ON THE JORDAN STRUCTURE OF C^*-ALGEBRAS

BY
ERLING STØRMER

1. **Introduction.** The following problem presents itself in operator theory: how well does the Jordan structure of the self-adjoint operators determine the ring structure of a C^*-algebra? We shall be concerned with two aspects of this; first we shall give an intrinsic characterization for a Jordan algebra of self-adjoint operators to be the self-adjoint part of a C*-algebra (Theorem 2.16), then we shall show that a C^*-homomorphism from one C^*-algebra into another is the sum of a *-homomorphism and a *-anti-homomorphism (Theorem 3.3). The first result roughly states that the Jordan algebras in question are those which are algebraically the same as the self-adjoint parts of C^*-algebras while at the same time not too real (cf. the real symmetric matrices which satisfy the first property but not the second). In the finite-dimensional case this result is immediate from a paper by Jordan, von Neumann and Wigner [8], in which they characterize all finite-dimensional Jordan algebras. C^*-homomorphisms, or rather Jordan homomorphisms, have been studied by several authors; for an exposition see [6]. The key result for our applications in Jacobson and Rickart's [7], which states that a Jordan homomorphism from an $n \times n$ matrix ring over a ring with identity is the sum of a homomorphism and anti-homomorphism. Kadison [9] used this to show that C^*-homomorphisms from von Neumann algebras onto C^*-algebras are in a strong sense sums of *-homomorphisms and *-anti-homomorphisms. We shall use his arguments in order to obtain our result.

The main technique used in this paper is the recognition by Sherman [13], see also Takeda [15] and Grothendieck [5] that the second dual of a C^*-algebra $\mathfrak{A}$ is a von Neumann algebra, and that $\mathfrak{A}$ is weakly dense in $\mathfrak{A}^{**}$. We may thus reduce our problems to those of von Neumann algebras, in which case they are readily available. For a neat exposition on the major facts on $\mathfrak{A}^{**}$ we refer the reader to the introduction of [3]. We are greatly indebted to E. Effros for conversations on this technique and for remarks which helped us to prove Theorem 3.3. We are also indebted to D. Topping for valuable correspondence on Jordan algebras.

By a C^*-algebra we shall mean a uniformly closed self-adjoint algebra of operators on a Hilbert space. Following Topping [16] we call a Jordan algebra of self-adjoint operators a *JC-algebra* if it is uniformly closed; the Jordan product

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is denoted by $A \circ B$ ($= \frac{1}{2}(AB + BA)$). A *Jordan ideal* $\mathfrak{J}$ in a JC-algebra $\mathfrak{A}$ is a Jordan algebra $\mathfrak{J} \subset \mathfrak{A}$ such that $A \circ B \in \mathfrak{J}$ whenever $A \in \mathfrak{A}$ and $B \in \mathfrak{J}$. By a *C^*-homomorphism* of a C^*-algebra into another C^*-algebra we mean a linear self-adjoint map preserving squares of self-adjoint operators (viz.$\phi(A^2) = \phi(A)^2$). Since such a map ϕ is positive and carries the identity into a projection its norm equals 1 if its domain $\mathfrak{A}$ has an identity except if $\phi = 0$, in which case the norm is 0. If not we can extend ϕ to a C^*-homomorphism of $\mathfrak{A}$ with an identity adjoined to it by $\phi(\lambda I + A) = \lambda I + \phi(A)$, where I on the right side denotes the identity in the smallest C^*-algebra with identity containing $\phi(\mathfrak{A})$. Thus ϕ still has norm 1, hence is uniformly continuous.

2. **Jordan algebras**. In this section we shall give an intrinsic characterization for a JC-algebra to be the self-adjoint part of a C^*-algebra. We shall need some notation.

DEFINITION 2.1. Let $\mathfrak{A}$ be a JC-algebra. Then $\mathfrak{R}(\mathfrak{A})$ denotes the uniformly closed real algebra generated by $\mathfrak{A}$. $(\mathfrak{A})$ denotes the C^*-algebra generated by $\mathfrak{A}$.

Thus operators of the form $\sum_{i=1}^{n}\prod_{j=1}^{m_i}A_{ij}$ with A_{ij} in $\mathfrak{A}$ are uniformly dense in $\mathfrak{R}(\mathfrak{A})$. If we by $i\mathfrak{R}(\mathfrak{A})$ mean operators of the form iA, $A \in \mathfrak{R}(\mathfrak{A})$ then $\mathfrak{R}(\mathfrak{A}) + i\mathfrak{R}(\mathfrak{A})$ is uniformly dense in $(\mathfrak{A})$.

