SELF-UNLINKED SIMPLE CLOSED CURVES

BY
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1. **Discussion of results.** This paper is a sequel to [4] and all the definitions and notations of [4] will be assumed. In addition, the numbering of the theorems in the present paper has been made to follow the numbering of [4].

A simple closed curve J in a space M is said to be *self-unlinked*, if there exist a mapping $h: J \times [0,1] \to M$ such that

(a) $h \mid J \times \{0\} =$ inclusion of J in M,

(b) $h(J \times \{1\}) =$ a point, and

(c) $h(J \times (0,1]) \subset M - J$.

In [4] we proved, as a partial answer to Question IV.1, that (IV.2) every self-unlinked tame simple closed curve (scc) in a 3-manifold bounds a disk. In this paper we investigate this question when we allow the scc's to be wild.

First we give some pertinent definitions, for which it will be assumed that everything is in a 3-manifold M. A complex is *wild* if it is not tame (see I. 11 of [4]). A *0-dimensional set is tame* if, for every $\varepsilon > 0$, it can be covered by the interiors of a collection of disjoint 3-cells each of diameter less than ε. A set X is *locally tame* at p if p has a closed neighborhood in X which is a tame complex in M. If X is not locally tame at p then p is a *wild point* of X. A set is called *nicely wild* if the union of its wild points is a tame 0-dimensional set.

For J an arc or scc we make the following definitions, the first of which is used in [1]. The *penetration index* $P(J,x)$ *of* J *at a point* $x \in J$ is the smallest cardinal number n such that there are arbitrarily small 2-spheres enclosing x and containing no more than n points of J. The *penetration index* $P(J)$ *of* J is the least upper bound of the cardinal numbers $P(J,x)$, for all $x \in J$. If J is nicely wild, then the *nice penetration index* $NP(J)$ *of* J is the smallest integer n such that, for every $\varepsilon > 0$, the set of wild points of J can be covered by the interiors of a collection of disjoint 3-cells each with diameter less than ε and such that the boundary of each 3-cell intersects J in no more than n points. (The union of members of this collection is called a *taming* ε*-set of* J *of index* n.)

Conjecture. *There is a nicely wild scc* J *such that* $NP(J) \neq P(J)$.

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The author expects such an example because he knows of a nicely wild scc J which has a point x such that $P(J,x)=3$; and, for any J, $NP(J)$ is even.

In the definition of nice penetration index we may require that the 3-cells are tame, because of the following:

THEOREM V.1. *Suppose every set of diameter less than* ε *in* M *lies in the interior of a convex* 3-*cell.* (*For instance, metrize* M *with the barycentric metric and let* ε *be less than* 1.) ***If*** *J is a nicely wild scc that is locally polyhedral* mod *its wild points, and if* T *is a taming* ε-*set of* J *of finite index,* ***then*** *there is a polyhedral taming* ε-*set* T' *of* J *with the same number of components as* T, *such that* $\mathrm{Bd}T' \cap J$ *has no more points than* $\mathrm{Bd}T \cap J$ *and* J *pierces* $\mathrm{Bd}T'$ *at each point of intersection.*

The principal results of this section are the following theorems.

In each *J is a self-unlinked, nicely wild scc in a* 3-*manifold* M, *and we further suppose that* J *is locally polyhedral* mod W ($W \equiv$ *set of wild points of* J).

THEOREM V. 2. *J bounds an s-disk D which is locally polyhedral* mod W, *and* $|J| \cap |\mathrm{int}\, D| = \varnothing$.

THEOREM V.3. ***If*** *either*

(a) $NP(J)=2$, *or*

(b) $NP(J)$ *is finite and J has only finitely many wild points,*

then *there is an s-disk D' and a sequence $\{T_i\}$ such that*

(a) *for each i, T_i is a taming $\frac{1}{2}i$-set of A of index $NP(J)$,*

(b) $|\mathrm{Bd}\, D'| = J$ *and* $[D' | D'^{-1}(|D'| - W), \mathrm{Bd}\, D' | D'^{-1}(J - W)]$ *is in rnp in* $M - W$, *and*

(c) *for each i, there is an s-disk D_i such that*

(i) $(D_i, \mathrm{Bd}\, D_i)$ *is in rnp in* $(M - \mathrm{int}\ T_i, J + \mathrm{Bd}\, T_i)$(2),

(ii) $|D_i| \supset |D_{i-1}|$, *and*

(iii) *D' equals the limit of the D_i's, as maps.*

THEOREM V.4. *If J bounds an s-disk D' satisfying the stated conclusion of Theorem* V.3, *then J bounds a nonsingular disk D.*

THEOREM V.5. *If* $NP(J) \leqq 4$ *and J has only finitely many wild points, then* $NP(J)=2$.

