KÜNNETH FORMULAS FOR BORDISM THEORIES(1)

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1. **Introduction.** Conner and Floyd have noted [3, §44] the lack of a general Künneth formula for the oriented bordism groups $\Omega_n(X \times Y)$, and Atiyah has remarked [1] that the argument he employs to obtain a Künneth formula for K-theory cannot be applied to all cohomology theories. We show that Atiyah's argument can be made to yield Künneth formulas for $\Omega_*(X \times Y)$ and the complex bordism groups $U_*(X \times Y)$, provided that the spaces X and Y satisfy certain conditions.

In particular, we can extend the result of Conner and Floyd [3, Theorem (44.3)], that the bordism product

$$\Omega_*(B(Z_p)) \otimes_\Omega \Omega_*(Y) \xrightarrow{\alpha} \Omega_*(B(Z_p) \times Y)$$

is a monomorphism for any CW-complex Y and odd prime p, to an exact sequence

$$(1) \quad 0 \to \Omega_*(B(Z_p)) \otimes_\Omega \Omega_*(Y) \xrightarrow{\alpha} \Omega_*(B(Z_p) \times Y) \xrightarrow{\beta} \Omega_*(B(Z_p)) *_\Omega \Omega_*(Y) \to 0.$$

Here "$*_\Omega$" denotes the torsion product of graded Ω-modules. If $Y = B(Z_p)$, then the exact sequence will be shown to split into a direct sum decomposition for $\Omega_*(B(Z_p) \times B(Z_p))$.

We shall make the standing assumption that X and Y are CW-complexes, and that X is of "finite type" in the sense of having a finite number of cells in each dimension. Thus $X \times Y$ is again a CW-complex. If nothing is said to the contrary, homology and cohomology have coefficient group Z. We now state the main theorems.

THEOREM A. *Let X and Y be CW-complexes as above. Assume further that*
(a) *the oriented bordism spectral sequence of X collapses;*
(b) *either $\tilde{H}_*(X)$ or $\tilde{H}_*(Y)$ consists entirely of odd torsion.*
Then there is a natural exact sequence of Ω-modules

$$(2) \qquad 0 \to \Omega_*(X) \otimes_\Omega \Omega_*(Y) \xrightarrow{\alpha} \Omega_*(X \times Y) \xrightarrow{\beta} \Omega_*(X) *_\Omega \Omega_*(Y) \to 0$$

where α is the bordism product and β has degree -1.

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THEOREM B. *Let X and Y be CW-complexes as above. Suppose that the complex bordism spectral sequence for X collapses. Then there is a natural exact sequence of U-modules*

$$0 \to U_*(X) \otimes_U U_*(Y) \xrightarrow{\alpha} U_*(X \times Y) \xrightarrow{\beta} U_*(X) *_U U_*(Y) \to 0 \tag{3}$$

where α is again the bordism product and β has degree -1.

As an example, we apply Theorem A with $X = B(Z_p)$, for p an odd prime. The oriented bordism spectral sequence of $B(Z_p)$ collapses [3, (34.1)], and of course $B(Z_p)$ satisfies the standing finiteness condition and condition (b) of Theorem A. Thus we obtain the Künneth formula (1) for $\Omega_*(B(Z_p) \times Y)$.

It is convenient to first obtain "reduced" Künneth formulas, from which the Künneth formulas (2) and (3) are easily derived; e.g., the reduced analogue of (3) is the sequence

$$0 \to \tilde{U}_*(X) \otimes_U \tilde{U}_*(Y) \xrightarrow{\tilde{\alpha}} \tilde{U}_*(X \wedge Y) \xrightarrow{\tilde{\beta}} \tilde{U}_*(X) *_U \tilde{U}_*(Y) \to 0. \tag{4}$$

The proof of Theorem A runs into difficulties with two-torsion; a number of statements are only true modulo the class $\mathscr{C}_2$ of two-torsion groups. In particular, if the assumption (b) of Theorem A is weakened to the requirement that $H_*(X)$ have no two-torsion, there is a $\mathscr{C}_2$ Künneth formula (Theorem C), which is used to prove one case of Theorem A.

2. **Preliminaries.** The bordism groups $\Omega_n(X, A)$ may be identified with the homology groups $H_n(X, A; MSO)$ resulting from the Thom spectrum MSO, and the cobordism groups $\Omega^n(X, A)$ are by definition the cohomology groups $H^n(X, A; MSO)$; see [3, §§12, 13]. There is a multiplicative transformation of of spectra $\mu: MSO \to K(Z)$ of MSO into the integral Eilenberg-MacLane spectrum, given by maps $\mu_n: MSO(n) \to K(Z, n)$ which represent the fundamental (Thom) cohomology classes of the Thom spaces $MSO(n)$. There results a natural multiplicative transformation from oriented (co)bordism to integral (co)homology, still denoted μ. The bordism-to-homology transformation is just the "fundamental class" homomorphism defined in [3, §6]. When convenient, we write $\mu(X, A)$ for $\mu: \Omega_*(X, A) \to H_*(X, A)$, $\tilde{\mu}(X)$ for $\mu: \tilde{\Omega}_*(X) \to \tilde{H}_*(X)$, etc. We recall [3, Theorem (15.1)] that the oriented bordism spectral sequence of a CW-pair (X, A) collapses if and only if $\mu(X, A)$ is an epimorphism. We shall use the fact that the Alexander duality of G. Whitehead [8] for the spectra MSO and $K(Z)$ commutes with μ.