REMARK 2.2. If $\mathfrak{A}$ is a JC-algebra, then $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A})$ is an ideal in $(\mathfrak{A})$. In fact, if $A, B \in \mathfrak{R}(\mathfrak{A})$ and $C = iD \in \mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A})$ then

$$(A + iB)C = AC + iBiD = AC - BD \in \mathfrak{R}(\mathfrak{A}),$$

and similarly $(A + iB)C \in i\mathfrak{R}(\mathfrak{A})$. Thus $(A + iB)C \in R(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A})$. Since

$$\mathfrak{R}(\mathfrak{A}) \cap iR(\mathfrak{A})$$

is uniformly closed, and operators of the form $A + iB$ with $A, B \in \mathfrak{R}(\mathfrak{A})$ are uniformly dense in $(\mathfrak{A})$, $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A})$ is a left ideal in $(\mathfrak{A})$, and symmetrically a right ideal.

DEFINITION 2.3. If $\mathfrak{F}$ is a family of operators we denote by $\mathfrak{F}_{SA}$ the self-adjoint operators in $\mathfrak{F}$.

DEFINITION 2.4. If $\mathfrak{A}$ is a JC-algebra then $\mathfrak{A}$ is *reversible* if $\prod_{i=1}^{n}A_i + \prod_{i=n}^{1}A_i \in \mathfrak{A}$ whenever $A_1, \cdots, A_n \in \mathfrak{A}$, where by $\prod_{i=1}^{n}A_i$ we mean the product $A_1A_2 \cdots A_n$, in the indicated order.

As pointed out to us by D. Topping, reversible Jordan algebras have been studied by P.M. Cohn [1]. He showed that it suffices to show that products of form $\prod_{i=1}^{4}A_i + \prod_{i=4}^{1}A_i$ are in $\mathfrak{A}$ in order that $\mathfrak{A}$ be reversible. We shall not make use of this fact.

REMARK 2.5. If $\mathfrak{A}$ is a reversible JC-algebra then $\mathfrak{A} = \mathfrak{R}(\mathfrak{A})_{SA}$. In fact if

$$A = \sum_{i=1}^{n}\prod_{j=1}^{m_i} A_{ij} \in \mathfrak{R}(\mathfrak{A})_{SA}$$

then

$$A = \frac{1}{2}(A + A^*) = \frac{1}{2}\sum_{i=1}^{n}\left(\prod_{j=1}^{m_i} A_{ij} + \prod_{j=m}^{1} A_{ij}\right) \in \mathfrak{A}.$$

Thus $\mathfrak{R}(\mathfrak{A})_{SA} \subset \mathfrak{A}$, and they are equal.

In [4] it is shown that if $\mathfrak{A}$ is a JC-algebra and $\mathfrak{J}$ a uniformly closed Jordan ideal in $\mathfrak{A}$, then $(\mathfrak{J})$ is an ideal in $(\mathfrak{A})$ and $\mathfrak{J} = (\mathfrak{J}) \cap \mathfrak{A}$. Thus the following definition makes sense.

Definition 2.6. Let $\mathfrak{A}$ and $\mathfrak{J}$ be as above. Then by $\overline{\mathfrak{A}/\mathfrak{J}}$ we shall mean the image of $\mathfrak{A}/\mathfrak{J}$ in $(\mathfrak{A})/(\mathfrak{J})$ under the Jordan isomorphism $A + \mathfrak{J} \to A + (\mathfrak{J})$.

Lemma 2.7. *Let $\mathfrak{A}$ be a JC-algebra and $\mathfrak{J}$ a uniformly closed Jordan ideal in $\mathfrak{A}$. Then $(\overline{\mathfrak{A}/\mathfrak{J}}) = (\mathfrak{A})/(\mathfrak{J})$.*

Proof. From Definition 2.6 it is clear that $(\overline{\mathfrak{A}/\mathfrak{J}}) \subset (\mathfrak{A})/(\mathfrak{J})$. Let

$$A + (\mathfrak{J}) \in (\mathfrak{A})/(\mathfrak{J}).$$

Then $A = \lim \sum_{i=1}^{n} \prod_{j=1}^{m_i} \lambda_{ij} A_{ij}$ with $A_{ij} \in \mathfrak{A}$, $\lambda_{ij} \in \boldsymbol{C}$. Since the map $(\mathfrak{A}) \to (\mathfrak{A})/(\mathfrak{J})$ is a continuous $*$-homomorphism

$$\begin{aligned} A + (\mathfrak{J}) &= \lim \sum_{i=1}^{n} \prod_{j=1}^{m_i} (\lambda_{ij} A_{ij} + (\mathfrak{J})) \\ &= \lim \sum_{i=1}^{n} \prod_{j=1}^{m_i} \lambda_{ij}(A_{ij} + (\mathfrak{J})) \in (\overline{\mathfrak{A}/\mathfrak{J}}). \end{aligned}$$

Lemma 2.8. *Let $\mathfrak{A}$ be a JC-algebra. Then the map $\rho \to \rho \mid \mathfrak{A}$ is a one-one correspondence between $*$-representations of $(\mathfrak{A})$ onto $\boldsymbol{C}$ and Jordan representations of $\mathfrak{A}$ onto $\boldsymbol{R}$.*

