COHOMOLOGY AND HOMOLOGY THEORIES FOR CATEGORIES OF PRINCIPAL G-BUNDLES

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Introduction. In [4] G. W. Whitehead gives a method by which all extraordinary homology and cohomology theories may be generated by fundamental homotopy constructions. Unfortunately the definition of the extraordinary theories used is not sufficiently general so as to include all theories that would normally be considered as (co) homology theories. An example of a theory not included is (co) homology with local coefficients. In this paper, we will show how to extend the methods of Whitehead so as to include such theories.

Let G be a fixed continuous group. If we consider the category of principal G-bundles over C.W. complexes, we may define a connected sequence of functors from this category to the category of abelian groups, satisfying the usual axioms of extraordinary (co) homology theory, with principal G-bundles replacing spaces, and bundle maps and homotopies replacing maps and homotopies. It is the purpose of this paper to show that such theories may be generated by homotopy constructions. In particular, we will indicate how local coefficient theory may be generated in this manner.

In order to develop the homology theories spoken of above, it will first be necessary to define a class of spaces closely related to C.W. complexes but with just enough additional generality so as to include spaces for which a useful C.W. decomposition may not be available, but which are structured enough to give us a good deal of the type of information usually associated with a C.W.-decomposition.

The definition of cohomology groups used in this paper was proposed by Professor Israel Berstein of Cornell University and will be exploited by him in a forthcoming paper on the Thom Isomorphism. This paper represents part of a doctoral dissertation written under Professor Berstein at Cornell University.

1. **Piecewise-C.W. complexes.** We will need three lemmas from homotopy theory. The first two will not be proved as they are quite easy. In the following, if $X_0 \subseteq X$ and $f: X_0 \to Y$, then $M(f,X) = (0 \times X \cup_{0\times 1} I \times X_0) \cup_{1\times f} Y$, the partial mapping cylinder.

Received by the editors September 22, 1964.

(1) During the writing of this paper the author was partially supported by the National Science Foundation, Contract NSF-GP-3685.

LEMMA 1.1. *Let X and Y be two spaces. Let $X_0 \subseteq X$, and let $f \sim g: X_0 \to Y$ (i.e. f homotopic to g). $M(f, X)$ and $M(g, X)$ have the same homotopy type with a homotopy equivalence that is the identity on Y.*

LEMMA 1.2. *Let X be a space, and let X_0, f, Y be as above. Let $Y\lambda' Y'$ be a homotopy equivalence. $M(f, X)$ and $M(\lambda f, X)$ have the same homotopy type by a homotopy equivalence that restricts to $Y\lambda' Y'$.*

LEMMA 1.3. *Let Y be a topological space. Let (X, X_0) be a C.W. pair. Let $f: X_0 \to Y$ be a map. $Y \cup_f X$ and $M(f, X)$ have the same homotopy type by a homotopy equivalence that is the identity on Y.*

Proof. There is a natural map $\alpha: M(f, X) \to Y \cup_f X$ defined as follows:

$$\alpha : \begin{cases} y = y, & y \in Y, \\ (x, t) = f(x), & x \in X_0, \\ x = x, & x \in X - X_0. \end{cases}$$

We now construct $\beta: Y \cup_f X \to M(f, X)$. From the theory of C.W. complexes [5] we know there exists an open set $U \subseteq X$, with $X_0 \subseteq U$, and a homotopy $\tilde{F}: U \times I \to U$ with $\tilde{F}_0 = 1$, $\tilde{F}_1: U \subseteq X_0$ and $\tilde{F} / X_0 = 1$. Again, we know X is normal so we may choose two open sets V_1 and V_2 with

$$X_0 \subseteq V_1 \subseteq \bar{V}_1 \subseteq V_2 \subseteq \bar{V}_2 \subseteq U.$$

Consider a Urysohn function $h: X \to I$, with $h: V_1 = 1$ and $h: (X - V_2) = 0$. Restricting h to U, we may compose h with $\tilde{F}$ to define $\bar{F}: U \times I \to U$ by the formula $\bar{F}(u, t) = \tilde{F}(u, h(u)t)$. We note the following

$$\bar{F}(u, t) = \begin{cases} u, & u \in X - \bar{V}_2 \subseteq X - V_2, \\ \in X_0, & t = 1, \quad u \in V_1. \end{cases}$$