It is evident that the preceeding paragraph can be done over for complex bordism and cobordism in place of the oriented theories, in which case the Milnor spectrum MU replaces MSO.

We now prove two lemmas which allow us to take the CW-complex X to be finite in the proofs of the Künneth formulas.

Lemma 2.1. *Let W be a CW-complex, and denote by W^k the k-skeleton of W. If the oriented* (*resp., complex*) *bordism spectral sequence of W collapses, then this is also true for each W^k.*

Proof. We give the proof for oriented bordism. Thus we assume that $\mu(W)\colon \Omega_*(W) \to H_*(W)$ is an epimorphism, and must show the same for each $\mu(W^k)$.

Consider the commutative diagram

$$\begin{tikzcd} \Omega_n(W^k) \arrow[r, "\mu(W^k)"] \arrow[d, "i_*"] & H_n(W^k) \arrow[d, "i_*"] \\ \Omega_n(W) \arrow[r, "\mu(W)"] & H_n(W). \end{tikzcd}$$

If $n > k$, then $H_n(W^k) = 0$, so there is nothing to prove. If $n < k$, then both maps i_* are isomorphisms, so $\mu(W^k)$ is also an epimorphism. There remains the case $n = k$.

The space W^{k+1}/W^k is just a bunch of $(k+1)$-spheres joined at a point, so its integral homology has no torsion. Thus $\tilde{\mu}(W^{k+1}/W^k)$ is an epimorphism by [3, Theorem (15.2)], whence the same holds for $\mu(W^{k+1}, W^k)$. From the commutative diagram

$$\begin{tikzcd} \Omega_{k+1}(W^{k+1}, W^k) \arrow[r, "\mu"] \arrow[d, "j_*"] & H_{k+1}(W^{k+1}, W^k) \arrow[d, "j_*"] \\ \Omega_{k+1}(W, W^k) \arrow[r, "\mu"] & H_{k+1}(W, W^k) \end{tikzcd}$$

with vertical epimorphisms, we see that $\mu(W, W^k)$ is also an epimorphism. Finally, we see that $\mu(W^k)$ is an epimorphism by inspecting the following commutative diagram with exact columns:

$$\begin{tikzcd} \Omega_{k+1}(W, W^k) \arrow[r, "\mu(W,W^k)"] \arrow[d, "\partial"] & H_{k+1}(W, W^k) \arrow[d, "\partial"] \\ \Omega_k(W^k) \arrow[r, "\mu(W^k)"] \arrow[d, "i_*", "\cong"'] & H_k(W^k) \arrow[d, "i_*"] \\ \Omega_k(W) \arrow[r, "\mu(W)"] \arrow[d] & H_k(W) \arrow[d] \\ 0 & 0\,. \end{tikzcd}$$

Lemma 2.2. *Let W be a CW-complex, and assume that $H_*(W)$ has no two-torsion. Then for each k, $H_*(W^k)$ also is without two-torsion.*

Proof. Since $H_n(W^k)=0$ if $n>k$, $H_n(W^k)\cong H_n(W)$ if $n<k$, and $H_k(W^k)$ is free abelian, the lemma is obvious.

3. **Homology resolutions.** The results of this section are central to the proofs of the Künneth formulas, although the bordism resolutions in §5 have more intrinsic interest. The homology resolutions are obtained by mixing an idea of Atiyah with G. Whitehead's Alexander duality.

LEMMA 3.1. *Let X be a finite CW-complex with base point, and assume that the oriented bordism spectral sequence of X collapses. Then there is a positive integer m and a finite CW-subcomplex A of the suspension S^mX of X so that*

(a) $\tilde{H}_*(A)\to\tilde{H}_*(S^mX)$ *is an epimorphism;*

(b) $\tilde{H}_*(A)$ *has no odd torsion.*

LEMMA 3.2. *Let X be a finite CW-complex with base point, and assume that the complex bordism spectral sequence of X collapses. Then there is a positive integer m and a finite CW-subcomplex A of S^mX so that*

(a) $\tilde{H}_*(A)\to\tilde{H}_*(S^mX)$ *is an epimorphism;*

(b) $\tilde{H}_*(A)$ *has no torsion.*

These lemmas may be proved by the same argument, so it is enough to prove one of them.

