ_B is void, then $B_+ = B$. If a closed semi-algebra A satisfies $A_1 = B_+ = B$, then $-1 = x + \alpha$ for some $x \in A$ and $\alpha \geqq 0$, with the result that $1 = (1 + \alpha)^{-2} x^2 \in A$ and $A = B$. Now let Φ_B and A be as specified in (ii) and let $x \in B_+$. Since $x = (x - \phi(x)) + \phi(x)$, x belongs to A_1.

If Φ_B is nonvoid, suppose that A is a proper closed subsemi-algebra of B_+ with $A_1 = B_+$. Then there exists a functional $\mu \in A'$ with $\mu(1) = -1$. Let $\phi \in \Phi_B$. Then $\mu + \phi \in A'$, $(\mu + \phi)(1) = 0$ so that $\mu + \phi$ is a functional in B'_+ vanishing at the interior point 1. Hence $\mu + \phi = 0$, i.e. $\phi = -\mu$. Thus Φ_B consists of a single element and μ is the sole functional in A' with $\mu(1) = -1$. It follows that

$$B'_+ = \{\lambda\phi : \lambda \in \boldsymbol{R}^+\},$$

$$A' = \{\lambda\mu : \lambda \in \boldsymbol{R}^+\} \cup B'_+ = \{\lambda\phi : \lambda \in \boldsymbol{R}\},$$

so that $A = \{x : x \in B,\ \phi(x) = 0\}$.

Proof of Theorem 1.2. If A is a maximal closed subsemi-algebra of B_+, either $1 \in A$ or, A_1 being closed, $A_1 = B_+$. Lemma 1.2 shows that the only possibility for the latter case is that ϕ_B consists of a single element, Φ say, and that $A = \{x : x \in B,\ \phi(x) = 0\}$. Suppose this to be so; then we show that, for each $z \in B_+ \setminus A$, the least closed semi-algebra (A, z) containing A and the element z is in fact B_+, so that A is indeed maximal.For, if $y \in B_+$, then $y - (\phi(y)/\phi(z))z \in A$, so that $y = [y - (\phi(y)/\phi(z))z] + [(\phi(y)/\phi(z))z] \in (A, z)$. Then $B_+ \subseteq (A, z)$, and the theorem follows.

Remark. Case (ii) may actually occur; for example, $\{0\}$ is maximal closed in $\boldsymbol{R}^+$. See also §1.4, Example 1.1.

Theorem 1.3. *Let A be a proper closed subsemi-algebra of B_+ which contains an interior point a. Then A is contained in a maximal closed subsemi-algebra of B_+.*

Proof. It follows from Lemmata 1.1 and 1.2 that A_1 is also a proper closed subsemi-algebra of B_+. Choose $\lambda > 0$ sufficiently large that $b = a + \lambda$ is regular. Since $\lambda \in A_1$ and $a \in \operatorname{int} A_1$, it follows that b is an interior point of A_1. Choose an open neighborhood U of b which consists entirely of regular elements and is contained in A_1.

Let P be any proper closed subsemi-algebra of B_+ which contains A_1. Suppose, if possible, that $u \in U^{-1} \cap P$. Then $1 = uu^{-1} \in uU \subseteq \operatorname{int} P$, so that $P = B_+$, contrary to assumption. Hence $U^{-1} \cap P = \varnothing$. Zorn's Lemma permits the choosing of a maximal chain (ordered by inclusion) of such semi-algebras P, and the closure of the union of the members of this chain is a maximal closed subsemi-algebra of B_+ containing A_1 (and hence A), but not intersecting the open set U^{-1}.

1.3. **Geometric semi-algebras.** In this section, suppose that B is a real Banach algebra with $\Phi_B \neq \varnothing$.

Definition. Let $\sigma \in B^*$. σ is a *positive linear functional* iff $\sigma(x) \geqq 0\ (\forall x \in B_+)$. σ is a *positive normalized linear functional* (*p.n.l.f.*) iff σ is positive and satisfies $\sigma(1) = 1$.

Suppose Φ_B to have the Gelfand topology. For $x \in B$, the function f_x defined on Φ_B by

$$f_x(\phi) = \phi(x) \qquad (\forall \phi \in \Phi_B)$$

is continuous. The mapping $x \to f_x$ is a continuous homomorphism of B onto a uniformly dense subalgebra C_B of $C(\Phi_B)$. For a p.n.l.f. σ on B, define the functional $\bar{\sigma}$ on C_B by

$$\bar{\sigma}(f_x) = \sigma(x) \qquad (\forall x \in B).$$

It is easily verified that, if $f_x = f_y$, then $\sigma(x) = \sigma(y)$, so that the definition is meaningful. The positivity of $\bar{\sigma}$ on $C_B \cap C^+(\Phi_B)$ means that $\bar{\sigma}$ can be uniquely extended to a Radon probability measure defined on the whole of $C(\Phi_B)$.