An immediate consequence of V.4, V.5 and the characterization of tame scc's by O. G. Harrold, H. C. Griffith, and E. E. Posey in [3] is the following:

THEOREM V.6. ***If*** *either*

(a) $NP(J)=2$, *or*

(b) $NP(J) \leqq 4$ *and J has only finitely many wild points,* ***then*** *J is tame.*

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(2) Bd $T_i + J$ is not a 2-manifold, but everything makes sense since $S(D_i) \subset \mathrm{Bd}\, T_i$.

If J is a scc on Alexander's Horned Sphere, S, which contains all the wild points of S, then

(a) J is a wild, nicely wild scc,

(b) $NP(J) = P(J) = 4$, and

(c) J bounds a disk.

In addition, by "tying the Fox'-Artin knot with a pointed ribbon" one can obtain a scc J such that

(a) J is a wild, nicely wild scc with one wild point,

(b) $NP(J) = P(J) = 6$, and

(c) J bounds a disk.

Finally, by tying a convergent sequence of knots in a scc, one obtains a scc J such that

(a) J is a nicely wild scc with one wild point,

(b) $NP(J) = P(J) = 2$,

(c) but J is wild.

2. **Proof of V.1.** Let C be a component of T. Theorem V.1 will follow if we produce a polyhedral 3-cell B such that $W \cap \text{int } C = W \cap \text{int } B$ (W= set of wild points of J), J pierces Bd B at each point of $J \cap \text{Bd } B$, diameter of $B < \varepsilon$, Bd $B \cap J$ has no more points than Bd $C \cap J$, and B does not intersect any other components of T.

Let δ be a positive number less than each of (ε − (diameter of C)), (1 / 3) (distance from C to $T - C$), and (1 / 3) (distance from Bd C to W). By the approximation theorems of [2] we may assume that Bd C is locally polyhedral mod $J \cap \text{Bd } C$.

Enclose each point p of Bd $C \cap J$ by a polyhedral 2-sphere S_p such that each S_p is so small that

(a) the diameter of S_p is less than δ,

(b) the S_p's are disjoint,

(c) $S_p \cap J$ is two points at each of which J pierces S_p,

(d) S_p is in general position with respect to Bd C, and

(e) there is a component K of Bd $C - \sum S_p$ which separates the (Bd $C \cap S_p$)'s on Bd C.

cl(K) (cl = closure) is a disk with holes and each component of Bd(cl(K)) is a scc on some S_p. For each p, only one scc of cl(K) $\cap S_p$ bounds a disk in Bd $C - K$ that intersects J. Therefore, since by hypothesis $C + \sum S_p$ is contained in the interior of a 3-cell, we may use linking arguments in E^3 to show that, for each p, all components but one of cl(K) $\cap S_p$ bounds a disk on $S_p - J$, and that the other one bounds a polyhedral disk on S_p that intersects J at most once. We can make these disks disjoint by pushing their interiors slightly to one side. Then K plus the above disks is a polyhedral 2-sphere S in a convex 3-cell of M. Let B be the 3-cell bounded by S.

Clearly Bd $B \cap J$ has no more points than Bd $C \cap J$, diameter of $B < \varepsilon$, and B does not intersect any other components of T. Let $w \in W \cap C$ and let λ be a general position arc from w to $M - (C + \sum S_p)$ which misses the S_p's. [This arc is possible since the 3-cells bounded by S_p have diameter less than $(1/3)$ (distance from $\operatorname{Bd} C$ to W).] Then $\lambda \cap \operatorname{Bd} B = \lambda \cap K = \lambda \cap \operatorname{Bd} C$; thus, since $B + C$ is in the interior of a 3-cell and $w \in C$, $\lambda \cap \operatorname{Bd} B$ is an odd number of points and, therefore, $w \in B$. B is the desired 3-cell.

3. **Proof of V.2.** The proof of V.2 parallels the proof of III.4 of [4] and thus will only be sketched here.

$|J| - W$ is an infinite 1-dimensional polyhedral graph in M. Since J is self-unlinked we may assume that J is the boundary of an s-disk D and that D is polyhedral mod J. Consider $|J| - W$ as a subcomplex of some subdivision α of $M - W$.

Let Δ be the standard disk and let $\Omega \equiv D^{-1}(W)$.

Now go through the proof of III.4 replacing M by $M - W$, $|L|$ by $|J| - W, D$ by $D|(\Delta - \Omega)$, Δ by $\Delta - \Omega$, et cetera. Choose Δ' so that $\operatorname{Bd} \Delta' \cap \operatorname{Bd} \Delta = \Omega$.

4. **Proof of V.3.** We shall assume that M is so metrized that every set of diameter no more than 1 lies in a convex 3-cell (for example, the barycentric metric).

Let D be the disk promised by V.2 and (using II.2 of [4] in $M - W$) suppose that $(D|D^{-1}(|D| - W), \operatorname{Bd} D|D^{-1}(J - W))$ is in rnp in $M - W$.