We now define $F: X \times I \to X$ by

$$F(x, t) = \begin{cases} \bar{F}(x, t), & x \in U, \\ x, & x \in X - V_2. \end{cases}$$

This map is well defined and continuous on U and $X - V_2$, and therefore it is continuous on X. We note that, as with $\bar{F}_1$, $F_1: V_1 \subseteq X_0$. We again define a Urysohn function, with $k(X_0) = 1$ and $k(X - V_1) = 0$. We define β as follows:

$$\beta: \begin{cases} y = y, & y \in Y, \\ k(x) \times F_1(x), & x \in X. \end{cases}$$

This map is well defined and continuous. It is easy to show $\alpha\beta \sim 1$ and $\beta\alpha \sim 1$, as is the rest of the lemma.

REMARK. We now introduce a definition for a type of space. We show this type to be of the homotopy type of C.W. complexes. We will show that this type is preserved under basic topological operations (e.g. suspension). Finally we will show that bundles have this type under fairly general conditions.

DEFINITION 1.4. A *Piecewise-C.W. complex* is a topological space X that is the union of a countable number of closed subspaces X_i, each having a fixed C.W. complex structure, and such that $X_i \cap X_j$ is a subcomplex in the C.W. structure of X_i and in the structure of X_j. In addition, the topology on X is the weak topology with respect to the subspaces X_i. Note that the decomposition of $X_i \cap X_j$ with respect to X_i may differ from that with respect to X_j. We call the X_i the *pieces* of X.

LEMMA 1.5. *Let p: $E \to B$ be a fibre bundle over a locally finite simplicial complex. Let the fibre F be a countable C.W. complex. Then E is a P.C.W. complex.*

Proof. For each closed simplex σ in B, $p^{-1}(\sigma)$ is homeomorphic to $F \times \sigma$. Therefore, we may decompose $p^{-1}(\sigma)$ as a C.W. complex using the product decomposition of $F \times \sigma$. Let σ_i and σ_j be two closed simplexes.

$$p^{-1}(\sigma_i) \cap p^{-1}(\sigma_j) = p^{-1}(\sigma_i \wedge \sigma_j)$$

is a subcomplex in the decomposition of $p^{-1}(\sigma_i)$ and $p^{-1}(\sigma_j)$. We complete the proof by checking that the topology is right.

THEOREM 1.6. *Any P.C.W. complex has the homotopy type of a C.W. complex.*

Proof. We prove this theorem by induction on the number of pieces in X by first proving the finite case and then proceeding to the infinite case by direct limit using weak topology arguments.

By induction on i: Let $i = 1$, then X is a C.W. complex by definition. Suppose we have proved the theorem for $i = n - 1$. Let X be "n-piecewise". Consider the space $X_1 \cup \cdots \cup X_{n-1} = \tilde{X}$, this is an $(n-1)$-P.C.W. complex so there exists a C.W. complex $\tilde{K}$, and maps $\zeta\colon \tilde{K} \to \tilde{X}$ and $\xi\colon \tilde{X} \to \tilde{K}$ with $\zeta\xi \sim 1$ and $\xi\zeta \sim 1$.

The space $\tilde{X} \cap X_n$ is a subcomplex of X_n since it is the union of the subcomplexes $(X_j \cap X_n)$, $1 \leqq j \leqq n-1$. Applying Lemma 1.3 X has the homotopy type of $M(1, X)$, where $1\colon \tilde{X} \cap X_n \to \tilde{X}$ is the inclusion. By Lemma 1.2, $M(1, X_n)$ has the homotopy type of $M(\xi 1, X_n)$. On the other hand, $\xi 1\colon \tilde{X} \cap X_n \to \tilde{K}$ is a map of C.W. complexes so we may apply the cellular approximation theorem to get $\xi 1$ homotopic to $\bar{\xi}$, a cellular map. We now apply Lemma 1.1 to show $M(\xi 1, X_n)$ has the homotopy type of $M(\bar{\xi}, X_n)$. Since $\bar{\xi}$ is cellular, $M(\bar{\xi}, X_n)$ is a C.W. complex and we are done with the first part of this theorem.

We proceed to the infinite case by using the last of our lemmas to check that the homotopy equivalences we obtain remain fixed at each level for all levels of the induction. We now apply the usual direct limit argument for the weak topology.