Proof. Let ρ be a $*$-representation of $(\mathfrak{A})$ onto $\boldsymbol{C}$. Then $\rho \mid \mathfrak{A}$ is a Jordan representation of $\mathfrak{A}$ into $\boldsymbol{R}$. If $\rho(\mathfrak{A}) = 0$ then by [4] $\rho((\mathfrak{A})) = 0$, contrary to assumption. Thus $\rho(\mathfrak{A}) = \boldsymbol{R}$. Conversely, let ρ be a Jordan representation of $\mathfrak{A}$ onto $\boldsymbol{R}$. Let $\mathfrak{J}$ be its kernel. Then $\mathfrak{A}/\mathfrak{J} \cong \boldsymbol{R}$. Thus $\overline{\mathfrak{A}/\mathfrak{J}} \cong \boldsymbol{R}$, and by Lemma 2.7, $(\mathfrak{A})/(\mathfrak{J}) = (\overline{\mathfrak{A}/\mathfrak{J}}) \cong \boldsymbol{C}$; ρ induces a $*$-representation of $(\mathfrak{A})$ onto $\boldsymbol{C}$ whose restriction to $\mathfrak{A}$ is ρ.

Lemma 2.9. *Let $\mathfrak{A}$ be a JC-algebra with a separating family of Jordan representations of $\mathfrak{A}$ onto $\boldsymbol{R}$. Then $\mathfrak{A}$ is abelian.*

Proof. If $A, B \in \mathfrak{A}$ and ρ is a Jordan representation of $\mathfrak{A}$ onto $\boldsymbol{R}$ then $\rho(ABA) = \rho(A)\rho(B)\rho(A) = \rho(A)^2 \circ \rho(B) = \rho(A^2) \circ \rho(B) = \rho(A^2 \circ B)$. Thus $ABA = A^2 \circ B$, $AB = BA$ by [16, Proposition 1]. A simpler proof of this latter fact was suggested to us by the referee. In fact, if $ABA = \frac{1}{2}(A^2B + BA^2)$ for all A in $\mathfrak{A}$, then it holds for each spectral projection E of A in place of A, by strong continuity of multiplication on bounded sets. Thus $EBE = \frac{1}{2}(EB + BE)$, and

$0=(I-E)EBE=\frac{1}{2}(I-E)(EB+BE)=\frac{1}{2}(I-E)BE$. Since B is self-adjoint, $EB=BE$ for each such E, and $AB=BA$, as asserted.

LEMMA 2.10. *Let $\mathfrak{A}$ be a reversible JC-algebra and $\mathfrak{J}$ a uniformly closed Jordan ideal in $\mathfrak{A}$. Then $\mathfrak{J}$ is a reversible JC-algebra.*

Proof. Let $A_i \in \mathfrak{J}$, $i=1,\cdots,n$. Then $\prod_{i=1}^n A_i+\prod_{i=n}^1 A_i \in \mathfrak{A} \cap (\mathfrak{J})=\mathfrak{J}$, by [4, Theorem 2].

LEMMA 2.11. *Let $\mathfrak{A}$ be a JC-algebra and $\mathfrak{J}$ a uniformly closed Jordan ideal in $\mathfrak{A}$. If $\overline{\mathfrak{A}/\mathfrak{J}}=(\overline{\mathfrak{A}/\mathfrak{J}})_{SA}$ and $\mathfrak{J}=(\mathfrak{J})_{SA}$, then $\mathfrak{A}=(\mathfrak{A})_{SA}$.*

Proof. Let A be self-adjoint in $(\mathfrak{A})$. If $A\in(\mathfrak{J})$ then $A\in\mathfrak{J}$, hence in $\mathfrak{A}$. In the general case $A+(\mathfrak{J})$ is self-adjoint in $(\mathfrak{A})/(\mathfrak{J})=(\overline{\mathfrak{A}/\mathfrak{J}})$ (Lemma 2.7). Hence, by hypothesis there exists a self-adjoint operator B in $\mathfrak{A}$ such that $A+(\mathfrak{J})=B+(\mathfrak{J})$. Thus $A-B\in(\mathfrak{J})_{SA}=\mathfrak{J}$. $A\in\mathfrak{A}$, and $(\mathfrak{A})_{SA}=\mathfrak{A}$.

We remark that the converse is also true in the above lemma.

LEMMA 2.12. *Let $\mathfrak{A}$ be a JC-algebra. If $\mathfrak{A}$ has no Jordan representations onto **R** let $\mathfrak{M}=\mathfrak{A}$. Otherwise let $\mathfrak{M}$ be the intersection of the kernels of all Jordan representations of $\mathfrak{A}$ onto **R**. Assume*

(1) *$\mathfrak{A}$ is reversible,*

(2)$\mathfrak{R}(\mathfrak{M})=i\mathfrak{R}(\mathfrak{M})$. *Then* $\mathfrak{A}=(\mathfrak{A})_{SA}$.

