LIMIT THEOREMS FOR MARKOV PROCESSES[1]

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Summary. Let $P(x, A)$ be the transition probability of a Markov Process that satisfies a "Doeblin Condition" and is irreducible (these notions are defined below). Then there are two possibilities:

1. *The process has a finite invariant measure, $\lambda \neq 0$, and there exists an an integer k such that the limit of $P^{nk+j}(x, A)$ exists for every x, A and $0 \leqq j < k$.*

2. *There exists a sequence of sets A_j with $\bigcup_{j=0}^{\infty} A_j = X$ such that $\lim_{n\to\infty} P^n(x, A_j) = 0$, $x \in X$.*

1. **Notation.** Let (X, Σ) be a measurable space. Let $P(x, A)$ be transition probabilities:

1.1. *$P(x, A)$ is defined for $x \in X$ and $A \in \Sigma$ and $0 \leqq P(x, A) \leqq 1$.*

1.2. *For a fixed x the set function $P(x, \cdot)$ is a measure on Σ.*

1.3. *For a fixed $A \in \Sigma$ the function $P(\cdot, A)$ is measurable.*

By measure we shall mean a countably additive, positive, finite measure. When we deal with finitely additive bounded measures we shall write $\mu \in \boldsymbol{ba}(X, \Sigma)$. On occasions we shall deal with σ-finite, countably additive positive measures.

Let us use the terminology of [2, p. 240]. It is well known that the transition probabilities induce an operator P on $B(X, \Sigma)$ and on its conjugate space $\boldsymbol{ba}(X, \Sigma)$ by:

1.4. If $f \in B(X, \Sigma)$, then $(Pf)(x) = \int f(y) P(x, dy)$.

1.5. If $\mu \in \boldsymbol{ba}(X, \cdot)$, then $(\mu P)(A) = \int P(x, A)\mu(dx)$, where

1.6. $\int (Pf)(x)\mu(dx) = \int f(x)(\mu P)(dx)$.

The iterates of these operators are given by the same expressions where P is replaced by P^n:

$$P^n(x, A) = \int P^{n-k}(x, dy) P^k(y, A), \qquad 0 < k < n.$$

Note that if μ is countably additive, so is μP.

2. **The limit theorems.** Throughout this section we assume:

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There exists a σ-finite measure ν with

2.1. *Doeblin's Condition: There exists an integer d such that if $\nu(A)=0$ then* $\sup\{P^d(x,A)\colon x\in X\}<1$.

2.2. *There exists a σ-finite measure λ that is stronger than ν and subinvariant:*

$$\lambda(A) \geqq \int P(x,A)\,\lambda(dx).$$

2.3. *The space X is a locally compact Hausdorff space and Σ consists of its Baire sets.*

Thus by Theorem G on p. 52 of [4] every measure is regular.

DEFINITION. *The process will be called ν-irreducible if:*

2.4. *If $P^n(x,A)=0$, $n=1,2,\cdots$, for some x, then $\nu(A)=0$.*

REMARKS. Condition 2.1 is weaker than the classical Doeblin Condition (see [1, p. 192, hypothesis D]). There one assumes the conclusion whenever $\nu(A)\leqq\varepsilon$ for some fixed $\varepsilon>0$; also uniformity in the sets A is assumed.

The σ-finite measure ν can be replaced by a finite measure ν_1 equivalent to it. Let $\nu_2=\sum \nu_1 P^n/2^n$ then $\nu\ll\nu_2$ and 2.1 holds with respect to ν_2. We shall see below that if $\mu\ll\tau$ then $\mu P\ll\tau P$; thus $\nu_1 P^n\ll\lambda P^n\leqq\lambda$ and $\nu_2\ll\lambda$. Finally let us show that if the process is ν-irreducible then it is ν_2-irreducible.

Note first that if $0\leqq f\in B(X,\Sigma)$ and $(P^n f)(x_0)=0$, $n=1,2,\cdots$ for some x_0, then

$$P^n(x_0,\{x\colon f(x)\geqq\varepsilon\}) \leqq \frac{1}{\varepsilon}(P^n f)(x_0) = 0.$$

Thus $\int f\,d\nu=0$. Apply this to $f(y)=P^k(y,A)$ to conclude:

$$0 = P^n(x_0,A) = \int P^k(y,A)\,P^{n-k}(x_0,dy)$$

implies

$$\int P^k(y,A)\,\nu(dy) = 0.$$

Hence $\nu_2(A)=0$ whenever $P^n(x_0,A)=0$ for all n.