Proof of Lemma 3.1. By assumption, $\mu:\Omega_*(X)\to H_*(X)$ is an epimorphism. Equivalently, $\mu:\tilde{\Omega}_*(X)\to\tilde{H}_*(X)$ is an epimorphism. Now consider a weak n-dual D_nX of X and the commutative diagram expressing the Alexander duality:

$$\begin{array}{ccc}\tilde{\Omega}_k(X) & \xrightarrow{\mu} & \tilde{H}_k(X)\\ \downarrow\cong & & \downarrow\cong\\ \tilde{\Omega}^{n-k-1}(D_nX) & \xrightarrow{\mu} & \tilde{H}^{n-k-1}(D_nX).\end{array}$$

Thus the homomorphism $\mu:\tilde{\Omega}^*(D_nX)\to\tilde{H}^*(D_nX)$ is also an epimorphism.

Choose a homogeneous generating set $c_1,\cdots,c_p$ for the abelian group $\tilde{H}^*(D_nX)$, and then select γ_i in $\tilde{\Omega}^*(D_nX)$ so that $\mu(\gamma_i)=c_i$. Let n_i denote the common dimension of γ_i and c_i. Thus $\gamma_i\in\tilde{\Omega}^{n_i}(D_nX)=[S^k(D_nX),\ MSO(k+n_i)]$, $i=1,\cdots,p$, for a suitably large k, which we hold fixed. Now let M_i be a finite CW-complex which (a) approximates $MSO(k+n_i)$ to a degree sufficiently high that the induced map

$$[S^k(D_nX),M_i]\to[S^k(D_nX),MSO(k+n_i)]$$

is an isomorphism ($i=1,\cdots,p$), and (b) has no odd torsion in integral cohomology. (That (b) is possible follows from the fact, proved in [2], that $H^*(BSO(m))$ has no odd torsion, and from the Thom isomorphism $\Phi: H^*(BSO(m))\cong H^*(MSO(m))$.) Thus each γ_i is represented by a map $f_i: S^k(D_nX)\to M_i$, which we combine to obtain a map $f: S^k(D_nX)\to M=\prod_{i=1}^p M_i$.

Now consider the commutative triangle

$$\begin{array}{ccc} \tilde{H}^*(M) & \xrightarrow{f^*} & \tilde{H}^*(S^k(D_nX)) \\ & {}_{\pi_i^*}\nwarrow \quad \nearrow_{f_i^*} & \\ & \tilde{H}^*(M_i). & \end{array}$$

We want to show that f^* is an epimorphism. The inclusion of M_i in $MSO(k+n_i)$ gives rise to an element $\delta_i \in \tilde{\Omega}^{k+n_i}(M_i)$ satisfying

$$f_i^*(\delta_i) \;=\; \sigma^k\gamma_i \in \tilde{\Omega}^{k+n_i}(S^k(D_nX)),$$

σ^k denoting the k-fold suspension. It follows that $\sigma^k c_i = f_*(\pi_i^*(\mu\delta_i))$, which shows that f^* is an epimorphism.

Recall that $S^k(D_nX) = D_{n+k}(X)$. Thus we have found a map $f: D_{n+k}(X) \to M$ so that $f^*: \tilde{H}^*(M) \to \tilde{H}^*(D_{n+k}(X))$ is an epimorphism. We want to apply the duality once more.

Choose an integer m so large that M admits a weak $(n+k+m)$-dual; we also suppose that $f: D_{n+k}(X) \to M$ is regarded as an inclusion of CW-complexes. Then there is the dual inclusion $i: D_{n+k+m}(M) \to D_{n+k+m}(D_{n+k}X)$. Now $D_{n+k+m}(S^mX) = D_{n+k}X$, so we have at hand an inclusion $i: A \to S^mX$, where we have put $A = D_{n+k+m}(M)$. By Alexander duality again, $i_*: \tilde{H}_*(A) \to \tilde{H}_*(S^mX)$ is an epimorphism. Also $\tilde{H}_*(A) \cong \tilde{H}^*(M) = \tilde{H}^*(\pi M_i)$ has no odd torsion, since this is true for all $H^*(M_i)$.

4. **Generators of bordism modules.** From now on, the complex bordism theory will be much easier to treat than the oriented theory; in the latter case, there are difficulties concerning two-torsion. We therefore make use of the class $\mathscr{C}_2$ of abelian two-groups [7, Ch. 10].

Let $\mathscr{C}$ be a class of abelian groups. In addition to the usual notions of $\mathscr{C}$-monomorphism, $\mathscr{C}$-epimorphism and $\mathscr{C}$-isomorphism, we call subgroups B_1 and B_2 of an abelian group *$\mathscr{C}$-equal* if both inclusions $B_1 \cap B_2 \to B_1$ and $B_1 \cap B_2 \to B_2$ are $\mathscr{C}$-isomorphisms. A sequence of homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called *$\mathscr{C}$-exact* if $\mathrm{im}(f)$ and $\ker(g)$ are $\mathscr{C}$-equal subgroups of B.