Definition. Let σ be a p.n.l.f. on B; denote by $\bar{\sigma}$ the probability measure induced by σ on Φ_B. The functional γ_σ is defined for $x \in B_+$ by

$$\gamma_\sigma(x) = GM_{\bar{\sigma}} f_x.$$

Lemma 1.3. *Let σ be a p.n.l.f. on B. For $x, y \in B_+$, $\lambda \geqq 0$:*

(a) $0 \leqq \gamma_\sigma(x)$; $\gamma_\sigma(\lambda) = \lambda$;

(b) $y - x \in B_+ \Rightarrow \gamma_\sigma(x) \leqq \gamma_\sigma(y)$;

(c) $\lim_{\lambda \to 0^+} \gamma_\sigma(x + \lambda y) = \gamma_\sigma(x)$;

(d) $0 \leqq \gamma_\sigma(x) \leqq \sigma(x) \leqq \| x \|$;

(e) $\gamma_\sigma(xy) = \gamma_\sigma(x)\gamma_\sigma(y)$;

(f) $\gamma_\sigma(x + y) \geqq \gamma_\sigma(x) + \gamma_\sigma(y)$;

(g) $\gamma_\sigma(\lambda x) = \lambda \gamma_\sigma(x)$;

(h) *if $x \in G$, then $\gamma_\sigma(x^{-1}) = (\gamma_\sigma(x))^{-1}$.*

For $\phi \in \Phi_B$, it is true that $\gamma_\phi(x) = \phi(x)$.

Let σ_1 and σ_2 be two p.n.l.f.'s on B and suppose that $\sigma = \alpha\sigma_1 + (1-\alpha)\sigma_2$ $(0 < \alpha < 1)$. Then $\gamma_\sigma(x) = (\gamma_{\sigma_1}(x))^\alpha (\gamma_{\sigma_2}(x))^{1-\alpha}$ for all $x \in B_+$.

Proof. These properties are consequences of the corresponding properties of $GM_{\bar{\sigma}}$ given in §0.2 and the homomorphic character of the mapping $x \to f_x$. For example, (b) is proved by the following chain of implications:

$$y - x \in B_+ \Rightarrow (f_y - f_x)(\phi) = (f_{y-x})(\phi) \geqq 0 \qquad (\forall \phi \in \Phi_B)$$
$$\Rightarrow f_y \geqq f_x \Rightarrow \gamma_\sigma(y) = GM_{\bar{\sigma}} f_y \geqq GM_{\bar{\sigma}} f_x = \gamma_\sigma(x).$$

Lemma 1.4. *Let σ be a p.n.l.f. on B. Then γ_σ is continuous on the interior of B_+; in particular, γ_σ is continuous on $G \cap B_+$.*

Proof. Note that $x \to \gamma_\sigma(x)$ is the composition of the continuous mappings $x \to f_x$ (of $\operatorname{int} B_+$ into $C^{\#}(\Phi_B)$), $f \to \log f$ (of $C^{\#}(\Phi_B)$ into $C(\Phi_B)$), $f \to \bar{\sigma}(f)$ (of $C(\Phi_B)$ into $\boldsymbol{R}$), $t \to e^t$ (of $\boldsymbol{R}$ into $\boldsymbol{R}^+$).

THEOREM 1.4. *Let σ be a p.n.l.f. on B and let $\psi \in \Phi_B$. Define*:

$$H_{\sigma,\psi} \equiv \{x : x \in B_+,\ \psi(x) \leqq \gamma_\sigma(x)\}.$$

If $\sigma = \psi$, then $H_{\sigma,\psi} = B_+$. If $\sigma \neq \psi$, then $H_{\sigma,\psi}$ is a proper closed subsemi-algebra of B_+ which contains the identity.

Proof. If $\sigma = \psi$, then $\gamma_\sigma = \psi$, by Lemma 1.3, so that the first statement is true. Henceforth, assume that $\sigma \neq \psi$. If $H_{\sigma,\psi}$ were not proper in this case, then $\psi(x) \leqq \gamma_\sigma(x)$ for each $x \in B_+$, whereupon $(\log f)(\psi) \leqq \bar{\sigma}(\log f)$ for each function f in $C^{\#}(\Phi_B) \subseteq \mathrm{Cl}(C_B \cap C^{\#}(\Phi_B))$, and $f(\psi) \leqq \bar{\sigma}(f)$ for each function f in $C(\Phi_B)$. Since $C(\Phi_B)$ is closed under the taking of additive inverses, it follows that $f(\psi) = \bar{\sigma}(f)$ $(\forall f \in C(\Phi_B))$, and a contradiction is obtained.