DEFINITION 1.7. A *P.C.W. subcomplex*, X_0, of a P.C.W. complex, X, is a subspace of X, with $X_0 \cap X_i$ a subcomplex of X_i for all i. Note

$$(X_0 \cap X_i) \cap (X_0 \cap X_j) = (X_0 \cap X_i) \cap (X_i \cap X_j)$$

is a subcomplex of X_i: Hence $X_{0,i} = X_0 \cap X_i$ gives X_0 the structure of a P.C.W. complex. (X, X_0) is called a *P.C.W. pair*. A *base point* $x \in X$ is a point such that $x \in X_i$ implies x is a vertex in the decomposition of X_i.

THEOREM 1.8. *A P.C.W. pair has the homotopy type of a C.W. pair. This means that if (X, X_0) is a P.C.W. pair, then there is a C.W. pair (K, K_0) and a homotopy equivalence of X and K that restricts to a homotopy equivalence of X_0 and K_0.*

Proof. The proof is similar to Theorem 1.6, only checking that after each stage we may perform the construction for a pair of spaces.

LEMMA 1.9. *Let (X, X_0) be a P.C.W. pair, then all of the following hold*:
(a) *X / X_0 is a P.C.W. complex.*
(b) *$X \cup CX_0$ is a P.C.W. complex (the cone over X_0).*
(c) *$\tilde{S}X$ is a P.C.W. complex (nonreduced suspension).*
(d) *If X has a base point, then SX is a P.C.W. complex (reduced suspension).*

Proof. (a) We know that $X_i / X_0 \cap X_i$ has the structure of the quotient complex. From the definition of a P.C.W. subcomplex it is easy to check that $(X_i / X_0 \cap X_i) \cap (X_j / X_0 \cap X_j) = (X_i \cap X_j) / X_0$ is a subcomplex of $X_i / X_0 \cap X_i$ in the quotient structure. The rest is a matter of checking topology.

(b), (c), (d) These all follow from (a) and the fact that if X is a P.C.W. complex then, so is $X \times I$ in the obvious way.

DEFINITION 1.10. For $n > 0$, we say that a P.C.W. complex X is *n-piecewise connected* if X is connected and X_i contains no m-cells for any i and $0 < m \leqq n$.

LEMMA 1.11. *Let $n > 0$, and let X be an n-piecewise connected P.C.W. complex. Then $\pi_j(X) = 0$ for $0 < j \leqq n$.*

Proof. Again we perform an induction on i, the number of pieces. For $i = 1$, the hypothesis gives the result since X is a C.W. complex. Suppose we have the result for $i = k - 1$, we know from the theory of C.W. complexes that we may select a model for $\tilde{K}$, in the proof of Theorem 1.6, with the n-skeleton $\tilde{K}^n = *$. Consider the space $M(\xi, X_k)$. Using Lemma 1.3, we know that this space has the type of $\tilde{K} \cup_\xi X_k$. Since ξ is cellular, we know $K \cup_\xi X_k$ is a C.W. complex. This complex has no cells in dim m, $0 < m \leqq n$: Hence by the theory of C.W. complexes the conclusion of the theorem holds. We proceed to the infinite case by the usual direct limit argument.

DEFINITION 1.12. Let $f:(X,X_0) \to (X',X'_0)$ be a map of P.C.W. pairs. Let (K,K_0) be a C.W. pair, and

$$(X,X_0) \underset{\beta}{\overset{\alpha}{\leftrightarrows}} (K,K_0)$$

a homotopy equivalence given by Theorem 1.8. Let (K',K'_0), $\alpha'\beta'$ be the same for (X',X'_0). We define $\bar{f} = \beta' f \alpha : (K,K_0) \to (K',K'_0)$. It follows at once from the preceding that we have the following three commutative diagrams:

$$\begin{array}{ccc} \pi_n(K) & \xrightarrow{\bar{f}*} & \pi_n(K') \\ \alpha \cong \uparrow & & \uparrow \cong \alpha', \\ \pi_n(X) & \xrightarrow{f*} & \pi_n(X') \end{array}$$

$$\begin{array}{ccc} \pi_n(K_0) & \xrightarrow{\bar{f}*} & \pi_n(K'_0) \\ \alpha \cong \uparrow & & \uparrow \cong \alpha', \\ \pi_n(X_0) & \xrightarrow{f*} & \pi_n(X'_0) \end{array} \qquad \begin{array}{ccc} \pi_n(K/K_0) & \xrightarrow{\bar{f}*} & \pi_n(K'/K'_0) \\ \alpha \cong \uparrow & & \uparrow \cong \alpha', \\ \pi_n(X/X_0) & \xrightarrow{f*} & \pi_n(X'/X'_0) \end{array}$$

REMARK. In all that follows the symbol "p-" before any term involving the word "complex" will imply that the term refers to a "piecewise object". All other uses of the term "complex" refer to "C.W. objects".