Let $\varepsilon_1 = 1$, if $NP(J) = 2$; otherwise let ε_1 be a positive number less than 1 and so small that, if k is the number of points in W, then there is a positive integer $n \leqq NP(J) \times k$ such that

(4.1) no taming ε_1-set T of J of index $\leqq NP(J)$ has fewer than k components nor does $\operatorname{Bd} T \cap J$ have fewer than n points.

Let $\delta(\varepsilon)$ be a positive number less than $\varepsilon/3$ so small that

(4.2) if Δ' is a subdisk of Δ (the standard disk) and $\operatorname{diam}(D(\operatorname{Bd} \Delta')) < \delta(\varepsilon)$, then $\operatorname{diam} D(\Delta') < \varepsilon/3$.

Let $\Delta_1, \Delta_2, \cdots, \Delta_i, \cdots$ be an expanding sequence of proper subdisks of $\operatorname{int} \Delta$ so that $\{\operatorname{Bd} \Delta_i\}$ converges uniformly to $\operatorname{Bd} \Delta$.

Choose T_1 so that

(4.3) T_1 is a polyhedral (see VI.1) taming $\delta(\varepsilon_1)$-set of J of index $NP(J)$,

(4.4) all components of T_1 intersect W,

(4.5) if $NP(J) \neq 2$, T_1 has only k components and $\operatorname{Bd} T_1 \cap J$ has n points (see (4.1)), and

(4.6) $T_1 \subset M - D(\Delta_1)$.

We may suppose that $\operatorname{Bd} T_1$ and D are in general position so that

$$D^{-1}(\operatorname{Bd} T_1 \cap |D|)$$

is a finite collection of disjoint scc's and spanning arcs of Δ in $\Delta - \Delta_1$.

Let K be the component of $\Delta - D^{-1}(\operatorname{Bd} T_1 \cap |D|)$ containing Δ_1. The boundary of K is a finite collection of scc's in $D^{-1}(\operatorname{Bd} T_1 \cap |D|) + \operatorname{Bd} \Delta$. Let E_1 be the smallest disk in Δ containing K. (Note that $\operatorname{Bd} K \cap \operatorname{Bd} \Delta \subset \operatorname{Bd} E_1 \subset \operatorname{Bd} K$.)

If A is a member of $\mathfrak{A}$ (those scc's of $D^{-1}(\operatorname{Bd} T_1) \cap \operatorname{int} E_1$ which can be shrunk to a point in $\operatorname{Bd} T_1 - J$) and E_a is the disk that A bounds in E_1, then we can replace $D(E_a)$ by the singular disk which $D(A)$ bounds on $\operatorname{Bd} T_1 - J$. By pushing this disk slightly to one side of $\operatorname{Bd} T_1$ we can remove a component of $D^{-1}(\operatorname{Bd} T_1 \cap |D|)$. If we apply the above "disk-switching and pushing" only to outermost (in E_1) members of $\mathfrak{A}$ then no point of Δ will have its image changed more than once.

Thus, by applying the "disk-switching and pushing" to each outermost (in E) member of $\mathfrak{A}$ and then II.2 of [4] we obtain an s-disk D_1' such that

(4.7) $D_1'|\Delta_1 + (\Delta - \operatorname{int} E_1) + \operatorname{Bd} \Delta = D|\Delta_1 + (\Delta - \operatorname{int} E_1) + \operatorname{Bd} \Delta$,

(4.8) $D_1'^{-1}(\operatorname{Bd} T_1 \cap |D_1'|) \cap \operatorname{int} E_1$ is a finite collection of scc's whose images under D_1' cannot be shrunk to a point on $\operatorname{Bd} T_1 - J$, and

(4.9) $(D_1' - W, \operatorname{Bd} D_1' - W)$ is in rnp in $M - W$.

Let $\mathfrak{B}$ be the collection of all components (scc's) of $D_1'^{-1}(\operatorname{Bd} T_1 \cap |D_1'|) \cap \operatorname{int} E_1$. If $\mathfrak{B} \neq \varnothing$, let A be an innermost (in E_1) scc of $\mathfrak{B}$. A bounds a disk $E_a \subset E_1$ and $D_1'(E_a) \subset |D_1'| - \operatorname{int} T_1$, or $T_1 - J$. We shall treat these two cases separately.

If $D_1'(E_a) \subset T_1 - J$, then, since $D_1'(A)$ cannot be shrunk on $\operatorname{Bd} T_1 - J$, we can use the loop theorem to get a scc J_a such that J_a bounds a disk D_a in $T_1 - J$ but each of the two disks which J_a bounds on $\operatorname{Bd} T_1$ contain points of $J \cap \operatorname{Bd} T_1$. Thus D_a separates $J \cap C$, where C is the component of T_1 containing J_a. If we "cut" C apart along D_a (this cut could be accomplished by removing from C the interior of a regular neighborhood of D_a that misses J), we obtain a new taming δ-set T' of J.