The properties of the functional γ_σ listed in Lemma 1.3 can be used to show that $H_{\sigma,\psi}$ is a semi-algebra containing the identity. Suppose that y belongs to the closure of $H_{\sigma,\psi}$. Then there exists a sequence $\{y_n\}$ such that $y_n \in H_{\sigma,\psi}$ and $\| y - y_n \| < n^{-1}$ for each positive integer n. Therefore $|\phi(y - y_n)| < n^{-1}$, so that

$$\phi(y - (1/n)) < \phi(y_n) < \phi(y + (1/n))$$

for each $\phi \in \Phi_B$, each positive integer n. Since $\phi(y + (1/n) - y_n) > 0$ $(\forall \phi \in \Phi_B)$, it is true that $y + (1/n) - y_n \in B_+$, so that, by Lemma 1.3(b), $\gamma_\sigma(y_n) \leqq \gamma_\sigma(y + (1/n))$. Hence:

$$\psi(y - (1/n)) \leqq \psi(y_n) \leqq \gamma_\sigma(y_n) \leqq \gamma_\sigma(y + (1/n))$$

for each positive integer n. Letting n tend to infinity and making use of Lemma 1.3(c), we obtain that $\psi(y) \leqq \gamma_\sigma(y)$, so that $y \in H_{\sigma,\psi}$. Thus, $H_{\sigma,\psi}$ is closed and the proof is complete.

DEFINITION. A *geometric semi-algebra* contained in B_+ is a semi-algebra of the form $H_{\sigma,\psi}$ where σ is a p.n.l.f. on B and $\psi \in \Phi_B$. It is implicit in the definition that $\sigma \neq \psi$.

For any geometric semi-algebra, it can always be arranged that the p.n.l.f. involved in the definition is not a nontrivial convex combination of the homomorphism involved and another p.n.l.f. For, if $\sigma = \alpha\psi + (1 - \alpha)\omega$ where $0 < \alpha < 1$ and ω is a p.n.l.f., then, by the final part of Lemma 1.3, $H_{\sigma,\psi} = H_{\omega,\psi}$. The remainder of this section will be devoted to showing that the geometric semi-algebras are maximal closed subsemi-algebras of the principal semi-algebra. For H a closed subsemi-algebra of B_+ and v an element of B_+, the least closed subsemi-algebra of B_+ containing H and the element v will be denoted by (H, v). The maximality of H is equivalent to the property that $(H, v) = B_+$ whenever $v \in B_+ \setminus H$.

LEMMA 1.5. *Let $H_{\sigma,\psi}$ be a geometric semi-algebra contained in B_+. If u is an element of B_+ belonging to $G \setminus H_{\sigma,\psi}$, then $(H_{\sigma,\psi},\ u) = B_+$.*

Proof. It can be seen that $\gamma_\sigma(u^{-1}) > \psi(u^{-1})$, so that by the continuity of γ_σ and ψ on $B_+ \cap G$, there exists a neighbourhood V of u^{-1} contained in $H_{\sigma,\psi}$. Then uV is a neighbourhood of 1 contained in $(H_{\sigma,\psi}, u)$, and the result follows by the definition of B_+.

LEMMA 1.6. *Let $H_{\sigma,\psi}$ be a geometric semi-algebra contained in B_+. If $v \in B_+$ and $\psi(v) > \sigma(v)$, then $(H_{\sigma,\psi}, v) = B_+$.*

Proof. Choose $\lambda \in (0,1]$ such that $\lambda < (\|1 - v\|)^{-1}$. Then $u = \lambda v + (1-\lambda)$ satisfies the conditions of Lemma 1.5. Hence $B_+ = (H_{\sigma,\psi}, u) \subseteq (H_{\sigma,\psi}, v) \subseteq B_+$.

LEMMA 1.7. *Let $H_{\sigma,\psi}$ be a geometric semi-algebra contained in B_+. If $v \in B_+ \setminus H_{\sigma,\psi}$, and if $\psi(v) \leqq \sigma(v)$, then $(H_{\sigma,\psi}, v) = B_+$.*

Proof. Choose a positive real λ sufficiently small that $u = v + \lambda$ does not belong to $H_{\sigma,\psi}$. Then f_u is a regular element of $C^+(\Phi_B)$ and $(f_u)(\psi) = \psi(u) > \gamma_\sigma(u) = GM_{\bar\sigma} f_u$, so that $(f_u)^{-1}(\psi) < GM_{\bar\sigma}(f_u)^{-1}$ for the multiplicative inverse of f_u. Choose a uniform neighbourhood V of f_u^{-1} with $V \subseteq C^{\#}(\Phi_B)$ and $f(\psi) < GM_{\bar\sigma} f$ $(\forall f \in V)$. Since $C_B \cap C^+(\Phi_B)$ is uniformly dense in $C^+(\Phi_B)$, it is possible to find in V a function f_z, for some $z \in B_+$, such that $(f_u f_z)(\psi) - \bar\sigma(f_u f_z) > 0$. (Note that f_u^{-1} lies in the proper closed hyperplane $\{f : f \in C(\Phi_B), (f_u f)(\psi) - \bar\sigma(f_u f) = 0\}$.) Then $z \in H_{\sigma,\psi}$ and $\psi(uz) > \sigma(uz)$. Hence, $uz \in (H_{\sigma,\psi}, u)$ and, by Lemma 1.6,