The following simple applications of these notions are left for the reader to verify.

LEMMA 4.1. *Let T be a graded Ω-module, $T\in\mathscr{C}_2$. Then for any graded Ω-module S, both $T\otimes_\Omega S$ and $T*_\Omega S$ belong to $\mathscr{C}_2$.*

LEMMA 4.2. *Let T be a graded Ω-module which is an odd torsion group. Then for any graded Ω-module S, both $T\otimes_\Omega S$ and $T*_\Omega S$ are odd torsion groups.*

LEMMA 4.3. *There is a five-lemma for $\mathscr{C}_2$ exact sequences of abelian groups.*

Let (X,A) be a CW-pair for which the complex bordism spectral sequence collapses. Then the homomorphism $\mu: U_*(X,A) \to H_*(X,A)$ is an epimorphism. In fact, generators of the U-module $U_*(X,A)$ may be obtained by selecting preimages under μ of generators of the graded abelian group $H_*(X,A)$, and if $H_*(X,A)$ is free abelian we get a better result.

LEMMA 4.4. *Suppose that the complex bordism spectral sequence of the CW-pair (X,A) collapses. Let (c_i) be a homogeneous set of generators of $H_*(X,A)$, and select homogeneous γ_i in $U_*(X,A)$ so that $\mu(\gamma_i) = c_i$. Then (γ_i) generates the U-module $U_*(X,A)$.*

LEMMA 4.5. *Suppose that $H_*(X,A)$ is free abelian, for (X,A) a CW-pair. Then the complex bordism spectral sequence of (X,A) collapses, and $U_*(X,A)$ is a free U-module. Moreover, in the notation of Lemma* 4.4, *if (c_i) is a free basis of $H_*(X,A)$, then (γ_i) is a free basis of $U_*(X,A)$.*

These results are due to Conner and Floyd. See [3, §18] for the techniques involved (in the oriented case), and [4, §5] or [5] for statements involving the complex case.

Now let (X,A) be a CW-pair for which the oriented bordism spectral sequence collapses, and suppose also that $X - A$ has only a finite number of cells in each dimension. The following result is proved by the method of [3, §18].

LEMMA 4.6. *Let (X,A) be as above, and let (c_i) be a homogeneous base for the free part of $H_*(X,A)$, and (d_j) a homogeneous generating set for the odd torsion part of $H_*(X,A)$. Then select homogeneous γ_i and δ_j in $\Omega_*(X,A)$ so that $\mu(\gamma_i) = c_i$, $\mu(\delta_j) = d_j$. Then* (a) *$\sum \Omega\gamma_i$ is a free Ω-submodule of $\Omega_*(X,A)$ on (γ_i), and* (b) *the inclusion $\sum \Omega\gamma_i + \sum \Omega\delta_j \to \Omega_*(X,A)$ is a $\mathscr{C}_2$-isomorphism.*

Proof. We shall modify the proof of [3, Theorem (18.1)] as needed. The independence of the γ_i is a direct consequence of the final paragraph of the proof in [3]. Thus $\sum \Omega\gamma_i$ is a freely generated Ω-module with (γ_i) as base.

To prove (b), it suffices to show that, for each positive n, there is a positive integer m so that

$$2^m\Omega_n(X,A) \subset \sum \Omega\gamma_i + \sum \Omega\delta_j.$$

First we remark that

$$H_k(X,A;\Omega_{n-k}) \cong H_k(X,A) \otimes \Omega_{n-k} \oplus \mathrm{Tor}_z(H_{k-1}(X,A),\Omega_{n-k})$$

by the universal coefficient theorem. Now the torsion in Ω_{n-k} is all of order two, and so the torsion term above is also of order two. Let $2^{m(k)}$ be chosen so large that $2^{m(k)}H_k(X,A)$ has no two-torsion. Then also $2^{m(k)}H_k(X,A;\Omega_{n-k})$ has no two-torsion, and furthermore

$$2^{m(k)}H_k(X,A;\Omega_{n-k}) \subset (\sum Zc_i^k + \sum Zd_j^k) \otimes \Omega_{n-k}$$

since the c_i^k and d_j^k (the superscript indicates dimension) generate all but the two-torsion of $H_k(X,A)$.

The inductive step in the proof of [3, Theorem (18.1)] is now easily replaced by

$$2^{m(k)}J_{k,n-k} \subset (\sum \Omega\gamma_i + \sum \Omega\delta_j) + J_{k-1,n-k+1},$$

and the result of the induction is to establish the inclusion

$$2^m\Omega_n(X,A) \subset \sum \Omega\gamma_i + \sum \Omega\delta_j$$

with $m = m(0) + \cdots + m(n)$. (Recall that there is a filtration

$$\Omega_n(X,A) = J_{n,0} \supset \cdots J_{k,n-k} \cdots J_{0,n} \supset 0$$

and that μ induces isomorphisms $J_{k,n-k}/J_{k-1,n-k+1} \cong H_k(X,A;\Omega_{n-k})$.) This completes the proof.