If $NP(J) = 2$, then $J \cap \operatorname{Bd} C$ is two points and J intersects the boundary of each part of the "cut apart" C in only one point. But a scc that intersects a 2-sphere only once is contained wholly in one complementary domain or the other; therefore, $J \cap C$ is two points and C contains no points of W. This is a contradiction of (4.4).

If $NP(J) \neq 2$, then T' is a taming $\delta(\varepsilon_1)$-set of index $NP(J)$ and with $k+1$ components. But since W has only k points one of the components, C' say, of T' does not intersect W. But then $T' - C'$ is a taming $\delta(\varepsilon_1)$-set of index $NP(J)$ and with k components such that $\operatorname{Bd}(T' - C') \cap J$ has fewer points than $\operatorname{Bd} T \cap J$ which contradicts (4.1), (4.3), and (4.5).

Thus $D_1'(E_a)$ is not contained in $T_1 - J$.

If $D_1'(E_a) \subset |D_1'| - (\operatorname{int} T_1 + J)$, then by the loop theorem there is a real disk E_a such that $\operatorname{int} E_a$ is contained in $M - (T_1 + J)$. Also each of the disks E_a' and E_a'' which $\operatorname{Bd} E_a$ bounds on $\operatorname{Bd} T_1$ contains points of $J \cap \operatorname{Bd} T_1$. Because of (4.2) and (4.3), the diameter of E_a is less than $\varepsilon_1/3$. Thus one of $E_a + E_a'$ or $E_a + E_a''$, say $E_a + E_a'$, is a 2-sphere of diameter less than $2\varepsilon_1/3$ not containing C (the component of T_1 containing $\operatorname{Bd} E_a$) in its small complementary domain. Thus $E_a + E_a'$ lies in a convex 3-ball of M (see note at beginning of §4) and thus bounds a 3-cell B of diameter less than $2\varepsilon_1/3$.

$C + B$ is a 3-cell and $J \cap \mathrm{Bd}(C+B) = J \cap E_a''$ has fewer points than $J \cap \mathrm{Bd}\, C$. Thus, if $NP(J) = 2$, $J \cap \mathrm{Bd}(C+B)$ is one point and $C \subset C + B$ does not intersect W, which contradicts (4.4). If $NP(J) \neq 2$, then $\mathrm{Bd}(T_1 + B) \cap J$ has fewer points than $\mathrm{Bd}\, T_1 \cap J$ which contradicts (4.1), (4.2), (4.3), and (4.5).

Thus we conclude that $\mathfrak{B}$ is empty and that $D_1 = D_1' | E_1$ is an s-disk satisfying (c) (i) of V.3, if D_1' is substituted for D'. With the same substitution T_1 and D_1' satisfy (a) and (b) of V.3.

We now repeat the above process letting ε_2 be a positive number less than ε_1 and $\frac{1}{2}$ and with the following substitutions: ε_2 for ε_1, D_1' for D, D_2' for D_1', E_2 for E_1, $\Delta_2 + E_1$ for Δ_1, D_2 for D_1, and T_2 for T_1. We can choose T_2 to satisfy $T_2 \subset M - D_1'(\Delta_2 + E_1)$ since $D_1'^{-1}(W) = D^{-1}(W) \subset \mathrm{Bd}\, \Delta - E_1$. Thus, T_1, T_2, D_1, D_2, D_2' satisfy (a), (b), and (c) (i) and (ii) of V.3 with D' replaced by D_2'.

We repeat the process at the ith stage after letting ε_i be a positive number less than ε_{i-1} and $1/2^{i-1}$ and then substituting ε_i for ε_1, D_{i-1}' for D, D_i' for D_1', E_i for E_1, $\Delta_i + E_{i-1}$ for Δ_1, D_i for D_1, and T_i for T_1. Thus for each $i, T_1, T_2, \cdots, T_i$, $D_1, D_2, \cdots, D_i, D_i'$ satisfy (a), (b), and (c) (i) and (ii) of V.3 with D' replaced by D_i'.

By (4.7)

$$D_i' | \Delta_i + E_{i-1} + (\Delta - \mathrm{int}\, E_i) + \mathrm{Bd}\, \Delta = D_{i-1} | \Delta_i + E_{i-1} + (\Delta - \mathrm{int}\, E_i) + \mathrm{Bd}\Delta$$

and, since $E_i \supset \Delta_i + E_{i-1}$ and $\{\mathrm{Bd}\, \Delta_i\}$ converges to $\mathrm{Bd}\, \Delta$, every $p \in \mathrm{int}\, \Delta$ is in $\Delta_i + E_{i-1}$ for some i and thus $D_j'(p) = D_{i-1}'(p)$, for all $j \geqq i$. In addition, for each i, $D_i' | \mathrm{Bd}\, \Delta = D | \mathrm{Bd}\, \Delta$. Also the diameter of each component of $\Delta - E_i$ approaches zero as i approaches infinity and, for all p, the distance between $D_i(p)$ and $D_{i+1}'(p)$ is less than $\varepsilon_{i+1} < 1/2^i$. Thus $D' = \lim D_i' = \lim D_i$ is the s-disk desired for V.3.