Proof. By Lemmas 2.7 and 2.9 $(\mathfrak{A})/(\mathfrak{M})=(\overline{\mathfrak{A}/\mathfrak{M}})$ is abelian. Thus $(\overline{\mathfrak{A}/\mathfrak{M}})_{SA}=\overline{\mathfrak{A}/\mathfrak{M}}$. In order to show $\mathfrak{A}=(\mathfrak{A})_{SA}$ it suffices by Lemma 2.11 to show $\mathfrak{M}=(\mathfrak{M})_{SA}$. By Lemma 2.10 $\mathfrak{M}$ is reversible. Since $\mathfrak{R}(\mathfrak{M})=i\mathfrak{R}(\mathfrak{M})$, $\mathfrak{R}(\mathfrak{M})$ is a C^*-algebra containing $\mathfrak{M}$, hence $\mathfrak{R}(\mathfrak{M}) \supset(\mathfrak{M})$. Clearly $\mathfrak{R}(\mathfrak{M})\subset(\mathfrak{M})$, so $\mathfrak{R}(\mathfrak{M})=(\mathfrak{M})$. By Remark 2.5 $\mathfrak{M}=(\mathfrak{M})_{SA}$.

We shall now show that conditions (1) and (2) in Lemma 2.12 are necessary in order that $\mathfrak{A}=(\mathfrak{A})_{SA}$.

LEMMA 2.13. *Let $\mathfrak{A}$ be a von Neumann algebra with no central abelian projections. Then there exist eight self-adjoint operators S_i, $i=1,\cdots,8$, in the unit ball of $\mathfrak{A}$ such that $S_1S_2S_3S_4+S_5S_6S_7S_8=iI$.*

Proof. We first assume the identity I can be halved, i.e. there exist orthogonal projections E and F with $E+F=I$ and a partial isometry V in $\mathfrak{A}$ such that $VV^*=E$, $V^*V=F$. Then $E-F$, $i(V-V^*)$, $V+V^*$ are self-adjoint unitaries in $\mathfrak{A}$, and

$$iI=(E-F)i(V-V^*)(V+V^*).$$

In the general case there exists a central projection P in $\mathfrak{A}$ such that $\mathfrak{A}P$ has no finite type I portion and $\mathfrak{A}(I-P)$ is finite and type I. Thus $\mathfrak{A}(I-P)$ is the direct sum of $n\times n$ matrix algebras M_n over abelian von Neumann algebras, where $n\geqq 2$, by assumption, say $\mathfrak{A}(I-P)=\sum_{n\in J'}M_n$, $J'\subset\{2,3,\cdots\}$. Denote $\mathfrak{A}P$ by

M_0, and let $J = J' \cup \{0\}$. Thus $\mathfrak{A} = \sum_{n \in J} M_n$. Let E_n be the central projection in $\mathfrak{A}$ such that $\mathfrak{A}E_n = M_n$. If the identity in M_n can be halved then there exist three self-adjoint operators in the unit ball of M_n with product iE_n. This is the case with n even. If n is odd, $n \geqq 3$ and $n-1$ is even and $\geqq 2$. Thus there exist self-adjoint operators T_i, $i = 1, \cdots, 6$ in the unit ball of M_n such that

$$T_1T_2T_3 = \begin{bmatrix} i & & & \\ & \ddots & & \\ & & i & \\ & & & 0 \end{bmatrix}, \quad T_4T_5T_6 = \begin{bmatrix} 0 & & & \\ & i & & \\ & & \ddots & \\ & & & i \end{bmatrix}.$$

Let

$$S = \begin{bmatrix} 1 & & & & \\ & \frac{1}{2} & & & \\ & & \ddots & & \\ & & & \frac{1}{2} & \\ & & & & 1 \end{bmatrix}$$

be in M_n. Then $iE_n = S(T_1T_2T_3 + T_4T_5T_6)$. Hence for each M_n there exist self-adjoint operators S_i^n, $i = 1, \cdots, 8$ in the unit ball in M_n such that $S_1^nS_2^nS_3^nS_4^n + S_5^nS_6^nS_7^nS_8^n = iE_n$. Let $S_i = \sum_{n \in J} S_i^n$. Then S_i is in the unit ball of $\mathfrak{A}$ and is self-adjoint. Moreover $S_1S_2S_3S_4 + S_5S_6S_7S_8 = iI$. The proof is complete.

If $\mathfrak{A}$ is a C^*-algebra we denote by $(\mathfrak{A}_{SA})_1^n$ the set of operators of the form $\prod_{i=1}^n A_i$ with A_i self-adjoint in the unit ball of $\mathfrak{A}$. We denote by $\mathfrak{A}_1$ the unit ball of $\mathfrak{A}$.