$$B_+ = (H_{\sigma,\psi}, uz) \subseteq (H_{\sigma,\psi}, u) \subseteq (H_{\sigma,\psi}, v) \subseteq B_+.$$

Combining Lemmata 1.6 and 1.7, and noting that for distinct elements ψ_1 and ψ_2 in Φ_B, H_{ψ_1,ψ_2} is a geometric semi-algebra, we obtain the following result:

THEOREM 1.5. *Let B be a real Banach algebra with identity such that Φ_B possesses at least two elements. Then B_+ contains geometric semi-algebras and each such semi-algebra is a maximal closed subsemi-algebra of B_+.*

REMARK. If Φ_B has a single element, then this element is the only p.n.l.f.

1.4. **Other examples of maximal closed subsemi-algebras.** In general, not every maximal closed subsemi-algebra of the principal semi-algebra of a Banach algebra is geometric. This section indicates other possibilities.

THEOREM 1.6. *Let ψ be a complex homomorphism on B not contained in Φ_B. Then*

$$K_\psi \equiv \{x : x \in B_+,\ \psi(x) \text{ is real}\}$$

is a maximal closed subsemi-algebra of B_+.

Proof. Evidently K_ψ is a proper closed subsemi-algebra of B_+ which contains the identity. For $u \in B_+ \setminus K_\psi$, let (K_ψ, u) be the least closed semi-algebra containing K_ψ and u, and suppose that $\psi(u) = \lambda e^{i\theta}$. Note that θ is not an integral multiple of π.

Suppose $v \in B_+$ and that $\psi(v) = \alpha e^{i\omega}$. If ω is an integral multiple of π, then $v \in K_\psi$. If ω is not an integral multiple of π, then positive real β and ρ, and a positive integer m can be chosen so that $\lambda e^{i\theta} + \beta = \rho e^{i\omega/m}$ (for ω suitably modified if necessary) and $u + \beta$ is regular. It can be checked that $u + \beta \in (K_\psi, u)$ and that $v(u+\beta)^{-m} \in K_\psi \subseteq (K_\psi, u)$. Hence $v = v(u+\beta)^{-m}(u+\beta)^m \in (K_\psi, u)$. The required result follows.

Remark. A semi-algebra of the form K_ψ may certainly be strict (i.e. $x, -x \in K_\psi \Rightarrow x = 0$). For, if B be the real Banach algebra of functions continuous on the closed unit disc, analytic on its interior and taking real values on the real axis, then B_+, and, a fortiori, the K_ψ (corresponding to points of the disc not on the real axis) are strict.

The sort of situation which may occur even when there are no strictly complex homomorphisms on B is illustrated by two examples.

Example 1.1. Let $K^{(2)}$ be the set of all real triples $x \equiv (x_0, x_1, x_2)$ with the norm and algebraic operations defined by:

$$\begin{aligned} \|x\| &= \sum_{i=0}^{2} |x_i|, \\ (x+y)_i &= x_i + y_i, \\ (xy)_i &= \sum_{j=0}^{i} x_j y_{i-j}, \qquad i = 0,1,2 \\ (\lambda x)_i &= \lambda x_i \end{aligned}$$

for elements x and y, and real λ. The multiplicative identity is the element $1 \equiv (1,0,0)$. Since $\Phi_{K^{(2)}}$ consists of the single homomorphism $x \to x_0$, the princinal semi-algebra $K_+^{(2)}$ is the set $\{x\colon x \in K^{(2)},\ x_0 \geqq 0\}$.

Define the sets:

$$\begin{aligned} P &\equiv \{x\colon x \in K_+^{(2)},\ x_1 \geqq 0\}, \\ Q &\equiv \{x\colon x \in K_+^{(2)},\ x_1 \leqq 0\}, \\ R &\equiv \{x\colon x \in K_+^{(2)},\ x_0 = 0\}, \\ S_\beta &\equiv \{x\colon x \in K_+^{(2)};\ 2x_0x_2 - x_1^2 \geqq \beta x_0 x_1;\ x_1 = 0,\ x_2 \geqq 0 \text{ when } x_0 = 0\}, \end{aligned}$$

where β is a real number. Each of these sets is a maximal closed subsemi-algebra of $K_+^{(2)}$. The maximality of R is a consequence of Theorem 1.2. The maximality of the others can be proved by an argument similar to that used for geometric semi-algebras.