5. **Proof of V.4.** Let D', $\{D_i\}$, $\{T_i\}$ be as given in the conclusion to V.3. Suppose α is a subdivision of $M - W$ so that $|D'| - W + \sum T_i$ is a subcomplex of $\alpha(M - W)$.

For $i = 1, 2, \cdots$, Theorem III.5 [applied to $(M - \mathrm{int}\, T_i, \mathrm{Bd}\, T_i + J)$] (see previous footnote) shows that there is an s-disk D_i' such that $(D_i', \mathrm{Bd}\, D_i')$ is a conservative δ_i-alteration of $(D_i, \mathrm{Bd}\, D_i)$, and $|D_i'|$ is related to $|D_i|$ as $|D'|$ is related to $|D^*|$ in the Addendum. We choose δ_i and $n(i)$ so that

$$(\delta_i\text{-neighborhood of } S(D_i')) \subset \mathrm{st}[S(D_i'), \alpha^{n(i)}(M - \mathrm{int}\, T_i)] \subset M - J.$$

Thus, since $S(D_i')$ contains, if anything, only crossing pinch points, $S(D_i')$ is empty because $|\mathrm{int}\, D_i'| \subset M - \mathrm{int}\, T_i$. We also assume that each $|D_i'|$ is in general position with respect to each $\mathrm{Bd}\, T_j$.

For each i there is a positive integer $k(i)$ such that $|D_j| \supset |D'| \cap (M - \mathrm{int}\, T_i)$ for all $j \geqq k(i)$. Let $U_i = \mathrm{st}[S(D_i'), \alpha^{n(i)}(M - \mathrm{int}\, T_i)]$. Then, for all i and for all $j \geqq k(i)$

(a) $(|D_j'| - (U_i + \mathrm{int}\, T_i)) = (|D'| - (U_i + \mathrm{int}\, T_i))$, and

(b) $|D_j'| - \mathrm{int}\, T_i$ is related to $|D'| - \mathrm{int}\ T_i$ as $|D'|$ is related to $|D^*|$ in the Addendum.

There are only finitely many ways of putting things in U_i so that the Addendum is satisfied. Thus for some strictly increasing sequence of positive integers $\{n(1,i)\}$, $n(1,1) \geqq k(1)$, the pairs $[U_1, |D_{n(1,i)}| \cap U_1]$ are all pwl homeomorphic for $i = 1,2,3,\cdots$. Likewise there is a subsequence of $\{n(1,i)\}$ which we call $\{n(2,i)\}$ such that $n(2,i) \geqq k(n(1,1))$ and, for $i = 1,2,3,\cdots$, the pairs

$$[U_{n(1,1)}, |D'_{n(2,i)}| \cap U_{n(1,1)}]$$

are all pwl homeomorphic. In this way we get a sequence of sequences $\{n(1,i)\}$, $\{n(2,i)\}$, $\{n(3,i)\}, \cdots$ such that $\{n(j,i)\}_{i=1}^{\infty}$ is a subsequence of $\{n(k,i)\}_{i=1}^{\infty}$ for all $k < j$, and, for each fixed k, the pairs $[U_{n(k,1)}, |D'_{n(k+1,i)}| \cap U_{n(k,1)}]$ are pwl homeomorphic for $i = 1,2,\cdots$.

Set $m(i) = n(i,1)$, for $i = 1,2,\cdots$. By moving things slightly in $\sum U_{m(i)}$ we can suppose that

$$|D'_{m(i)}| - \operatorname{int} T_{m(j)} = |D'_{m(k)}| - \operatorname{int} T_{m(j)}, \text{ for all } i,k > j.$$

The (nonsingular) s-disks $D'_{m(i)}$ are not nice enough because their limit might not be a disk. However, we shall choose certain subdisks and alter them to produce a nonsingular disk with boundary J.

Let E_1 be a sub-s-disk of $D'_{m(1)}$ such that

$$(5.2)_1 \qquad J + \operatorname{Bd} T_{m(1)} \text{ contains } |\operatorname{Bd} E_1|.$$

Let E_2 be a sub-s-disk of $D'_{m(2)}$ such that

$$(5.2)_2 \qquad J \cap |E_2| \subset |\operatorname{Bd} E_2| \subset J + \operatorname{Bd} T_{m(1)} \text{ and } |\operatorname{Bd} E_1| \subset |E_2|.$$

By induction, pick E_n to be a sub-s-disk of $D'_{m(n)}$ such that

$$(5.2)_n \qquad J \cap |E_n| \subset |\operatorname{Bd} E_n| \subset J + \operatorname{Bd} T_{m(n-1)} \text{ and } |\operatorname{Bd} E_{n-1}| \subset |E_n|.$$

PROPOSITION V.7. $J \subset \liminf\{|E_i|\}$.