LEMMA 2.14. *Let $\mathfrak{A}$ be as in Lemma* 2.13. *Then*

$$\mathfrak{A}_1 \subseteq (\mathfrak{A}_{SA})_1^5 + (\mathfrak{A}_{SA})_1^5 + (\mathfrak{A}_{SA})_1^1.$$

Proof. Let $A \in \mathfrak{A}_1$. Then $A = B + iC$ with $B, C \in (\mathfrak{A}_{SA})_1^1$. Since

$$iC = (iI)C \in (\mathfrak{A}_{SA})_1^5 + (\mathfrak{A}_{SA})_1^5$$

by Lemma 2.13, the proof is complete.

LEMMA 2.15. *Let $\mathfrak{A}$ be a C^*-algebra with no nonzero abelian representations. Then $\mathfrak{R}(\mathfrak{A}_{SA}) = \mathfrak{A}$.*

Proof. We may consider $\mathfrak{A}$ as a weakly dense subalgebra of its second dual $\mathfrak{A}^{**}$. By assumption $\mathfrak{A}^{**}$ has no central abelian projections; indeed, if E were one, then the map $A \to AE$ is an abelian representation of $\mathfrak{A}^{**}$, hence an abelian representation of $\mathfrak{A}$, nonzero since $\mathfrak{A}$ is weakly dense in $\mathfrak{A}^{**}$. By Lemma 2.14 $\mathfrak{A}_1^{**} \subseteq (\mathfrak{A}_{SA}^{**})_1^5 + (\mathfrak{A}_{SA}^{**})_1^5 + (\mathfrak{A}_{SA}^{**})_1^1$. Thus $\mathfrak{R}(\mathfrak{A}_{SA})^- = \mathfrak{A}^{**}$. In fact, let $A \in \mathfrak{A}^{**}$. Then by Lemma 2.14

$$A = \prod_{i=1}^{5} A_i + \prod_{i=6}^{10} A_i + A_{11} \text{ with } A_i \in \mathfrak{A}_{SA}^{**} \text{ and } \|A_i\| \leqq \|A\|, \quad i = 1, \cdots, 11.$$

Since $(\mathfrak{A}_{SA})_1^1$ is strongly dense in $(\mathfrak{A}_{SA}^{**})_1^1$ by the Kaplansky density theorem [2, Théorème 3, p. 46], we can choose nets $(A_{i\alpha})$ in $\mathfrak{A}_{SA}$ with $\|A_{i\alpha}\| \leqq \|A\|$ such that

$A_{i\alpha} \to A_i$ strongly with α. Since multiplication is strongly continuous on $\mathfrak{A}_1^{**}$ [2, p. 32], $\prod_{i=1}^{5} A_{i\alpha} + \prod_{i=6}^{10} A_{i\alpha} + A_{11\alpha} \to A$ strongly. Thus $A \in \mathfrak{R}(\mathfrak{A}_{SA})^-$ as asserted. In $\mathfrak{A}^{**}$ the weak and the weak* topologies coincide. Since $\mathfrak{R}(\mathfrak{A}_{SA})$ is convex its strong closure equals its weak closure [2, Théorème 1, p. 40]. Thus $\mathfrak{R}(\mathfrak{A}_{SA})^-$ equals the w^*-closure of $\mathfrak{R}(\mathfrak{A}_{SA})$. Since $\mathfrak{R}(\mathfrak{A}_{SA})$ is also uniformly closed

$$\mathfrak{R}(\mathfrak{A}_{SA}) = \mathfrak{R}(\mathfrak{A}_{SA})^- \cap \mathfrak{A} = \mathfrak{A}^{**} \cap \mathfrak{A} = \mathfrak{A}.$$

The proof is complete.

THEOREM 2.16. *Let $\mathfrak{A}$ be a JC-algebra. If $\mathfrak{A}$ has no Jordan representations onto the reals let $\mathfrak{M} = \mathfrak{A}$. Otherwise let $\mathfrak{M}$ be the intersection of the kernels of all Jordan representations of $\mathfrak{A}$ onto the reals. Then $\mathfrak{A}$ is the self-adjoint part of a C^*-algebra if and only if*

(1) *$\mathfrak{A}$ is reversible,*

(2) $\mathfrak{R}(\mathfrak{M}) = i\mathfrak{R}(\mathfrak{M})$.