Example 1.2. Let $C^{(2)}[0,1]$ be the Banach algebra of all real-valued continuous twice continuously differentiable functions defined on the closed unit interval, with the algebraic operations defined pointwise and the norm $\|\cdot\|$ given by $\|f\| \equiv \sup\{|f(t)| + |f'(t)| + \frac{1}{2}|f''(t)| : 0 \leqq t \leqq 1\}$ for any member f. The principal semi-algebra $C_+^{(2)}[0,1]$ is precisely the set of functions which take only non-negative values.

For s a fixed point of $[0,1]$, β some real number and α some *nonpositive* real number, define the linear functional ρ on $C^{(2)}[0,1]$ by

$$\rho(g) \equiv g(s) + \beta g'(s) + \alpha g''(s) \qquad (\forall g \in C^{(2)}[0,1]).$$

The functional ρ can be extended in an obvious manner so that when $f \in C_+^{(2)}[0,1]$ and $f(s) > 0$, $\log f$ exists in a neighbourhood of s and $\rho(\log f)$ is defined. Define τ_ρ for $f \in C_+^{(2)}[0,1]$ by

$$\tau_\rho(f) \equiv \begin{cases} \exp \rho(\log f) & \text{whenever } f(s) > 0 \\ \lim_{\lambda \to 0^+} \tau_\rho(f + \lambda) & \text{whenever } f(s) = 0. \end{cases}$$

τ_ρ may be positively infinite when $f(s) = 0$. Leaving the products $\infty \cdot \infty$ and $0 \cdot \infty$ undefined and using the usual conventions otherwise in dealing with infinity, we can show that, for $f, g \in C_+^{(2)}[0,1]$:

$$0 \leqq \tau_\rho(f); \; \tau_\rho(f+g) \leqq \tau_\rho(f) + \tau_\rho(g);$$
$$\tau_\rho(\lambda) = \lambda, \; \tau_\rho(\lambda f) = \lambda \tau_\rho(f) \; (\forall \lambda \geqq 0);$$
$$\tau_\rho(fg) = \tau_\rho(f)\tau_\rho(g),$$

whenever the latter product is defined. For σ a p.n.l.f. on $C^{(2)}[0,1]$ not equal to ρ, define the set

$$H_{\sigma,\rho} \equiv \{f \in C_+^{(2)}[0,1], \quad \tau_\rho(f) \leqq \gamma_\sigma(f)\}.$$

$H_{\sigma,\rho}$ is a maximal closed subsemi-algebra of $C_+^{(2)}[0, 1]$.

CHAPTER 2

2.1. **Intersections of geometric semi-algebras contained in $C^+(E)$.** In this section a characterization of those subsemi-algebras of $C^+(E)$ which are the intersections of families of geometric semi-algebras will be given. Define:

$$H_{\sigma,\xi} \equiv \{f : f \in C^+(E), \, f(\xi) \leqq GM_\sigma f\}$$

for a point ξ in E and a probability measure σ on E. Since every homomorphism of $C(E)$ into the real numbers arises from a point of E, each geometric semi-algebra has the form $H_{\sigma,\xi}$.

DEFINITION. A subset S of $C^+(E)$ is *power closed* iff, for each positive real λ, S contains, along with any member f, the function f^λ, defined for $\xi \in E$ by

$$f^\lambda(\xi) \equiv \text{principal value } (f(\xi))^\lambda.$$

A uniformly closed power closed subsemi-algebra of $C^+(E)$ is called a *cornet.*

Any geometric semi-algebra is a cornet with identity. Further, since for f, $g \in C^+(E)$, $\lim_{n\to\infty}(f^n + g^n)^{1/n} = f \cup g$, each cornet is an upper semi-lattice. (The term "lim" refers to the taking of the uniform limit.)

THEOREM 2.1. *Let P be a uniformly closed subset of $C^+(E)$ which satisfies the conditions*:

(i) *P is closed under multiplication*;

(ii) *$\lambda P \subseteq P$ for each positive real λ*;

(iii) *P is an upper semi-lattice*;

(iv) *P is power closed*;

(v) *P contains the identity function* 1.

Let $\mathscr{F}$ be the set of all pairs (σ, ξ) with σ a probability measure and ξ a point of E, such that $P \subseteq H_{\sigma,\xi}$. Then:

$$P = \bigcap_{(\sigma,\xi)\in\mathscr{F}} H_{\sigma,\xi}.$$

REMARK. It is not stipulated that P is closed under addition. This more general form of the theorem will be useful later. The theorem is valid even when $\mathscr{F}$ is void.