Proof. By (5.2), $J \cap |E_i| \subset J \cap |E_{i+1}|$. Therefore, we need only show that every point of $J - W$ belongs to some $|E_i|$. Let q be any point of $J \cap |E_1|$ and let $p \in J - W$. For some positive integer r, $p \in M - T_{m(r)}$. Now suppose that $p \notin |E_{r+j}|$, for every $j \geqq 1$. Then, for each $j \geqq 1$, $T_{m(r+j-1)} \cap D'_{m(r+j)}$ separates p from q in $D'_{m(r+j)}$ and, because a disk is unicoherent, one component of $T_{m(r+j-1)} \cap D'_{m(r+j)}$ separates p from q. But $(p+q) \not\subset T_{m(r+j-1)}$ and each component of $T_{m(r+j-1)}$ has diameter less than $1/2^{m(r+j-1)}$. We conclude that, for every ε, there is a subset R of J which is of diameter less than ε and which is within ε of W, such that R separates p from q. But, since neither p nor q belong to W, some point of W must separate p from q in J. This is a contradiction since no scc is separated by a single point. This proves V.7.

PROPOSITION V.8. *For every positive integer r, there is a positive integer $s(r)$, such that, for all $i,j \geqq s(r)$,*

$$|E_j| - \operatorname{int} T_{m(r)} = |E_i| - \operatorname{int} T_{m(r)}.$$

Proof. $D'_{m(r+1)} - \operatorname{int} T_{m(r)}$ has finitely many components and if, for some i, $|E_i|$ intersects one of these components, then it contains the whole component. For each component C of $(D'_{m(r+1)} - \operatorname{int} T_{m(r)})$, let $n(C)$ be the least integer such that $C \subset |E_{n(C)}|$ and set $n(C) = 0$ if C intersects no $|E_i|$. The s desired by V.8 is the maximum of the $n(C)$'s over all components C of $D'_{m(r+1)} - \operatorname{int} T_{m(r)}$.

Define $s^n(r) = s(s^{n-1}(r))$.

We now change the E_i's into an expanding sequence of disks in a countable number of steps.

Step 1. Let F_1 be the singular s-disk gotten by removing from $E_{s(1)}$ the interior of $\operatorname{Bd} E_1$ in $E_{s(1)}$ (see (5.2)) and replacing it by E_1. Formally, let Δ' be the subdisk of Δ bounded by $E_{s(1)}^{-1}(\operatorname{Bd} E_1)$; and let f be a homeomorphism of Δ onto Δ' such that

$$(E_{s(1)} | \operatorname{Bd} \Delta') \circ (f | \operatorname{Bd} \Delta) = E_1 | \operatorname{Bd} \Delta.$$

Then F_1 equals $E_{s(1)}$ on $\Delta - \operatorname{int} \Delta'$ and $E_1 \circ f^{-1}$ on Δ'. The singularities $S(F_1)$ are contained in $M - T_{m(1)}$. Let $\delta_1 = \frac{1}{2}$(distance from $S(F_1)$ to $T_{m(1)}$) and apply IV.3 of [4] to get a nonsingular s-disk F'_1 which is a conservative δ_1-alteration of F_1 such that $\operatorname{Bd} F'_1 = \operatorname{Bd} F_1 = \operatorname{Bd} E_{s(1)}$. Note that $F'_1 \subset M - T'_{m(s(1))}$.

Step 2. Let F_2 be the singular disk gotten by removing from $E_{s^2(1)}$ the interior of $\operatorname{Bd} E_{s(1)}$ in $E_{s^2(1)}$ and replacing it by F'_1. Since

$$|E_{s(1)}| - \operatorname{int} T_{m(1)} = |E_{s^2(1)}| - \operatorname{int} T_{m(1)}, \qquad |F_2| - |F'_1| \subset \operatorname{int} T_{m(1)}.$$

Thus, because

$$F'_1 \subset M - T_{m(s(1))}, \qquad S(F_2) \subset \operatorname{int} T_{m(1)} - T_{m(s(1))}.$$

Let $\delta_2 = \frac{1}{2}$(distance from $S(F_2)$ to $T_{m(s(1))}$) and apply IV.3 of [4] to get a nonsingular s-disk F'_2 which is a conservative δ_2-alteration of F_2. F'_2 has the following properties:

$(5.3)_2$ $\qquad E_1$ is a sub-s-disk of F'_2.

$(5.4)_2$ $\qquad \operatorname{Bd} F'_2 = \operatorname{Bd} F_2 = \operatorname{Bd} E_{s^2(1)}$.