Proof. By Lemma 2.12, Conditions (1) and (2) are sufficient. Conversely, assume $\mathfrak{A} = (\mathfrak{A})_{SA}$. It is then clear that $\mathfrak{A}$ is reversible. By Lemma 2.8, $\mathfrak{A}$ has no Jordan representations onto $\boldsymbol{R}$ if and only if $(\mathfrak{A})$ has no representations onto $\boldsymbol{C}$, in which case the theorem is immediate from Lemma 2.15. Otherwise let $\mathfrak{J}$ be the intersection of the kernels of all one dimensional representations of $(\mathfrak{A})$. Then $\mathfrak{J}$ has no one dimensional representations, in fact if ρ were a one dimensional representation of $\mathfrak{J}$ let $\bar{\rho}$ be an irreducible extension of ρ to $(\mathfrak{A})$. Then $\bar{\rho}(\mathfrak{J})$ is an irreducible C^*-algebra, since it is an ideal in $\bar{\rho}((\mathfrak{A}))$, and is abelian since isomorphic to $\rho(\mathfrak{J})$. Thus $\bar{\rho}$ is a one-dimensional representation of $\mathfrak{A}$, hence $\bar{\rho}(\mathfrak{J}) = 0$, a contradiction. By Lemma 2.8, $\mathfrak{J}_{SA} = \mathfrak{M}$, hence $\mathfrak{M}$ is in particular reversible (a fact which also follows from Lemma 2.10). By Lemma 2.15, $\mathfrak{R}(\mathfrak{M}) = i\,\mathfrak{R}(\mathfrak{M})$. The proof is complete.

A simple related result is the following.

THEOREM 2.17. *Let $\mathfrak{A}$ be a JC-algebra with identity I. Assume $\mathfrak{A}$ is reversible and $iI \in \mathfrak{R}(\mathfrak{A})$. Then $\mathfrak{A} = (\mathfrak{A})_{SA}$.*

Proof. By Remark 2.2 $\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A})$ is an ideal $\mathfrak{J}$ in $(\mathfrak{A})$. Since $\mathfrak{A}$ is reversible $\mathfrak{J}_{SA} \subset \mathfrak{A}$ by Remark 2.5. By hypothesis $iI \in \mathfrak{J}$, hence $\mathfrak{J} = (\mathfrak{A})$ and $(\mathfrak{A})_{SA} = \mathfrak{A}$.

3. **C^*-homomorphisms.** In this section we show that C^*-homomorphisms from C^*-algebras into C^*-algebras are sums of *-homomorphisms and *-anti-homomorphisms. We shall first modify Sherman's result that a representation of a C^*-algebra has an ultra-weakly continuous extension to a representation of the second dual of the C^*-algebra.

LEMMA 3.1. *Let ϕ be a C^*-homorphism of the C^*-algebra $\mathfrak{A}$ into the von Neumann algebra $\mathfrak{B}$. Then ϕ has an ultra-weakly continuous extension*

$$\phi^{**}: \mathfrak{A}^{**} \to \mathfrak{B},$$

which is also a C^*-homomorphism.

Proof. If $\mathfrak{A}$ is given the weak* topology, i.e. the $\sigma(\mathfrak{A}, \mathfrak{A}^*)$ topology, and $\mathfrak{B}$ the ultra-weak topology then ϕ is continuous. We thus obtain a continuous map

$$\phi^*: \mathfrak{B}_* \to \mathfrak{A}^*$$

defined by $\phi^*(\rho) = \rho \circ \phi$, where $\mathfrak{B}_*$ is the predual of $\mathfrak{B}$, the set of ultra-weakly continuous linear functionals of $\mathfrak{B}$. Now the weak* and the ultra-weak topologies coincide on $\mathfrak{A}^{**}$. Moreover, by [2, Théorème 1, p. 40] $\mathfrak{B} = (\mathfrak{B}_*)^*$. Thus ϕ^* induces an ultra-weakly continuous map

$$\phi^{**}: \mathfrak{A}^{**} \to \mathfrak{B},$$

which extends ϕ. We show ϕ^{**} is a C^*-homomorphism. Since the left and right multiplications coincide in $\mathfrak{A}^{**}$, $X \circ Y = \frac{1}{2}(XY + YX)$ is a well-defined Jordan product in $\mathfrak{A}^{**}$. In order to show $\phi^{**}(X \circ Y) = \phi^{**}(X) \circ \phi^{**}(Y)$ we first consider $A \in \mathfrak{A}$ and $Y \in \mathfrak{A}^{**}$. Since $\mathfrak{A}$ is strongly dense in $\mathfrak{A}^{**}$, $\mathfrak{A}$ is ultra-weakly dense in $\mathfrak{A}^{**}$. Hence there exists a net (B_α) in $\mathfrak{A}$ such that $B_\alpha \to Y$ ultra-weakly. Thus, since multiplication is ultra-weakly continuous in one variable,

$$\begin{aligned}\phi^{**}(A \circ Y) &= \phi^{**}(\text{u.w.} \lim A \circ B_\alpha) = \text{u.w.} \lim \phi^{**}(A \circ B_\alpha) \\ &= \text{u.w.} \lim \phi(A \circ B_\alpha) = \text{u.w.} \lim \phi(A) \circ \phi(B_\alpha) \\ &= \phi(A) \circ \phi^{**}(Y) = \phi^{**}(A) \circ \phi^{**}(Y).\end{aligned}$$

If $X \in \mathfrak{A}^{**}$ choose a net (A_α) in $\mathfrak{A}$ such that $A_\alpha \to X$ ultra-weakly. Then

$$\begin{aligned}\phi^{**}(X \circ Y) &= \phi^{**}(\text{u.w.} \lim A_\alpha \circ Y) = \text{u.w.} \lim \phi^{**}(A_\alpha \circ Y) \\ &= \text{u.w.} \lim \phi^{**}(A_\alpha) \circ \phi^{**}(Y) = \phi^{**}(X) \circ \phi^{**}(Y).\end{aligned}$$

Thus ϕ^{**} is a C^*-homomorphism; the proof is complete.