Thus we shall assume, with no loss of generality, that ν is finite and $\nu P\ll\nu$.

LEMMA 1. *Let μ and τ be two σ-finite measures. If $\mu\ll\tau$, then $\mu P\ll\tau P$.*

Proof. Let $d\mu=f\,d\tau$ and $d\mu_k=\min(f,k)\,d\tau$. Then

$$(\mu_k P)(A) = \int P(x,A)\,d\mu_k \leqq k\int P(x,A)\,d\tau.$$

Thus

$$\mu_k P\ll\tau P \quad\text{and also}\quad \mu P\ll\tau P.$$

THEOREM 1. *Let μ be any measure. If $\mu P^n = \tau_n + \sigma_n$, where $\tau_n \ll \nu$ and $\sigma_n \perp \nu$, then* $\lim \sigma_n(X) = 0$.

Proof. Since $\tau_{n+1} + \sigma_{n+1} = \tau_n P + \sigma_n P$ and $\tau_n P \ll \nu$, then $\sigma_{n+1} \leqq \sigma_n P$.

Assume that $\lim \sigma_n(X) \neq 0$. Let σ be a weak * limit point of σ_n, where $\sigma \in \boldsymbol{ba}$. Let $Y \in \Sigma$ be such that $\nu(Y) = 0$ and $\sigma_n(X - Y) = 0$. Given $\varepsilon > 0$, choose n so that

$$\left|(\sigma P^d)(Y) - (\sigma_n P^d)(Y)\right| < \varepsilon,$$

thus

$$(\sigma P^d)(Y) \geqq (\sigma_n P^d)(Y) - \varepsilon \geqq \sigma_{n+d}(Y) - \varepsilon \geqq \lim \sigma_n(X) - \varepsilon,$$

and

$$\begin{aligned}\lim \sigma_n(X) \leqq (\sigma P^d)(Y) &= \int P^d(x, Y)\,\sigma(dx) \\ &\leqq \sup\{P^d(x, Y) : x \in X)\}\,\sigma(X) \\ &= \sup\{P^d(x, Y) : x \in X\} \lim \sigma_n(X) < \lim \sigma_n(X)\end{aligned}$$

by 2.1. This contradiction proves that $\lim \sigma_n(X) = 0$.

We may, and shall, assume that λ is equivalent to ν:

Put $\lambda = \lambda_1 + \lambda_2$ where $\lambda_1 \ll \nu$ and $\lambda_2 \perp \nu$; then $\lambda(A) \geqq (\lambda_1 P)(A) + (\lambda_2 P)(A)$ for every $A \in \Sigma$. Let X_1 be such that $\nu(X - X_1) = 0$ and $\lambda_2(X_1) = 0$ then $(\lambda_1 P)(A) = (\lambda_1 P)(A \cap X_1) \leqq \lambda(A \cap X_1) = \lambda_1(A)$. Thus λ_1 is subinvariant, too, and $\lambda_1 \ll \nu$. Finally since $\nu \ll \lambda$, then $\nu \ll \lambda_1$. If $\lambda_1(A) = 0$, then $\lambda(A \cap X_1) = \lambda_1(A) = 0$ and $\nu(A) = \nu(A \cap X_1) = 0$, too.

Let P be considered as an operator on $L_2(X, \Sigma, \lambda)$ by extending it from $B(X, \Sigma)$ as in [3, pp. 1–2]. For the next lemma we shall use the notation of [3, Theorem 1.1]. Thus there exists a subfield Σ_1 of Σ such that

2.5. *If $f \in L_2(\lambda)$ and $\int_A f\,d\lambda = 0$ for every $A \in \Sigma_1$, then* weak $\lim P^n f = 0$ *in $L_2(\lambda)$ sense.*

2.6. *The sets A in Σ_1 are defined in* [3] *as sets of finite λ-measure such that the functions $P^{n*}\chi_A$, $P^n\chi_A$ are all characteristic functions a.e. where χ_A denotes the characteristic function of A.*

LEMMA 2. *The σ field Σ_1 is generated by a countable collection of disjoint sets.*

Proof. It is enough to show that each set $A \in \Sigma_1$ contains an atom.