$(5.5)_2$ $\qquad F'_2 \subset M - T_{m(s^2(1))}$.

$(5.6)_2$ $\qquad F'_1 - T_{m(1)} = F'_2 - T_{m(1)}$.

Step n ($n = 3, 4, \cdots$). Let F_n be the singular s-disk gotten by removing from $E_{s^n(1)}$ the interior of $\operatorname{Bd} E_{s^{n-1}(1)}$ in $E_{s^n(1)}$ and replacing it by F'_{n-1} (see (5.2) and $(5.4)_{n-1}$). By V.8, $|F_n| - |F_{n-1}| \subset \operatorname{int} T_{m(s^{n-2}(1))}$. Thus, by $(5.5)_{n-1}$,

$$S(F_n) \subset \operatorname{int} T_{m(s^{n-2}(1))} - T_{m(s^{n-1}(1))}.$$

Let $\delta_n = \frac{1}{2}$(distance from $S(F_n)$ to $T_{m(s^{n-1}(1))}$) and apply IV.3 of [4] to get a nonsingular s-disk F' which is a conservative δ_n-alteration of F_n. F'_n has the following properties.

$$(5.3)_n \qquad F'_{n-2} \text{ is a sub-}s\text{-disk of } F'_n.$$

$$(5.4)_n \qquad \operatorname{Bd} F' = \operatorname{Bd} E_{s^n(1)}.$$

$$(5.5)_n \qquad F'_n \subset M - T_{m(s^n(1))}.$$

$$(5.6)_n \qquad F'_{n-1} - T_{m(s^{n-1}(1))} = F'_n - T_{m(s^{n-1}(1))}.$$

Define $E_1 \equiv F'_0$.

We now use the F'_i's to construct a nonsingular s-disk D whose boundary is J.

PROPOSITION V.9. *For all $m \geqq n \geqq 2$ and for all onto homeomorphisms $g: \Delta \to\to \Delta$, there is an onto homeomorphism $h^m_n(g): \Delta \to\to \Delta$, such that*

$$(F'_m \circ h^m_n(g)) \,|\, (F'_n \circ g)^{-1} |F'_{n-2}| = (F'_n \circ g) \,|\, (F'_n \circ g)^{-1} |F'_{n-2}|.$$

Proof. There is essentially only one way of extending a disk Δ' to a larger disk Δ when $\Delta' \cap \operatorname{Bd} \Delta$ is given. [That is to say, given $\Delta' \subset \Delta_1$ and $\Delta' \subset \Delta_2$ such that $\Delta' \cap \operatorname{Bd} \Delta_1 = \Delta' \cap \operatorname{Bd} \Delta_2$, there is a homeomorphism of Δ_1 onto Δ_2 fixed on Δ'.] From (5.2) and (5.4) we conclude that, for $m \geqq n$, $F'^{-1}_m(|\operatorname{Bd} F'_{n-2}|) \cap \operatorname{Bd} \Delta = F'^{-1}_m(|\operatorname{Bd} F'_{n-2}| \cap J)$. Proposition V.9 now follows.

Using V.9, define

$$F''_0 \equiv F'_0 = E_1,$$

$$F''_1 \equiv F'_1,$$

$$F''_2 \equiv F'_2,$$

and, for $n = 3, 4, 5, \cdots$,

$$F''_n = F'_n \circ h^n_{n-1}(F'^{-1}_{n-1} \circ F''_{n-1}).$$

The reader can check that F''_n, $n = 2, 3, \cdots$, satisfies $(5.3)_n$–$(5.6)_n$ with all primes (′) replaced by double-primes (″). In addition, if we define $\Delta_i = F''^{-1}_{i+2}(|F''_i|)$,

$$(5.7)_n \qquad \text{for all } m \geqq n+2 \geqq 4, \quad F''_m | \Delta_n = F''_{n+2} | \Delta_n.$$

This follows from V.9.

Define $D | \Delta_i \equiv F''_{i+2} | \Delta_i$. By (5.7), D is a 1-1, continuous map of $\sum_{i=1}^{\infty} \Delta_i$ into M. Since each component of T_i is of diameter less than $1/2^i$, (5.5) and (5.6) show that D can be extended to a 1-1, continuous map (and thus, an embedding) of Δ into M. It follows from V.7 that $J \subset D(\Delta)$ and from (5.2) and (5.4) that $J = |\operatorname{Bd} D|$.

This completes the proof of V.6.

6. **Proof of V.5.** Let D be the nonsingular s-disk promised by Theorem V.4 and let ε_1, $\delta(\varepsilon_1')$, and T_1 be as in (4.1)–(4.5) with the additional requirement that T_1 be a γ-set, where γ is less than $\frac{1}{2}\delta(\varepsilon_1)$ and so small that if p and q are points of J within γ of each other then one of the components of $J-(p+q)$ has diameter less than $\frac{1}{2}\delta(\varepsilon_1)$. Assume that D is polyhedral mod W and that D and $\operatorname{Bd} T_1$ are in general position.