Proof. Let $P^{\#} = P \cap C^{\#}(E)$. Then, since $f \cup \varepsilon \in P^{\#}$ for $f \in P$ and $\varepsilon > 0$, P is the uniform closure of $P^{\#}$. The set $W \equiv \log P^{\#}$ is a closed wedge in $C(E)$ which is an upper semi-lattice. Let $\mathscr{U}_1$ and $\mathscr{U}_2$ be defined with respect to W as in Theorem 0.1. Because the functions 1 and -1 belong to W, $\mathscr{U}_2$ consists of the zero measure alone and, for each $(\sigma,\xi) \in \mathscr{U}_1$, σ is a probability measure.

By Theorem 0.1, $W = \bigcap_{(\sigma,\xi)\in\mathscr{U}_1} U_{\sigma,\xi}$, so that $P^{\#} = (\bigcap_{(\sigma,\xi)\in\mathscr{U}_1} H_{\sigma,\xi}) \bigcap C^{\#}(E)$. It is straightforward to show that $P = \bigcap_{(\sigma,\xi)\in\mathscr{U}_1} H_{\sigma,\xi}$. Finally, noting that $\mathscr{U}_1 \subseteq \mathscr{F}$ and that $P \subseteq H_{\sigma,\xi}$ for $(\sigma,\xi) \in \mathscr{F}$, we obtain that $P = \bigcap_{(\sigma,\xi)\in\mathscr{U}_1} H_{\sigma,\xi} \supseteq \bigcap_{(\sigma,\xi)\in\mathscr{F}} H_{\sigma,\xi} \supseteq P$.

COROLLARY 1. *The set P described in the theorem is a semi-algebra* (*and hence a cornet*).

COROLLARY 2. *Let A be a cornet with identity, $\mathscr{F}$ be the set of all pairs (σ,ξ) with σ a probability measure, ξ a point of E, such that $A \subseteq H_{\sigma,\xi}$. Then $A = \bigcap_{(\sigma,\xi)\in\mathscr{F}} H_{\sigma,\xi}$.*

The next result is a useful application of Theorem 2.1. For a subsemi-algebra A of $C(E)$, define the subset of E:

$$N(A) \equiv \{\eta \colon \eta \in E,\, f(\eta) = 0 \quad (\forall f \in A)\}.$$

THEOREM 2.2. *Let A be a uniformly closed subsemi-algebra of $C^+(E)$ with $N(A)$ void. Define*:

$P_A \equiv \{f \colon f \in C^+(E), f^n \in A$ *for some positive integer* $n \equiv n(f)\}$

$\sqrt{A} \equiv \bar{P}_A$, *the uniform closure of* P_A.

Then $\sqrt{A}$ is the least cornet containing A.

Proof. Clearly, any cornet containing A must contain $\sqrt{A}$. It remains only to show that $\sqrt{A}$ is a cornet. We show that $\sqrt{A}$ satisfies the five conditions of Theorem 2.1 and apply Corollary 1.

(i) Evidently, P_A is closed under multiplication; hence, the closure of P_A is also.

(ii) P_A, and hence its closure, is closed under multiplication by non-negative scalars.

(iii) Let $f,\, g \in P_A$. Choose positive integers m, n such that f^m and g^n belong to A. Then for every positive integer k, $f^{mnk} + g^{mnk} \in A$, so that $(f^{mnk} + g^{mnk})^{1/mnk} \in P_A$. Then (letting k tend to infinity) we see that $f \cup g \in \sqrt{A}$. If f and g are functions in $\sqrt{A}$, then there exist sequences $\{f_n\}$ and $\{g_n\}$ contained in P_A with limits f and g respectively. Since, for each positive integer n, $f_n \cup g_n \in \sqrt{A}$, then $f \cup g = \lim_{n\to\infty}(f_n \cup g_n) \in \sqrt{A}$.

(iv) If $f \in P_A$, then for any pair p,q of positive integers $f^{p/q} \in P_A$. Then, for each positive real $\lambda, f^{\lambda} \in \sqrt{A}$. If $f \in \sqrt{A}, f^{\lambda}$ can be approximated by a sequence $\{f_n^{\lambda} : f_n \in P_A\}$, so that $f^{\lambda} \in \sqrt{A}$.

(v) For each point ξ in E, there exists a function $f_\xi \in A$ such that f_ξ is positive on a neighbourhood of ξ. Because of the compactness of E, it is possible to find a finite set of such functions f_ξ whose sum f is an everywhere positive function belonging to A. Then $1 = \lim_{\lambda\to 0^+} f^{\lambda} \in \sqrt{A}$.

The theorem now follows.