Let us assume, to the contrary, that some set A, in Σ_1, with $\lambda(A) \neq 0$ does not contain atoms of Σ_1. Let $\chi_B = P^{d*}\chi_A$ where P^{d*} is the $L_2(X, \Sigma, \lambda)$ adjoint of P^d. Then by Theorem 1.1 of [3], $P^d(x, B) = P^d(\chi_B) = \chi_A$ a.e.

Since λ is a regular measure, there exists a compact subset C_0, $\lambda(C_0) \neq 0$, of A, such that $P^d(x, B) = 1$ for *every* $x \in C_0$. Let A' be the set in Σ_1 which contains C_0 and has *minimal* λ-measure. Such a set is unique up to sets of measure zero and $A' \subset A$. Since A' is not an atom it contains a set A_1, in Σ_1, with $\lambda(A_1) \leqq \frac{1}{2}\lambda(A') \leqq \frac{1}{2}\lambda(A)$. Now $\lambda(C_0 \cap A_1) \neq 0$ for otherwise $A' - A_1$ would be smaller than A' and contain C_0. Let $\chi_{B_1} = P^{d^*}\chi_{A_1}$; then $P^d(x, B_1) = 1$, $x \in C_0 \cap A_1$ a.e. Thus there exists a compact subset C_1 of C_0 such that $P^d(x, B_1) = 1$ for *every* $x \in C_1$ and $\lambda(B_1) = \lambda(A_1) \leqq \frac{1}{2}\lambda(A)$. Using an induction argument, we find a decreasing sequence of sets $B_n \in \Sigma_1$, with $\lambda(B_n) \to 0$, and a decreasing sequence of compact sets C_n, such that $P^d(x, B_n) = 1$ for *every* $x \in C_n$. Let $x_0 \in \cap C_n$; then

$$P^d(x_0, \cap B_n) = \lim P^d(x_0, B_n) = 1$$

while $\lambda(\cap B_n) = 0$, which contradicts 2.1.

Let $W \in \Sigma_1$ be an atom and let PW denote the set whose characteristic function is $P\chi_W$.

Call W of the first kind if the sets $P^n W$ are a.e. disjoint. Otherwise W will be called of the second kind. If $P^n W$ intersects $P^k W$ for $k < n$, then $P^n W = P^k W$ a.e. since they are atoms and hence $P^{n-k} W = W$ a.e. Define:

2.7. $X_1 = \bigcup\{W \colon W \in \Sigma_1$ *and is of the first kind*$\}$.

2.8. $X_2 = \bigcup\{W \colon W \in \Sigma_1$ *and is of the second kind*$\}$.

2.9. $X_3 = X - X_1 \cup X_2$.

LEMMA 3. *If the process is ν-irreducible then either $X = X_3$ or $X = X_2$ and there exists an integer k such that $\Sigma_1 = \{W, PW, \cdots, P^{k-1}W\}$ where $P^k W = W$. In this case the measure λ is finite.*

Proof. If $W \in \Sigma_1$ and $W \subset X_1$, then $\int_W P^n \chi_W d\lambda = 0$, $n = 1, 2, \cdots$. Thus $P^n(x, W) = 0$, $x \in W$ a.e. Thus, since the process is ν-irreducible, $\nu(W) = 0$ and also $\lambda(W) = 0$. Therefore X_1 is empty. Now let us assume that X_2 contains the nonempty set W. Then $P^n(x, W) = 0$ a.e. for $x \in X - W \cup \cdots \cup P^{k-1}W$ and thus this difference is empty.

THEOREM 2. *Let μ be any measure. Let $A \in \Sigma$ and $\lambda(A) < \infty$.*

(a) *If $A \subset X_3$ then* $\lim(\mu P^n)(A) = 0$.

(b) *If $A \subset W \subset X_2$ where $W \in \Sigma_1$ and $P^k W = W$, then the limit of* $(\mu P^{nk+j})(A)$ *exists as $n \to \infty$ and $0 \leqq j < k$.*

(c) *If $A \subset W \subset X_1$ where $W \in \Sigma_1$, then* $\lim(\mu P^n)(A) = 0$.