Kadison [9, Theorem 10] has shown that a C^*-homomorphism of a von Neumann algebra onto a C^*-algebra is in a strong sense the sum of a *-homomorphism and a *-anti-homomorphism. However, he proved more. The proof of the next lemma is a slight modification of his proof.

LEMMA 3.2. *Let $\mathfrak{A}$ be a von Neumann algebra and ϕ a C^*-homomorphism of $\mathfrak{A}$ into the bounded operators on a Hilbert space. Let $\mathfrak{B}$ be the C^*-algebra generated by $\phi(\mathfrak{A})$. Then there exist two orthogonal central projections E and F in $\mathfrak{B}$ with $E + F = I$ such that the map $\phi_1: A \to \phi(A)E$ (resp. $\phi_2: A \to \phi(A)F$) is a *-homomorphism (resp. *-anti-homomorphism), and $\phi = \phi_1 + \phi_2$ as linear maps.*

Proof. As in Kadison's proof we may assume $\mathfrak{A}$ is an $n \times n$ matrix ring over a von Neumann algebra with $n \geqq 2$. Let $\mathfrak{R}$ be the (purely algebraic) ring generated by $\phi(\mathfrak{A})$. Then $\mathfrak{R}$ is a *-algebra. In fact, if $A \in \mathfrak{R}$ then $A = \sum_{i=1}^{n} \prod_{j=1}^{m_i} \phi(A_{ij})$ with $A_{ij} \in \mathfrak{A}$. Hence

$$A^* = \sum_{i=1}^{n} \prod_{j=m_i}^{1} \phi(A_{ij})^* = \sum_{i=1}^{n} \prod_{j=m_i}^{1} \phi(A_{ij}^*) \in \mathfrak{R}.$$

By [7, Theorem 7] there exist central idempotents E and F in $\mathfrak{R}$ such that $\phi_1 = E\phi$ is a homomorphism and $\phi_2 = F\phi$ is an anti-homomorphism, $E + F = I$, and $\phi = \phi_1 + \phi_2$. Since E and F are central and $\mathfrak{R}$ is a *-algebra, E^*, $F^* \in \mathfrak{R}$, hence commute with E and F. Thus E and F are projections. Thus $\phi_1(A^*) = E\phi(A^*) = E\phi(A)^* = (E\phi(A))^* = \phi_1(A)^*$, and ϕ_1 is a *-homomorphism. Similarly ϕ_2 is a *-anti-homomorphism. Since $\mathfrak{R}$ is uniformly dense in $\mathfrak{B}$, E and F are central in $\mathfrak{B}$. The proof is complete.

We are now in position to show that every C^*-homomorphism from one C^*-algebra into another is the sum of a *-homomorphism and a *-anti-homomorphism. More specifically we have

THEOREM 3.3. *Let $\mathfrak{A}$ be a C^*-algebra and ϕ a C^*-homomorphism of $\mathfrak{A}$ into the bounded operators on a Hilbert space. Let $\mathfrak{B}$ be the C^*-algebra generated by $\phi(\mathfrak{A})$. Then there exist two orthogonal central projections E and F in $\mathfrak{B}^-$ such that the map $\phi_1 : A \to \phi(A)E$ (resp. $\phi_2 : A \to \phi(A)F$) is a *-homomorphism (resp. *-anti-homomorphism), $E + F = I$, and $\phi = \phi_1 + \phi_2$ as linear maps.*

Proof. By Lemma 3.1, ϕ has an ultra-weakly continuous extension to a C^*-homomorphism $\phi^{**} : \mathfrak{A}^{**} \to \mathfrak{B}^-$. An application of Lemma 3.2 completes the proof, as it is clear that E and F will be central in $\mathfrak{B}^-$.

COROLLARY 3.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. Let ϕ be a C^*-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Assume $\mathfrak{B}$ equals the C^*-algebra generated by $\phi(\mathfrak{A})$. If ψ is an irreducible representation of $\mathfrak{B}$ then $\psi \circ \phi$ is either a homomorphism or an anti-homomorphism.*

Proof. Replacing ϕ by $\psi \circ \phi$ we may assume $\mathfrak{B}$ is irreducible. By Theorem 3.3 the central projections E and F in $\mathfrak{B}^-$ must be either 0 or I, hence ϕ must be either a homomorphism or an anti-homomorphism.