So far, the theorems of this section have given information only about cornets which contain the identity. Let A now be an arbitrary cornet, and denote by A_c the least *cornet* which contains A along with the identity function 1. Observe that $A = A_c$ if and only if $1 \in A$, and if and only if the set $N(A)$ is void. Note also that A_c will not in general be the same as the semi-algebra A_1 defined in §1.2. The investigation of cornets without the identity involves the following lemma.

LEMMA 2.1. *Let f, g be two functions in $C^+(E)$ and let β, ε be positive real numbers. Suppose that for some point η in E, $f(\eta) = 0$, and that for some positive integer n, $\|f - (g+\beta)^{1/n}\| < \frac{1}{2}\varepsilon$. Then $\|f - g^{1/n}\| < \varepsilon$.*

Proof. Clearly $0 < \beta \leqq g(\eta) + \beta < (\frac{1}{2}\varepsilon)^n$. Since $(t+\beta)^{1/n} - t^{1/n}$ is decreasing for non-negative values of t, it follows that for each point ξ in E,

$$0 \leqq (g+\beta)^{1/n}(\xi) - g^{1/n}(\xi) \leqq \beta^{1/n} < \frac{1}{2}\varepsilon.$$

Thus,

$$\|f - g^{1/n}\| \leqq \|f - (g+\beta)^{1/n}\| + \|(g+\beta)^{1/n} - g^{1/n}\| < \varepsilon.$$

There is a very simple relationship between the cornets A and A_c. In fact, we show that

$$A = \{f : f \in A_c,\ f(\eta) = 0\ \ (\forall \eta \in N(A))\}.$$

This is true if $N(A)=\varnothing$. Suppose henceforth that $N(A)\neq\varnothing$. Evidently, A is contained in the right-hand side. Since A_c is the least cornet containing the closed semi-algebra A_1, it follows from Theorem 2.2 that $A_c=\sqrt{A_1}$. Suppose that the function f belongs to A_c and vanishes on $N(A)$. For given positive ε, there exists a function $g\in A$, a positive real β and a positive integer n such that $\|f-(g+\beta)^{1/n}\|<\frac{1}{2}\varepsilon$. Since f vanishes on the nonvoid set $N(A)$, Lemma 2.1 shows that $\|f-g^{1/n}\|<\varepsilon$. Because A is a cornet containing g, A contains $g^{1/n}$: let $f_\varepsilon=g^{1/n}$. It is concluded that for any function f in A_c vanishing on $N(A)$, there exists a function f_ε in A with $\|f_\varepsilon-f\|<\varepsilon$; hence $f\in A$.

The connection between A and A_c leads directly to the following result.

THEOREM 2.3. *Let A be a cornet. Suppose that $\mathscr{F}$ is the set of pairs (σ,ξ) with σ a probability measure on E and ξ a point of E, such that $A\subseteq H_{\sigma,\xi}$. Then*

$$A=\{f:f\in H_{\sigma,\xi}\ (\forall(\sigma,\xi)\in\mathscr{F}),\ f(\eta)=0\ (\forall\eta\in N(A))\}.$$

THEOREM 2.4. *Let A be a uniformly closed subsemi-algebra of $C^+(E)$. Define $\sqrt{A}$ exactly as in Theorem 2.2. Then $\sqrt{A}$ is the least cornet containing A.*

Proof. Only the case $N(A)\neq\varnothing$ is to be considered. The arguments used in Theorem 2.2 apply here to show that $\sqrt{A}$ is closed under multiplication, $\lambda\sqrt{A}\subseteq\sqrt{A}$ and $\sqrt{A}$ is power closed. It remains to be verified that $\sqrt{A}$ is closed under addition.

$\sqrt{A}$ is contained in $\sqrt{A_1}$, which, by Theorem 2.2, is known to be a semi-algebra. Therefore, if $f,\ g\in\sqrt{A}$, then $f+g\in\sqrt{A_1}$. By the definition of $\sqrt{A_1}$, for given positive ε, there exists $h\in A$, $\alpha>0$, a positive integer n, with $\|(f+g)-(h+\alpha)^{1/n}\|<\frac{1}{2}\varepsilon$. Since $N(A)=N(\sqrt{A})$, f, g and $f+g$ all vanish on $N(A)$. By Lemma 2.1, $\|(f+g)-h^{1/n}\|<\varepsilon$. Thus, $f+g$ can be uniformly approximated by integral roots of elements of A, so that $f+g\in\sqrt{A}$. Hence $\sqrt{A}$ is a cornet, and the theorem follows.