Proof. By Theorem 1 it is enough to prove these results for a measure $\mu \ll \lambda$. We may assume that $d\mu = f d\lambda$ where $f \in L_2(\lambda)$ since any measure μ which is weaker than λ can be approximated by such measures. Thus:

If $A \subset X_3$, then $(\mu P)^n(A) = \int P^n \chi_A f d\lambda \to 0$ since χ_A is orthogonal to the sets in Σ_1 and 2.5.

If $A \subset W \subset X_1$ where $W \in \Sigma_1$, then $\lim \int P^n \chi_A f d\lambda \leqq \lim \int P^n \chi_W f d\lambda = 0$ since $\int_W P^n \chi_W d\lambda = 0$ and Theorem 2.1 of [3] applies.

Finally let $A \subset W \subset X_2$ where $W \in \Sigma_1$ and $P^k W = W$. Then

$$\lim_{n \to \infty} (\mu P^{nk+j})(A) = \lim_{n \to \infty} \int P^{nk+j} \chi_A \cdot f d\lambda = \frac{\lambda(A)}{\lambda(W)} \int P^j \chi_W \cdot f d\lambda$$

since the function $g = \chi_A - (\lambda(W)^{-1} / \lambda(A) \chi_W$ is orthogonal to all sets in Σ_1 and thus weak $\lim P^n g = 0$ or

$$\lim \int P^{nk+j} \chi_A \cdot f d\lambda = \lim \frac{\lambda(A)}{\lambda(W)} \int P^{nk+j} \chi_W \cdot d\lambda$$

$$= \frac{\lambda(A)}{\lambda(W)} \int P^j \chi_W \cdot f d\lambda.$$

COROLLARY. *If the process is v-irreducible, then either*

(a) $\lim_{n \to \infty} P^n(x, A) = 0$ *for every* $x \in X$ *and every set* A *with* $\lambda(A) < \infty$, π *or:*

(b) *The limit of* $P^{nk+j}(x, A)$ *exists for every* $x \in X$, $A \in \Sigma$ *and* $0 \leqq j < k$.

Proof. It is enough to note that we get (a) when $X = X_3$ and (b) when $X = X_2$ since every set $A \in \Sigma$ can be written as

$$A = (A \cap W) \cup (A \cap PW) \cup \cdots \cup (A \cap P_W^{k-1})$$

and the previous theorem applies to $A \cap P^i W$.

THEOREM 3. *If the process is v-irreducible and* $X = X_2$, *then for any measure* μ *and every* j *there are constants* $\gamma_1 \cdots \gamma_k$ *such that*

$$\lim_{n \to \infty} (\mu P^{nk+j})(A) = \sum_{i=0}^{k-1} \gamma_i \lambda(A \cap P^i W)$$

for all A.

Proof. It is enough to consider μP^{nk}. Let $\tau(A) = \lim (\mu P^{nk})(A)$ where the limit exists for any $A \in \Sigma$. Then, by Corollary III. 7.4. of [2] the set function τ is countably additive and clearly $\tau = \tau P^k$.

From Theorem 1 it follows that $\tau \ll \lambda$. Let $\tau = \tau^0 + \cdots + \tau^{(k-1)}$ where $\tau^{(i)}$ is the restriction of τ to $P^i W$. Thus $\tau^{(i)} P^k = \tau^{(i)}$ and so $\tau^{(i)} + \tau^{(i)} P + \cdots + \tau^{(i)} P^{k-1}$ is invariant under P. It is easy to see that the invariant measure is unique (Theorem 1 and the v-irreducibility) hence this sum is equal to $\gamma_i \lambda$ for some constant γ_i. Now $\tau^{(i)} P^j$ is zero on any subset of W_i:

If $A \subset W_i$, then $P^j \chi_A \cap W_i = \varnothing$ a.e. λ, hence a.e. τ, for $0 < j \leqq k-1$. Thus $\tau^{(i)}(A) = \gamma_i \lambda(A \cap W_i)$.

3. **Existence of a subinvariant measure for irreducible processes.** In this section we use a small modification of Harris' argument to find a subinvariant measure.