2. Extraordinary homology.

REMARK. In this section we modify the definitions of G. W. Whitehead in order to apply them to categories of principal G-bundles.

DEFINITION 2.1. Let $\phi: P \to B$ be a principal G-bundle. Let $b \in B$ be a fixed base point. Let G operate as a group of transformations on a space F, and let $x \in F$ be fixed under G. Let E be the bundle associated with P, having fibre F. There is a cross section s_x associated with the point x in the associated bundle. We now define a space called the *P-reduced join of F and B* and written $F \wedge_P B$. We set $F \wedge_P B = E/(\phi_E^{-1}(b) \cup s_x(B))$.

LEMMA 2.2. *In the setting of Definition* 2.1 *and Lemma* 1.5, *suppose F is a connected complex with x the only* 0*-cell. Suppose there are no cells in F in dimension m, $0 < m \leqq n$, then $\pi_m(F \wedge_P B) = 0$, $0 < m \leqq n$.*

Proof. This follows easily from Lemmas 1.9 and 1.11.

DEFINITION 2.3. In the setting of Definition 2.1, we may define an operation of G on SF as follows: G acts on $F \times I$ by $g(f,t) = (gf,t)$. This action carries over to SF since G carries $F \times \dot{I} \cup x \times I$ into itself.

LEMMA 2.4. $i: S(F \wedge_P B) = SF \wedge_P B$.

Proof. We must only check relations on identification maps.

DEFINITION 2.5. We define a *G-spectrum* to be a sequence of spaces E_i with base point acted upon by G which leaves the base points fixed, and a sequence of maps $n_i\colon SE_i \to E_{i+1}$ that are equivariant with respect to G.

DEFINITION 2.6. Let $\mathscr{E}$ be a G-spectrum. Let P be a principal G-bundle over B. For each space S^nE_i (nth-suspension), we may construct a bundle $P(S^nE_i)$, the associated bundle of P with fibre S^nE_i, and bundle maps

$$P(S^n n_i)\colon P(S^{n+1}E_i) \to P(S^nE_{i+1}).$$

Now consider the groups $\pi_{n+k}(E_k \wedge_P B)$. We have the natural suspension map $s\colon \pi_{n+k}(E_k \wedge_P B) \to \pi_{n+k+1}(S(E_k \wedge_P B))$. We may compose this map with the map i_* of Lemma 2.4 to give us a map into the group $\pi_{n+k+1}(SE_k \wedge_P B)$. Now the map $P(n_i)$ induces a map $\hat{P}(n_i)\colon SE_k \wedge_P B \to E_{k+1} \wedge_P B$. Therefore, we may define maps $r_{n,k} = \hat{P}(n_i)_* i_* s\colon \pi_{n+k}(E_k \wedge_P B) \to \pi_{n+k+1}(E_{k+1} \wedge_P B)$.

We define the *homology groups of P with coefficients in $\mathscr{E}$* to be inj lim $r_{n,k}$. We write these groups as $H_n(P;\mathscr{E})$.

Let (B, A) be a pair. Let $P(A)$ be the portion of the bundle over A. We define $H_n(P, P(A);\mathscr{E}) = \text{inj lim } \pi_{n+k}(E_k \wedge_P B \,/\, E_k \wedge_{P(A)} A)$. Note that all groups constructed are abelian.

REMARK. From this point it will be assumed that all spaces that are part of a spectrum are countable C.W. complexes, and that all bases are locally finite simplicial complexes.

DEFINITION 2.7. A G-spectrum is called *convergent* if there is an integer N such that E_{N+i} contains no j-cells $0 < j \leqq i$. The spaces E_i are connected. Note that this is a stronger use of the word than is usual.