In the notation of Lemma 4.6, we call $\sum \Omega\gamma_i$ the *free part* of $\Omega_*(X,A)$, and denote it $\Omega_*^f(X,A)$. Notice that $\Omega_*^f(X,A)$ depends on the choices of the c_i and the γ_i. We say that $\Omega_*(X,A)$ is a *$\mathscr{C}_2$-free* Ω-module whenever the inclusion $\Omega_*^f(X,A) \to \Omega_*(X,A)$ is a $\mathscr{C}_2$-isomorphism.

COROLLARY 4.7. *If $H_*(X,A)$ has no odd torsion, then $\Omega_*(X,A)$ is $\mathscr{C}_2$-free.*

Proof. By [3, (15.2)], if $H_*(X,A)$ has no odd torsion then the oriented bordism spectral sequence collapses, so Lemma 4.6 applies. Since there are no d_j's, we conclude that the inclusion $\sum \Omega\gamma_i \to \Omega_*(X,A)$ is a $\mathscr{C}_2$-isomorphism, i.e. that $\Omega_*(X,A)$ is $\mathscr{C}_2$-free.

LEMMA 4.8. *If $\Omega_*(X,A)$ is $\mathscr{C}_2$-free, than the functor $\Omega_*(X,A) \otimes \circ$ from Ω-modules to Ω-modules preserves $\mathscr{C}_2$-exactness.*

This is an easy exercise, making use of Lemma 4.1.

We now study $\Omega_*(X,A)$ for CW-pairs for which $H_*(X,A)$ has no two-torsion, in addition to the standing assumptions made above. Our first result sharpens Lemma 4.6, and the second studies the quotient of $\Omega_*(X,A)$ by the free submodule $\Omega_*^f(X,A)$.

LEMMA 4.9. *Suppose that $H_*(X,A)$ has no two-torsion. Then, in the notation of Lemma 4.6, $\Omega_*(X,A) = \sum \Omega\gamma_i + \sum \Omega\delta_j$.*

Proof. Since the Thom bordism ring Ω has no odd torsion, and since $H_*(X,A)$ is assumed to be without two-torsion, we have $H_k(X,A;\Omega_{n-k}) \cong H_k(X,A) \otimes \Omega_{n-k}$. The method of [3, §18], as was illustrated in the proof of Lemma 4.6, now shows that the γ_i and δ_j together generate $\Omega_*(X,A)$ as an Ω-module.

LEMMA 4.10. *Suppose that $H_*(X,A)$ has no two-torsion. Then the cokernel of the inclusion $\Omega_*^f(X,A) \to \Omega_*(X,A)$ is an odd torsion group.*

Once again, a proof may be given in the spirit of [3, §18].

5. **Bordism resolutions.** We now combine the homology resolutions of §3 with the results of §4 on generators of bordism modules to obtain bordism resolutions. We begin with the complex case.

LEMMA 5.1. *Let X be a finite CW-complex with base point, and assume that the complex bordism spectral sequence of X collapses. Then there is a positive integer m and a finite CW-subcomplex A of S^mX so that the reduced bordism exact sequence of the pair (S^mX, A) is an exact sequence.*

$$0 \to \tilde{U}_*(B) \overset{\partial}{\to} \tilde{U}_*(A) \overset{i_*}{\to} \tilde{U}_*(S^mX) \to 0 \tag{5}$$

$(B = S^mX / A)$ of U-modules with ∂ of degree -1. Furthermore $\tilde{U}_*(A)$ and $\tilde{U}_*(B)$ are free U-modules.

Proof. By Lemma 3.2, we know that an inclusion $i: A \to S^mX$ exists so that $\tilde{H}_*(A)$ has no torsion and so that $i_*: \tilde{H}_*(A) \to \tilde{H}_*(S^mX)$ is an epimorphism. From the homology sequence of (S^mX, A), we see that $\partial: \tilde{H}_*(B) \to \tilde{H}_*(A)$ is a monomorphism, so also $\tilde{H}_*(B)$ has no torsion. Since A and B are finite complexes, it follows that $\tilde{H}_*(A)$ and $\tilde{H}_*(B)$ are free abelian, and so by Lemma 4.5 both $\tilde{U}_*(A)$ and $\tilde{U}_*(B)$ are free U-modules. Thus it remains to show that $i_*: \tilde{U}(A) \to \tilde{U}_*(S^mX)$ is an epimorphism. To see this, consider the commutative diagram

$$\begin{array}{ccc} \tilde{U}_*(A) & \xrightarrow{i_*} & \tilde{U}_*(S^m X) \\ \downarrow \mu & & \downarrow \mu \\ \tilde{H}_*(A) & \xrightarrow{i_*} & \tilde{H}_*(S^mX) \to 0 \end{array}$$

and apply Lemma 4.4 as follows. Let (c_j) generate $\tilde{H}_*(A)$ and select (γ_j) so that $\mu(\gamma_j) = c_j$; then $(i_*(c_j))$ generates $H_*(S^mX)$, and $\mu\{i_*(\gamma_j)\} = i_*c_j$, so $(i_*(\gamma_j))$ generates $\tilde{U}_*(S^mX)$. Thus the upper i_* is an epimorphism, and so the reduced bordism sequence of (S^mX, A) breaks into the short exact sequence (5). Since ∂ is the boundary homomorphism, it has degree -1.