In [5] Harris constructs a σ-finite invariant measure for infinitely recurrent process. Here we find only subinvariant measure under weaker conditions. Throughout this section we assume:

3.1. *For every* x, $P(x,X) = 1$.

3.2. *The* σ*-field* Σ *is the Borel extension of a countable family of sets.*

3.3. *The process is* ν*-irreducible where* ν *is a given* σ*-finite measure.*

Notice that X is not assumed to be a topological space and 2.1 is not assumed. Let us just mention those parts of [5] that require a modification in this case.

THEOREM 4. *The process has a* σ*-finite subinvariant measure that is stronger than* ν.

Let P_A by defined as in [5]. Lemma 1 of [5] should be restated:

A. *Let* A *be a measurable set with* $0 < \nu(A) < \infty$. *If* λ_A *is a bounded subinvariant measure for* P_A, *then the measure* λ:

3.4. $$\lambda(E) = \int_A \lambda_A(dx) P_A(x,E)$$

is subinvariant for P *and is* σ*-finite.*

The proof is almost identical to Harris'. First if $E \subset A$, then $\lambda(E) \leqq \lambda_A(E)$. Also

$$\int \lambda(dy)P(y,E) = \int_A \lambda(dy)P(y,E) + \int_{X-A}\left[\int_A \lambda_A(dx)P_A(x,dy)\right]P(y,E)$$

$$\leqq \int_A \lambda_A(dx)\left[P(x,E) + \int_{X-A} P_A(x,dy)P(y,E)\right] = \int_A \lambda_A(dx)P_A(x,E)$$

$$= \lambda(E).$$

The proof that λ is σ-finite is the same as in [5] and also $\lambda(A) \neq 0$, for we will see that $P_A(x,A) > 0$ for every $x \in A$.

Lemma 2 and Lemma 3 of [5] are unchanged. Thus $P_A^1 + \cdots + P_A^n \geqq P^1 + \cdots + P^n$ see [5, Equation 4.17]. Now if $P_A(x,A) = 0$, $x \in A$, then

$$P_A^i(x,A) = \int_A P_A(x,dy)P_A^{i-1}(y,A) = 0.$$

Thus $P^i(x,A) = 0$, $i = 1,2,\cdots$, contrary to 3.3. Let us define

3.5. $$Q(x,E) = \frac{P_A(x,E)}{P_A(x,A)}, \qquad x \in A, \quad E \subset A.$$

Then clearly $Q^i \geqq P_A^i$.

Put

3.6. $$R(x,E) = \frac{Q^1(x,E) + \cdots + Q^k(x,E)}{k}, \qquad x \in A, \quad E \subset A,$$

where k is defined as in [5]. Then Lemmas 4 and 5 of [5] will show us that there exists a measure λ_A with

$$\text{3.7.} \qquad \lambda_A(E) = \int_A \lambda_A(dx)Q(x,E), \qquad E \subset A.$$

Finally, it follows from 3.7 that

$$\text{3.8} \qquad \lambda_A(E) = \int_A \lambda_A(dx)Q(x,E) \geqq \int_A \lambda_A(dx)P_A(x,E), \qquad E \subset A.$$

It remains to show that λ is stronger than v. Now if $\lambda(E) = 0$, then

$$\int \lambda(dx)P^n(x,E) \leqq \lambda(E) = 0.$$

Hence $P^n(x,E) = 0$ a.e., $n = 1,2,\cdots$. Since $\lambda \neq 0$, there exists an $x_0 \in X$ with $P^n(x_0,E) = 0$, $n = 1,2,\cdots$; hence $v(E) = 0$.

4. Existence of an invariant measure.

Throughout this section we assume:

4.1. *For every x, $P(x,X) = 1$.*

4.2. *There exists a σ-finite measure v, and an increasing sequence of sets X_n, in Σ, such that*:

a. $\bigcup X_n = X$.

b. $v(X_n) < \infty$.

c. If $A \in \Sigma$ and $A \subset X_k$, then for every $\varepsilon > 0$ there exists an integer $n = n(A,\varepsilon)$ such that

$$\sup \{P^n(x,A) : x \in X\} \leqq v(A) + \varepsilon.$$

Lemma 4. *Let $\mu \in \boldsymbol{ba}(X,\Sigma)$ be invariant. If $A \subset X_k$, then $\mu(A) \leqq v(A)$.*