EXAMPLE AND REMARK 2.8. An example of a spectrum with nontrivial group action is the spectrum of spheres $\mathscr{S}^n$ (beginning with the n-sphere). The group $O(n)$ acts on the equator of the n-sphere leaving, say, the north pole fixed. It acts on all the other spheres in the spectrum through suspension.

The process above may be generalized. Given a G-spectrum $\mathscr{E}$, we may form a G-spectrum $S\mathscr{E}$ by suspending each space E_i and each map n. We use Definition 2.3, to check that this gives a G-spectrum. We are now in a position to state the theorem which corresponds to the suspension isomorphism of homology.

THEOREM 2.9. *Let P be a principal bundle, then there is a natural isomorphism $\sigma^*\colon H_n(P;\mathscr{E}) \to H_{n+1}(P, S\mathscr{E})$, where $\mathscr{E}$ is convergent.*

Proof. The map σ^* is defined as follows:

$$H_n(P;\mathscr{E}) = \text{inj lim } \pi_{n+i}(E_i \wedge_P B) \xrightarrow{s^*} \text{inj lim } \pi_{n+i+1}(S(E_i \wedge_P B))$$

$$= \text{inj lim } \pi_{n+i+1}(SE_i \wedge_P B) = H_{n+1}(P; S\mathscr{E}).$$

The proof that this map is an isomorphism is similar to the proof of 4.5 in [4] in light of Lemma 2.2 and §1.

Definition 2.10. Let P be a principal bundle over B and let A be a subcomplex of B. It is possible to define a boundary operator $\partial_n\colon H_{n+1}(P, P(A)\mathscr{E} \to H_n(P(A);\mathscr{E})$. We do this by letting $\lambda^i_*\colon \pi_{n+i}(E_i \wedge_P B \cup C(\phi^{-1}(A))) \to \pi_{n+i}(E_i(E_i \wedge_P B/\phi^{-1}(A))$ be induced by the projection map. In light of §1, this map is a homotopy equivalence and therefore it induces an isomorphism which passes to the direct limit, and gives an isomorphism λ_*. We also have the projection map

$$\rho\colon \pi_{n+i+1}(E_i \wedge_P B \cup C(\phi^{-1}(A))) \to \pi_{n+i+1}(S(E_i \wedge_{P(A)} A)).$$

We define

$$\partial_n = \sigma_*^{-1}\rho_*\, \lambda_*^{-1}.$$

Theorem 2.11. *The following sequence is exact*:

$$\cdots \xrightarrow{\partial_{n+1}} H_n(P(A);\mathscr{E} \xrightarrow{i_*} H_n(P;\mathscr{E}) \xrightarrow{j_*} H_n(P, P(A);\mathscr{E}) \xrightarrow{\partial_n} \cdots,$$

where $\mathscr{E}$ *is a convergent spectrum.*

Proof. The proof is in two parts. We first show that three short sequences are exact; we then hook them together properly. The three sequences we need are as follows:

(1) inj lim: $\pi_{i+n}(E_i \wedge_{P(A)} A) \xrightarrow{\alpha} \pi_{i+n}(E_i \wedge_P B) \xrightarrow{\beta} \pi_{i+n}(E_i \wedge_P B/\phi^{-1}(A))$,

(2) inj lim: $\pi_{i+n}(E_i \wedge_P B) \to \pi_{i+n}(L) \to \pi_{i+n}(L/E_i \wedge_P B)$,

(3) inj lim: $\pi_{i+n}(L) \to \pi_{i+n}(L \cup C(E_i \wedge_P B)) \to \pi_{i+n}(L \cup C(E_i \wedge_P B)/L)$,

where $L = E_i \wedge_P B \cup C(\phi^{-1}(A))$, and "inj lim" applies to the entire line. The essence of the proof that each line is exact can be found in 5.2 of [4]. For example, the first line is exact because of the following diagram:

$$\begin{tikzcd} \text{inj lim: } \pi_{i+n}(\phi_i^{-1}(A)) \arrow[r] & \pi_{i+n}(E_i \wedge_P B) \arrow[r, "j"] \arrow[dr, "\beta"'] & \pi_{i+n}(E_i \wedge_P B, \phi_i^{-1}(A)) \arrow[d, "q"] \\ \pi_{i+n}(E_i \wedge_{P(A)} A) \arrow[u, "="] \arrow[ur, "\alpha"'] & & \pi_{i+n}(E_i \wedge_P B/\phi_i^{-1}(A)). \end{tikzcd}$$