The next corollary is known [9, Theorem 5].

COROLLARY 3.5. *A C^*-isomorphism of one C^*-algebra into another is an isometry.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and ϕ a C^*-isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ such that $\mathfrak{B}$ equals the C^*-algebra generated by $\phi(\mathfrak{A})$. Since ϕ is a *-isomorphism on every commutative C^*-subalgebra of $\mathfrak{A}$, $\phi \mid \mathfrak{A}_{SA}$ is in particular an isometry. Let $A \in \mathfrak{A}$. Then $\| A \|^2 = \| A^*A \| = \| \phi(A^*A) \| = \sup\{ \| \psi(\phi(A^*A)) \| : \psi$ irreducible representation of $\mathfrak{B}\}$. By [10, p. 234], or by [11] and [12], this sup is attained, say by ψ. By Corollary 3.4 $\psi \circ \phi$ is either a homomorphism or an anti-homomorphism; if it is a homomorphism then

$$\| A \|^2 = \| \psi(\phi(A^*A)) \| = \| \psi\phi(A^*)\psi\phi(A) \| = \|\psi\phi(A) \|^2 \leqq \| \phi(A) \|^2 \leqq \| A \|^2,$$

and similarly if $\psi \circ \phi$ is an anti-homomorphism. The proof is complete.

The next corollary should be compared with [14, Proposition 7.8]. We omit the proof as it is immediate from Corollary 3.4 and the first part of the proof of [14, Proposition 7.8].

COROLLARY 3.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and ϕ a C^*-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Then ϕ is a *-homomorphism if and only if there exists a real number $\alpha > 0$ such that ϕ satisfies the Cauchy-Schwarz inequality $\phi(A^*A) \geqq \alpha\phi(A^*)\phi(A)$ for all $A \in \mathfrak{A}$.*

Two algebraic properties of C^*-homomorphisms are considered in the next corollary. The proof is immediate from Theorem 3.3.

COROLLARY 3.7. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and ϕ a C^*-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Then ϕ satisfies the following two identities:*

(1) *If $A_1, \cdots, A_n \in \mathfrak{A}$ then*

$$\phi\left(\prod_{i=1}^{n} A_i + \prod_{i=n}^{1} A_i\right) = \prod_{i=1}^{n} \phi(A_i) + \prod_{i=n}^{1} \phi(A_i).$$

(2) *If $A, B, C, D \in \mathfrak{A}$ then*

$$(\phi(AB) - \phi(A)\phi(B))\,(\phi(CD) - \phi(D)\phi(C)) = 0.$$

Recall that the structure space of a C^*-algebra is the set of its primitive ideals equipped with the hull-kernel topology. Its influence on C^*-homomorphisms is seen in

COROLLARY 3.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and ϕ a C^*-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ such that $\mathfrak{B}$ equals the C^*-algebra generated by $\phi(\mathfrak{A})$. Assume $\mathfrak{B}$ has no nonzero abelian representations and that its structure space is connected. Then ϕ is either a homomorphism or an anti-homomorphism.*

Proof. Let Z denote the structure space of $\mathfrak{B}$. If $P \in Z$ denote by ρ_P the canonical map $\mathfrak{B} \to \mathfrak{B}/P$. Note that if two homomorphisms ψ and η of $\mathfrak{B}$ have the same kernels then $\psi \circ \phi$ is a homomorphism (resp. anti-homomorphism) if and only if $\eta \circ \phi$ is a homomorphism (resp. anti-homomorphism), because $\psi \circ \phi$ is a homomorphism (resp. anti-homomorphism) if and only if $\phi(AB) - \phi(A)\phi(B)$ (resp. $\phi(AB) - \phi(B)\phi(A)$) belongs to kernel ψ for all A, B in $\mathfrak{A}$. Thus the sets X (resp. Y) $= \{P \in Z : \rho_P \circ \phi$ is a homomorphism (resp. anti-homomorphism) of $\mathfrak{A}\}$ are well defined disjoint subsets of Z, since $\mathfrak{B}$ has no one-dimensional representations. By Corollary 3.4, $Z = X \cup Y$. If either X or Y is empty we are through. Assume $X \neq \Phi \neq Y$. Let $\bar{X} = \bigcap\{P : P \in X\}$. Let $P \in Z$, $P \supset \bar{X}$. Then $\rho_P \circ \phi$ is a homomorphism, since $\phi(AB) - \phi(A)\phi(B) \in \bar{X}$ for all $A, B \in \mathfrak{A}$. Thus $P \in X$, and X is closed. Similarly Y is closed. But $Y = X^C$ is open, hence $Y = Z$, since Z is connected and $Y \neq \Phi$, thus $X = \Phi$, a contradiction. Thus either X or Y is empty. The proof is complete.

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UNIVERSITY OF OSLO,
OSLO, NORWAY