Proof. Let $n = n(A,\varepsilon)$; then

$$\mu(A) = \int P^n(x,A)\,\mu(dx) \leqq (v(A) + \varepsilon) \int \mu(dx) = v(A) + \varepsilon.$$

Definition. *Let S be the collection of invariant measures with unit total measure.*

If $\mu \in S$, then $\mu \leqq v$ on subsets of X_k by Lemma 4. Since both are countably additive, $\mu \leqq v$. Thus $d\mu = f dv$ where $0 \leqq f \leqq 1$ and $f \in L_1(v)$.

Now

$$\text{4.3} \qquad (\mu P)(A) = \int P(x,A)\,\mu(dx) = \int P(x,A)f(x)\,v(dx).$$

Lemma 5. *Let $d\mu_1 = f_1 dv$, $d\mu_2 = f_2 dv$ where μ_1 and μ_2 are in S. If $d\mu = \max(f_1,f_2)dv$, then μ is invariant, too.*

Proof. Put $Y_1 = \{x : f_1(x) \geqq f_2(x)\}$, $Y_2 = X - Y_1$. Then

$$\int_A \max(f_1, f_2) dv = \int_{A \cap Y_1} f_1 dv + \int_{A \cap Y_2} f_2 dv$$

$$= \int P(x, A \cap Y_1) f_1(x) v(dx) + \int P(x, A \cap Y_2) f_2(x) v(dx)$$

$$\leqq \int \max(f_1(x), f_2(x)) P(x, A) v(dx).$$

We used 4.2 and the invariance of μ_1 and μ_2. Thus $\mu(A) \leqq (\mu P)(A)$ for every $A \in \Sigma$. But $(\mu P)(X) = \int P(x, X) \mu(dx) = \mu(X) < \infty$; hence $\mu(A) = (\mu P)(A)$.

Consider the collection of functions f such that $f dv \in S$. Since $0 \leqq f \leqq 1$, the supremum of this collection in $L_1(v)$ is the supremum of a sequence f_n in this collection (Theorem IV, 11.7 of [2]). Let $g = \sup f_n$ and $d\lambda = g dv$. If $S = \varnothing$, then take $g = 0$. Let $g_n = \max(f_1, \cdots, f_n)$, then $g = \lim g_n$ and by Lemma 5 and 4.3:

$$\int P(x, A) g_n(x) v(dx) = \int_A g_n(x) v(dx).$$

Passing to a limit, we see that λ is an invariant measure. Also $\lambda \leqq v$ since $0 \leqq g \leqq 1$; thus it is countably additive and finite on X_n.

HEOREM 5. *There exists a σ-finite measure λ with*

a. $\lambda \leqq v$.

b. *λ is invariant under P.*

c. *If $\mu \in S$, then $\mu \leqq \lambda$.*

d. *Let A be contained in some X_k and $\lambda(A) = 0$. For every $\tau \in \boldsymbol{ba}$*

$$\lim \frac{1}{n}(\tau(A) + (\tau P)(A) + \cdots + (\tau P^{n-1})(A)) = 0.$$

Proof. Parts a, b and c were proved above. Let $\tau_n = (\tau + \tau P + \cdots + \tau P^{n-1})/n$ and assume that for some subsequence n_i, $\tau_{n_i}(A) \geqq \delta > 0$. Since τ_n form a bounded sequence in $B(X, \Sigma)^* = \boldsymbol{ba}$, there exists a weak * limit point μ to the sequence τ_{n_i}. Thus $\mu \geqq 0$, $\mu(X) \leqq 1$ and $\mu(A) \geqq \delta > 0$. It is easily seen that $\mu P = \mu$. Let $\mu = \mu_1 + \mu_2$ where μ_1 is a measure (c.a.) and μ_2 is purely finitely additive (see [7, p. 52]). Then $\mu \leqq v$ on subsets of X_k by Lemma 4. Hence $\mu_2(X_k) = 0$, for the restriction of μ_2 to X_k is countably additive. It remains to show that μ_1 is invariant which will contradict part c. Now

$$\mu_1 + \mu_2 = \mu = \mu P = \mu_1 P + \mu_2 P.$$

Let $\mu_2 P = \sigma_1 + \sigma_2$ where σ_1 is c.a. and σ_2 is purely finitely additive. Then $\mu_1 = \mu_1 P + \sigma_1$ but $\mu_1(X) = (\mu_1 P)(X) + \sigma_1(X) = \mu_1(X) + \sigma_1(X)$ and $\sigma_1 = 0$.