The top line of this diagram is exact because it is just the homotopy sequence. The q is an isomorphism in the limit because by 1.8 the pairs $(E_i \wedge_P B, \phi_i^{-1}(A))$ have the type of C.W. pairs and by 2.2 in conjuction with the definition of convergence 2.4 we know that there is a cofinal subset of these pairs such that $q_i\colon \pi_{i+n}(E_i \wedge_P B, \phi^{-1}(A)) \to \pi_{i+n}(E_i \wedge_P B/\phi_i^{-1}(A))$ is already an isomorphism. This last remark is a direct application of the Blakers-Massey Theorem [3].

We are now able to draw a chart which shows how the short sequences are hooked together. The chart is as follows:

$$\begin{array}{ccccccc}
1a & \longrightarrow & 1b & \longrightarrow & 1c & & \\
 & & \wr\shortparallel & & \wr\shortparallel & & \\
 & & 2a & \longrightarrow & 2b & \longrightarrow & 2c \\
 & & & & \wr\shortparallel & & \wr\shortparallel \\
 & & & & 3a & \longrightarrow & 3b & \longrightarrow & 3c
\end{array}$$

where, for example, $3b$ refers to the second group on the third line, and $\wr\shortparallel$ stands for homotopy equivalence.

EXAMPLE 2.12. As an example of a homology theory given by these constructions, consider the $O(n)$-spectrum $\mathscr{S}^n$. Let $P \to B$ be a principal $O(n)$-bundle. Consider the principal $O(n)$-bundle P^+ over $B^+ = B \cup \{p_t\}$ (B with a discrete point added) which is P over B and $O(n)$ over p_t. It is easy to see $S^n \wedge_{P^+} B^+ = T^n_P$ (the Thom space of the n-dimensional vector bundle associated with P). We then have $H_m(P^+, \mathscr{S}^n) = \prod_m(T^n_P)$, the mth stable homotopy group of the Thom space.

3. Cohomology.

REMARK. In this section we develop the theory which is dual to that of the previous sections. However, we will do this in a more general setting. The reason for this is that the constructions and theorems of this section are, in a sense, of a more general nature than those of §2. We could generalize §2 in a way similar to the following, however, the enabling definitions would be very artificial.

We begin with some constructions. Let $p: E \to B$ be a fibre space. Let s^* be a fixed cross section of this fibre space. Let P_{s^*} be the space obtained by taking the full space of paths, $l(t)$, in E and restricting to those paths such that $p(l(0)) = p(l(t))$ for all t, and such that $s^*p(l(0)) = l(0)$. Let $L(E)$ be the full space of loops on E. Define $L_{s^*} = L(E) \cap P_{s^*}$. We again state a lemma which is easy and is left unproved.

LEMMA 3.1. *Let p: $E \to B$ be a fibre space with s^* a fixed cross section. Let $P(p): P_{s^*}(E) \to B$ and $L(p): L_{s^*}(E) \to B$ be the maps associated with p. These maps are fibre maps.*

DEFINITION 3.2. A *fibre spectrum* over B is a sequence of fibre spaces $p_i: E_i \to B$, and a sequence of maps $m_i: E_i \to L_{s^*}(E_{i+1})$ such that $L(p_{i+1})m_i = p_i$. We also insist that $m_i s_i^* = L s_{i+1}^*$, the section of constant loops.

EXAMPLES. (1) Let B be a space and let $\mathscr{E}$ be a spectrum in the usual sense [1]. It is immediate that $B \times \mathscr{E}$ is a fibre spectrum over B.

(2) A *G-Ω-spectrum* $\mathscr{F}$ is a sequence of spaces F_i each having a continuous group action with fixed point by the group G. In addition, there is a sequence of maps $m_i: F_i \to \Omega\, F_{i+1}$ such that the operation of G commutes with the maps m_i. The operation of the group G on the space ΩF_{i+1} is the standard pointwise operation on a loop. The sequence of spaces $P(F_i)$ and maps

$$P(m_i): P(F_i) \to P(\Omega F_{i+1})$$

of 2.6 form a fibre spectrum.