REMARK. Let X satisfy the hypotheses of Lemma 5.1. Since $\tilde{U}_*(X) = \tilde{U}_*(S^mX)$ as U-modules, Lemma 5.1 provides a free resolution of the U-module $\tilde{U}_*(X)$ of length at most one. Thus $\tilde{U}_*(X)$ has homological dimension at most one as a U-module.

LEMMA 5.2. *Let X be a finite CW-complex with base point, and assume that the oriented bordism spectral sequence of X collapses. Then there is a positive integer m and an inclusion $i: A \to S^mX$ of a finite subcomplex of S^mX so that the reduced bordism sequence of (S^mX, A) yields a $\mathscr{C}_2$-exact sequence*

$$0 \to \tilde{\Omega}_*(B) \xrightarrow{\partial} \tilde{\Omega}_*(A) \xrightarrow{i_*} \tilde{\Omega}_*(S^m X) \to 0 \tag{6}$$

($B = S^m X \mid A$) of Ω-modules with ∂ of degree -1. Furthermore, $\tilde{\Omega}_*(A)$ and $\tilde{\Omega}_*(B)$ are $\mathscr{C}_2$-free Ω-modules and the sequence (6) *is exact at $\tilde{\Omega}_*(A)$.*

Proof. The only difference from the proof of Lemma 5.1 is that we must now show that $i_*: \tilde{\Omega}_*(A) \to \tilde{\Omega}_*(S^m X)$ is a $\mathscr{C}_2$-epimorphism. This follows easily from Lemma 4.6, as in the proof of the preceeding lemma.

If $H_*(X)$ has no two-torsion, we may improve Lemma 5.2 slightly.

LEMMA 5.3. *Let X satisfy the hypotheses of Lemma* 5.2, *and assume that $H_*(X)$ has no two-torsion. Then the sequence* (6) *is genuinely exact.*

Proof. We must show that $i_*: \tilde{\Omega}_*(A) \to \tilde{\Omega}_*(S^m X)$ is an epimorphism. This follows, as in the previous proof, from Lemma 4.9.

6. **Proofs of the Künneth formulas.** We begin with an abstraction of the method to be used to derive Künneth formulas.

LEMMA 6.1. *Suppose given a commutative diagram with exact rows:*

$$\begin{tikzcd} 0 \arrow[r] & C'' \arrow[r, "J"] & B' \arrow[r, "\partial'"] \arrow[d, "\alpha_B"] & A' \arrow[r, "i'"] \arrow[d, "\alpha_A"] & C' \arrow[r] \arrow[d, "\alpha_C"] & 0 \\ A \arrow[r] & C \arrow[r, "j"] & B \arrow[r, "\partial"] & A \arrow[r, "i"] & C \arrow[r, "j"] & B. \end{tikzcd}$$

Then:

(a) *If α_A and α_B are isomorphisms, there is a unique homomorphism $\beta_C: C \to C'$ so that $\alpha_B J \beta_C = j$; the sequence*

$$0 \to C' \xrightarrow{\alpha_C} C \xrightarrow{\beta_C} C'' \to 0$$

is exact.

(b) *If α_A and α_B are $\mathscr{C}_2$-isomorphisms and C'' is an odd torsion group, there is a unique homomorphism $\beta_C: C \to C''$ so that $\alpha_B J \beta_C - J: C \to B$ has image lying in the two-torsion of B. The sequence*

$$0 \to C' \xrightarrow{\alpha_C} C \xrightarrow{\beta_C} C'' \to 0$$

is $\mathscr{C}_2$-exact; moreover, $\beta_C \cdot \alpha_C = 0$ and β_C is an epimorphism.

Proof. Case (a) is an easy exercise, as is the second case once the existence of β_C is established. We shall show how to define β_C in the latter case.

A bit of chasing shows that, given an element c of C, there exists for every sufficiently large positive integer n an element c''_n of C'' so that $2^n j(c) = \alpha_B J(c''_n)$. Since C'' is an odd torsion group, and since α_B is a $\mathscr{C}_2$-isomorphism, the elements c''_n are unique; also if $m \geqq n$, then $c''_m = 2^{m-n} c''_n$. Now multiplication by powers of 2 is isomorphism of C'' onto itself, so there is a unique element $c'' = \beta_C(c)$ in C'' so that, for n sufficiently large, $c''_n = 2^n c''$. This defines β_C, and we notice that for large n we have $2^n\{j(c) - \alpha_B J\beta_C(c)\} = 0$. The proof is now easily completed.