Let C be a component of T_1 such that $\operatorname{Bd} C \cap J$ has four points. Call these four points p_1, p_2, p_3, and p_4. $D^{-1}(\operatorname{Bd} C \cap |D|)$ is a finite collection of scc's and spanning arcs in Δ. Since the only possible end points for $D^{-1}(\operatorname{Bd} C \cap |D|)$ are $D^{-1}(\operatorname{Bd} C \cap J)$, $D^{-1}(\operatorname{Bd} C \cap |D|)$ has two spanning arcs which are situated as in Figure 1 or Figure 2. Note that $W \cap C$ is one point, which we call w; and

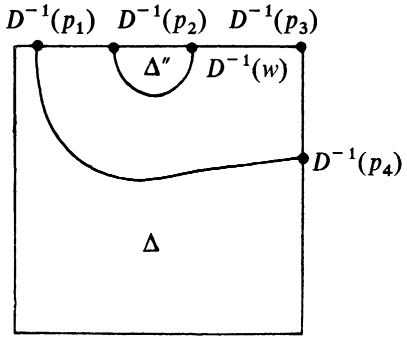

FIGURE 1

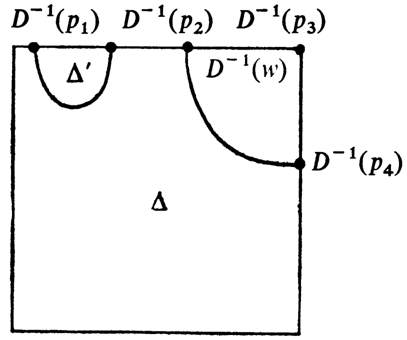

FIGURE 2

$D^{-1}(w)$ is between $D^{-1}(p_1)$ and $D^{-1}(p_2)$, or $D^{-1}(p_3)$ and $D^{-1}(p_4)$. We suppose the latter.

We shall look at the two cases depicted by Figure 1 and Figure 2.

Case depicted by Figure 1. The image of a neighborhood of $\operatorname{Bd}\Delta''$ is in $M - \operatorname{int} T_1$, and $(D|\Delta'')^{-1}(\operatorname{Bd} C \cap |(D|\Delta'')|)$ is a finite collection of scc's. By using arguments almost identical to those employed in the proof of V.3, we can remove the scc's that bound disks on $\mathrm{Bd}\,C - J$ and then show that no more are left. Thus we suppose that D has been altered so that

$$(6.1)\qquad D|\Delta'' \text{ is nonsingular, } \mathrm{int}(D|\Delta'') \subset M - T_i, \text{ and}$$
$$\mathrm{Bd}(D|\Delta'') \subset \mathrm{Bd}\,C + (\text{small part of } J \text{ between } p_2 \text{ and } p_3).$$

By our choice of γ, the diameter of $|\mathrm{Bd}(D|\Delta'')|$ is less than $\delta(\varepsilon_1)$ and thus the diameter of $|(D|\Delta'')|$ is less than $\varepsilon_1/3$. (See (4.2).) By thickening up $|(D|\Delta'')|$ we obtain a new taming ε_1-set T' of J of index $\leqq 4$ and such that $J \cap \mathrm{Bd}\,T'$ has two fewer points than $J \cap \mathrm{Bd}\,T_1$. This contradicts (4.1) and (4.5).

Case depicted by Figure 2. By repetition of previous arguments we can alter D on int Δ' to remove the components of intersection with $\mathrm{Bd}\,C$, and thus obtain a nonsingular s-disk D', such that $|\mathrm{int}\,D'| \subset \mathrm{int}\,C$ and $\mathrm{Bd}\,D' \subset \mathrm{Bd}\,C +$ (component of $J \cap C$ between p_1 and p_2). By splitting C apart along $|D'|$ we can obtain a new taming $\delta(\varepsilon_1)$-set T' of J of index $\leqq 4$ and such that $J \cap \mathrm{Bd}\,T'$ has two fewer points than $J \cap \mathrm{Bd}\,T_1$. This again is a contradiction.

Thus V.5 is proven.

References

1. B. J. Ball, *Penetration indices and applications*, Topology of 3-manifolds and related topics, Prentice-Hall, Englewood Cliffs, 1962.

2. R. H. Bing, *Approximating surfaces with polyhedral ones*, Ann. of Math. **61** (1957), 456–483.

3. O. G. Harrold, H. C. Griffith, and E. E. Posey, *A characterization of tame curves in* 3-*space*, Trans. Amer. Math. Soc. **79** (1955), 12–35.

4. D. W. Henderson, *Extensions of Dehn's Lemma and the Loop Theorem*, Trans. Amer. Math. Soc. **120** (1965), 448–469.

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