DEFINITION 3.3. Let $\mathscr{E}$ be a fibre spectrum over B. Let $\Gamma(B, -)$ be the covariant functor which carries a fibre space over B into the set of homotopy classes of cross sections where all homotopies are taken through cross sections. We have a sequence of maps $\Gamma(B, L^j m_i)\colon \Gamma(B, L^j E_i) \to \Gamma(B, L^{j+1} E_{i+1})$. We define

$$H^n(B, \mathscr{E}) = \text{inj lim } \Gamma(B, L^i m_{n+i}).$$

Let $\Gamma(B, A, -)$ be the covariant functor which carries a bundle over B into the set of homotopy classes of cross sections which when restricted to $A \subseteq B$ are the base cross section s^* /A. We define $H^n(B, A, \mathscr{E})$ as above using the $\Gamma(B, A, -)$ functor in place of $\Gamma(B, -)$.

It is easy to check that $H^n(B; \mathscr{E})$ and $H^n(B, A; \mathscr{E})$ are abelian groups.

THEOREM 3.4. *With the appropriate additional definitions we have*:
(1) *The following sequence is exact*:

$$\xrightarrow{j^*} H^n(B; \mathscr{E}) \xrightarrow{i^*} H^n(A; \mathscr{E}) \xrightarrow{\delta^*} H^{n+1}(B, A; \mathscr{E}) \xrightarrow{j^*}.$$

(2) *Let $B = A \cup D$. We then have the following isomorphism induced by inclusion*

$$i^*\colon H^n(B, A; \mathscr{E}) = H^n(D, A \cap D; \mathscr{E}).$$

Proof. The theorem, it's proof, and the additional definitions are so standard that they are left undone (see [1]).

REMARK. We now give an example of such cohomology theory. We first need some notation.

Let π be an abstract group. For $n > 1$, let π be abelian. We let $(\pi, \text{aut } \pi)$ stand for the category with one object (i.e. the group π) and as maps, the automorphisms of π.

If X is a space with base point. Let $(X, \text{b.p.h.}(X))$ stand for the category with one object namely X and as maps, the base pointed homeomorphisms of X to itself.

Let π_n be the nth-homotopy group functor. For a base pointed space X. Let $\bar{\pi}_n$ be the restriction of π_n to $\bar{\pi}_n\colon (X, \text{b.p.h.}(X)) \to (\pi_n(X), \text{ aut } \pi_n(X))$. We are now able to state a lemma.

LEMMA 3.5. *Let π be an abstract group. For $n > 1$, let π be abelian. For any n there is a model of $K(\pi, n)$ and a covariant functor*

$$F\colon (\pi, \text{aut}(\pi)) \Rightarrow (K(\pi, n), \, b.p.h.(K(\pi, n)))$$

such that $\bar{\pi}_n F$ is an isomorphism of functors.

Proof. Use the geometric realization of the semisimplicial $K(\pi, n)$.

REMARK. Let C be a local coefficient system on a simplicial complex B. Let the group of this system be π. Let $K(\pi, n)$ be the space of Lemma 3.5. We know that any local coefficient system over a simplicial complex can be considered an aut(π)-bundle over B. Call this bundle C. There is a bundle with fibre $K(\pi, n)$ associated with C. This is given by the functor F. We call this bundle $K_C(\pi, n)$.

THEOREM 3.6. *There is an isomorphism*:

$$\theta: H_C^n(B, \pi) \to \Gamma(B, K_C(\pi, n))$$

where H_C^n refers to local coefficient theory.

Proof. The map is constructed, and the theorem is proved in almost exactly the same way as they are done in the usual proof for trivial local coefficients. Note in this case $K_C(\pi, n) = B \times K(\pi, n)$ and the section homotopy classes of a product bundle are in a natural one-to-one correspondence with the homotopy classes of maps of the base into the fibre. Therefore, the statement of our theorem reduces to the statement that there exists the following isomorphism:

$$\theta: H^n(B, \pi) = [B, K(\pi, n)].$$

This well-known isomorphism carries over to our theorem with very little modification. A proof of the trivial case can be found in [2].

REMARKS. It should be noted that all of the formal constructions of the usual extraordinary theories may be carried out in our theories. We must only insist that all of the necessary maps are equivariant with respect to the group of the bundle. For example, we may develop a cup product following Whitehead's construction [4] with the only restriction being that the pairings of spectra are equivariant.

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