REMARK. *Part* d *can be replaced by*: *If A is contained in X_k, then $\lambda(A) = 0$ if and only if*

d^1: $\lim(P(x,A) + \cdots + P^n(x,A))/n = 0$ for every $x \in X$.

d^1 follows from d when we take τ to be a unit mass at x. Conversely given d^1 then for any $\mu \in S$

$$\mu(A) = \frac{1}{n} \int (P(x,A) + \cdots + P^n(x,A))\,\mu(dx) \to 0.$$

Thus $\lambda(A) = 0$, too.

An example. Let ν be a σ-finite measure and $P(x,A) = \int_A f(x,\xi)\,\nu(d\xi)$ where $0 \leqq f(x,\xi)$ and $\int_X f(x,\xi)\,\nu(d\xi) = 1$. It is easy to see that

$$P^n(x,A) = \int f^n(x,\xi)\,\nu(d\xi),\; f^n(x,\xi) = \int f^{n-k}(x,y) f^k(y,\xi)\,\nu(dy).$$

Put

$$g_n(\xi) = \sup\{f^n(x,\xi): x \in X\} \leqq \infty.$$

LEMMA 6. *For every $\xi \in X$, $g_{n+1}(\xi) \leqq g_n(\xi)$.*

Proof.

$$f^{n+1}(x,\xi) = \int f(x,y) f^n(y,\xi)\,\nu(dy) \leqq g_n(\xi) \int f(x,y)\,\nu(dy) = g_n(\xi).$$

Hence $g_{n+1}(\xi) \leqq g_n(\xi)$.

Let $g(\xi) = \lim g_n(\xi)$.

THEOREM 6. *Condition* 4.2 *holds with respect to a measure ν_1 equivalent to ν if $g(\xi) < \infty$ for every $\xi \in X$.*

Proof. Let $Y_k = \{\xi : g(\xi) < k\}$, then $Y_k \subset Y_{k+1}$ and $\bigcup_{k=1}^{\infty} Y_k = X$. Define ν_1 by: $\nu_1(A) = k\nu(A)$ if $A \subset Y_k - Y_{k-1}$. Then $\nu_1 \sim \nu$. If $f_1^n(x,\xi)$ is the Radon-Nikodym derivative of $P^n(x,A)$ with respect to ν_1, then $f_1^n(x,\xi) = (1/k) f^n(x,\xi)$ whenever $\xi \in Y_k - Y_{k-1}$. Hence if g_n^1 and g^1 are defined for f_1^n in the same way that g_n and g were defined for f^n, then $g_n^1(\xi) = (1/k) g_n(\xi)$, $g^1(\xi) = (1/k) g(\xi)$ for $\xi \in Y_k - Y_{k-1}$. Thus $g^1(\xi) < 1$ for every $\xi \in X$. Also ν_1 is σ-finite: if $\bigcup Z_k = X$ where $Z_k \subset Z_{k+1}$ and $\nu(Z_k) < \infty$; then $\nu_1(Z_k \cap Y_k) < k\nu(Z_k) < \infty$ and $\bigcup (Z_k \cap Y_k) = X$. Finally let $V_k = \{\xi : g_k^1(\xi) < 1\}$; then $V_k \subset V_{k+1}$ by Lemma 6 and with $X_k = Z_k \cap Y_k \cap V_k$ we get 4.2.

Let us conclude with a comparison between our results and Orey's [6]. In [6], Theorem 3 corresponds to part (b) of the corollary of Theorem 2. There it is assumed that the process is infinitely recurrent. We have to add a "Doeblin Condition," namely 2.1, but instead of assuming that whenever $\nu(A) > 0$, P [entering A at some time $|X_0 = x] = 1$, we only assumed that this quantity is not zero. Part (a) of Theorem 3 furnishes, under our conditions, a positive answer to the problem posed by Orey in [6 end of §3, p. 816].

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