We shall find it convenient to prove the existence of reduced Künneth formulas, as was mentioned in the introduction. It is an easy exercise to show that the reduced Künneth formulas lead to "absolute" Künneth formulas. Furthermore, the product map $\alpha\colon \Omega_*(X) \otimes_\Omega \Omega_*(Y) \to \Omega_*(X \times Y)$ is an isomorphism (or $\mathscr{C}_2$-isomorphism) if and only the reduced product map

$$\tilde{\alpha}\colon \tilde{\Omega}_*(X) \otimes_\Omega \tilde{\Omega}_*(Y) \to \tilde{\Omega}_*(X \wedge Y)$$

is.

Now we investigate those cases in which the bordism product α is an isomorphism or $\mathscr{C}_2$-isomorphism.

Lemma 6.2. *Let X and Y be CW-complexes, X of finite type. If $H_*(X)$ has no torsion, then the bordism product $\alpha\colon U_*(X) \otimes_U U_*(Y) \to U_*(X \times Y)$ is an isomorphism.*

Proof. It follows from Lemma 4.5 that $U_*(X)$ is a free U-module. Thus the functor $U_*(X) \otimes_U \circ$ from U-modules to U-modules preserves exactness. It is now an easy exercise in homology theory, using the five lemmas, to complete the proof by an induction up the skeleta of Y, at least if Y is finite; the general case follows easily (see [3, §44]).

Lemma 6.3. *Let X and Y be CW-complexes, X of finite type. If $H_*(X)$ has no odd torsion, then the bordism product $\alpha\colon \Omega_*(X) \otimes_\Omega \Omega_*(Y) \to \Omega_*(X \times Y)$ is a $\mathscr{C}_2$-isomorphism.*

Proof. By Lemma 4.8, the functor $\Omega_*(X) \otimes_\Omega \circ$ from Ω-modules to Ω-modules preserves $\mathscr{C}_2$-exactness. Furthermore, there is a $\mathscr{C}_2$ five lemma, Lemma 4.3. The proof may now be completed by an induction, as above.

Corollary 6.4. *In addition to the hypotheses of Lemma 6.3, assume that $H_*(Y)$ consists entirely of odd torsion. Then the bordism product*

$$\alpha\colon \Omega_*(X) \otimes_\Omega \Omega_*(Y) \to \Omega_*(X \times Y)$$

is an isomorphism.

Proof. We shall show that the reduced bordism product:

$$\tilde{\alpha}\colon \tilde{\Omega}_*(X) \otimes_\Omega \tilde{\Omega}_*(Y) \to \tilde{\Omega}_*(X \wedge Y)$$

is an isomorphism. Now $\tilde{H}_*(X \wedge Y)$ also consists entirely of odd torsion, by the Künneth theorem for ordinary homology. From Lemma 4.10, we see that $\tilde{\Omega}_*(Y)$ and $\tilde{\Omega}_*(X \wedge Y)$ are both odd torsion groups (notice that $\Omega^f_* = 0$ in both cases). Lemma 4.2 implies that the domain of $\tilde{\alpha}$ is also an odd torsion group. Thus $\tilde{\alpha}$ is a

$\mathscr{C}_2$-isomorphism between odd torsion groups, and is therefore a genuine isomorphism.

Proof of Theorem B. We are beginning with the complex case, since there is no problem with torsion. Thus X and Y are CW-complexes, X of finite type, and the complex bordism spectral sequence of X collapses. By Lemma 2.1, this is also the case for each skeleton X^k of X, and since X is of finite type each X^k is a finite CW-complex. So it suffices to prove the theorem with the first factor finite; by naturality, we may then pass to the general case by taking a direct limit.

From now on, X will be assumed to be finite and with base point. We shall obtain a reduced Künneth formula for $\tilde{U}_*(X \wedge Y)$.

We now apply Lemma 5.1, which provides a bordism resolution. Thus there is an inclusion $i: A \to S^m X$ of a CW-complex A in a suitably high suspension of X, so that, if $B = S^m X / A$, the reduced bordism sequence of $(S^m X, A)$ is an exact sequence

$$0 \to \tilde{U}_*(B) \xrightarrow{\partial} \tilde{U}_*(A) \xrightarrow{i_*} \tilde{U}_*(S^m X) \to 0,$$

and both $\tilde{U}_*(A)$, $\tilde{U}_*(B)$ are free U-modules.