2.2. **Every maximal closed subsemi-algebra of $C^+(E)$ is geometric.** Let μ and ν be two Radon measures on E. Following F. Riesz (see [14] or [3]), define for $f\in C^+(E)$ the functional

$$\sigma(f)\equiv\sup\{\mu(g)+\nu(f-g):0\leqq g\leqq f,\ g\in C^+(E)\}.$$

This functional is linear and continuous on $C^+(E)$, and can be extended uniquely to a Radon measure, which is denoted by $\mu\cup\nu$. Define: $\mu_+\equiv\mu\cup 0$, $\mu_-\equiv(-\mu)\cup 0$. Then μ_+ and μ_- are positive Radon measures and $\mu=\mu_+-\mu_-$. For $f\in C(E)$, the Radon measure $f\cdot\mu$ is defined by $(f\cdot\mu)(g)\equiv\mu(fg)$ $(\forall g\in C(E))$.

If K_1 and K_2 are two subcones of $C^+(E)$ and $\mu_1\in K_1'$, $\mu_2\in K_2'$, then $\mu_1\cup\mu_2\in(\mathrm{Cl}(K_1+K_2))'$.

LEMMA 2.2. *Let f be a function in $C^+(E)$, and δ a real number such that $0<\delta<f(\xi)<1$ $(\forall\xi\in E)$. Suppose that μ is a Radon measure on E*

Then $(f\cdot\mu)\cup\mu$ is a positive measure if and only if μ is a positive measure.

Proof. The positivity of μ implies that of $f\cdot\mu$, and hence that of $(f\cdot\mu)\cup\mu$. Suppose that μ is not positive. Then μ_- is nontrivial, so that a function $h\in C^+(E)$ can be found to satisfy

$$0<\mu(-h)\leqq\mu_-(h)\leqq(1-\delta)^{-1}\mu(-h).$$

Then

$$\begin{aligned}((f\cdot\mu)\cup\mu)(h) &= \sup\{(f\cdot\mu)(g)+\mu(h-g):0\leqq g\leqq h\}\\ &= \mu(h)+\sup\{\mu(-(1-f)g):\ 0\leqq g\leqq h\}\\ &\leqq -(1-\delta)\mu_-(h)+\mu_-((1-f)\,h)\\ &= \mu_-((\delta-f)\,h)<0,\end{aligned}$$

so that $(f\cdot\mu)\cup\mu$ is not positive.

Lemma 2.3. *Let A be a proper uniformly closed subsemi-algebra of $C^+(E)$. Suppose the function f in $C^+(E)$ is such that*

(i) *for some real δ, $0<\delta<f(\xi)<1$ $(\forall\xi\in E)$;*

(ii) *f^2 belongs to A.*

Then (A,f), the least closed semi-algebra containing A and the function f is also proper in $C^+(E)$.

Proof. If $f\in A$, the result is obvious. Suppose then that $f\notin A$. Since A is proper, there exists a nonpositive Radon measure μ belonging to A'. Then $f\cdot\mu$ belongs to $(Af)'$. Hence $(f\cdot\mu)\cup\mu$ is a nonpositive Radon measure in $(\mathrm{Cl}(A+Af))'$, so that the cone $\mathrm{Cl}(A+Af)$ is a proper cone of $C^+(E)$. Now $A+Af$ is a semi-algebra contained in every closed semi-algebra containing A and the function f. Hence $\mathrm{Cl}(A+Af)=(A,f)$ and the lemma follows.

Theorem 2.5. *Let A be a maximal closed subsemi-algebra of $C^+(E)$. Then A is power closed. If E contains at least two distinct points, then each maximal closed subsemi-algebra of $C^+(E)$ is a geometric semi-algebra.*

Proof. If E consists of a single point, then $C^+(E)$ is isomorphic to the semi-algebra of non-negative reals, so that the only proper closed subsemi-algebra of $C^+(E)$ is $\{0\}$, and this is power closed.

If E consists of more than one point, then the maximality of A implies, by Theorem 1.2, that $1\in A$. Let g be an element of A which is bounded away from zero. Define

$$f\equiv(2\,\|g^{1/2}\|)^{-1}g^{1/2};\ \delta\equiv\inf\{(3\,\|g^{1/2}\|)^{-1}g^{1/2}(\xi):\ \xi\in E\}.$$

Then f and δ satisfy the conditions of Lemma 2.3, so that, by the maximality of

A, $(A,f) = A$. Hence $g^{1/2} = 2\|g^{1/2}\|f \in A$. Similarly, it can be shown in turn that $g^{1/4}, g^{1/8}, \cdots$ all belong to A. Since any positive real λ can be approximated by the sum of an integer and various powers of $\frac{1}{2}$, and since A is closed, it follows that $g^{\lambda} \in A$ $(\forall \lambda > 0)$. If, now, h is any element in A, then for each positive integer n, $(h + n^{-1}) \in A \cap C^{*}(E)$, so that $(h + n^{-1})^{\lambda} \in A$ $(\forall \lambda > 0)$. Taking the limit as $n \to \infty$, we see that $h^{\lambda} \in A$. Thus A is a cornet, and, being maximal, must, by Theorem 2.1 Corollary 2, be geometric.

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