We now draw conclusions from the following commutative diagram with exact rows:

$$(7)\qquad \begin{array}{ccccccccc}
0 \to \tilde{U}_*(S^m X) *_U \tilde{U}_*(B) & \xrightarrow{J} & \tilde{U}_*(B) \otimes_U \tilde{U}_*(Y) & \xrightarrow{\partial'} & \tilde{U}_*(A) \otimes_U \tilde{U}_*(Y) & \xrightarrow{i'_*} & \tilde{U}_*(S^m X) \otimes_U \tilde{U}_*(Y) \to 0 \\
 & & \downarrow{\scriptstyle \tilde{\alpha}_B} & & \downarrow{\scriptstyle \tilde{\alpha}_A} & & \downarrow{\scriptstyle \alpha} \\
\cdots \to \tilde{U}_*(S^m X \wedge Y) & \xrightarrow{j_*} & \tilde{U}_*(B \wedge Y) & \xrightarrow{\partial} & \tilde{U}_*(A \wedge Y) & \xrightarrow{i_*} & \tilde{U}_*(S^m X \wedge Y) \to \cdots
\end{array}$$

(the lower row is isomorphic to the reduced bordism exact sequence of the pair $(S^m X \wedge Y,\ A \wedge Y)$). Since $\tilde{U}_*(A)$ and $\tilde{U}_*(B)$ are free U-modules, Lemma 6.2 implies that $\tilde{\alpha}_A$ and $\tilde{\alpha}_B$ are isomorphisms. From Lemma 6.1, we see that there exists a homomorphism $\tilde{\beta}$ determined uniquely by the diagram (7) and so that the sequence

$$0 \to \tilde{U}_*(S^m X) \otimes_U \tilde{U}_*(Y) \xrightarrow{\tilde{\alpha}} \tilde{U}_*(S^m X \wedge Y) \xrightarrow{\tilde{\beta}} \tilde{U}_*(S^m X) *_U \tilde{U}_*(Y) \to 0$$

is exact. Lowering the degrees by m units, we obtain the desired reduced Künneth formula for $\tilde{U}_*(X \wedge Y)$. The absolute Künneth formula for $U_*(X \times Y)$ is now a formal consequence. Since ∂' has degree -1, J must be assigned degree $+1$; therefore $\tilde{\beta}$ and β have degree -1.

It remains to prove the *naturality* of the Künneth formula; we dwell on the reduced formula for $\tilde{U}_*(X \wedge Y)$ with X finite. Since the reduced bordism product $\tilde{\alpha}$ is evidently natural with respect to maps of X and Y, it suffices to show that $\tilde{\beta} \equiv \tilde{\beta}(i)$ also has this property (and is so independent of the choice of the inclusion $i: A \to S^m X$). The naturality in the second variable Y is clear.

Now suppose given $f: X \to X'$ and suitable inclusions $i: A \to S^m X$ and $i': A' \to S^{m'} X'$. The diagram (7) is preserved under suspension (i.e., replacing $i: A \to S^m X$ by $Si: SA \to S^{m+1} X$), so we may assume $m = m'$. We now obtain a commutative diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"] & A \vee A' \arrow[d, "I"] & A' \arrow[l, "j'"'] \arrow[d, "i'"] \\ S^m X \arrow[r, "S^m f"] & S^m X' \arrow[r, equal] & S^m X'. \end{tikzcd}$$

Here $A \vee A'$ is the one-point union of A and A' (at their base points), j and j' are the obvious inclusions, and I is defined by the condition of commutativity. We assume that I has been made an inclusion (up to homotopy). From the fundamental diagrams (7) of the inclusions i, I and i', there results a commutative diagram

$$\begin{tikzcd} \tilde{U}_*(S^m X \wedge Y) \arrow[r, "\tilde{\beta}(i)"] \arrow[d] & \tilde{U}_*(S^m X) *_U U_*(Y) \arrow[d] \\ \tilde{U}_*(S^m X' \wedge Y) \arrow[r, "\tilde{\beta}(i')"] & \tilde{U}_*(S^m X') *_U U_*(Y) \end{tikzcd}$$

which expresses the desired naturality of $\tilde{\beta}$. Thus Theorem B is completely proved.

Proof of Theorem A: $\tilde{H}_*(Y)$ an odd torsion group. Again we take X finite and prove a reduced Künneth formula. Lemma 5.2 provides a bordism resolution: there is an inclusion $i: A \to S^m X$ and a $\mathscr{C}_2$-exact sequence

$$0 \to \tilde{\Omega}_*(B) \xrightarrow{\partial} \tilde{\Omega}_*(A) \xrightarrow{i_*} \tilde{\Omega}_*(S^m X) \to 0$$

which is exact at $\tilde{\Omega}_*(A)$; both $\tilde{\Omega}_*(A)$ and $\tilde{\Omega}_*(B)$ are $\mathscr{C}_2$-free Ω-modules.

There now results a fundamental diagram of the form (7), with U replaced by Ω in all occurrences. The diagram is commutative and the rows are exact; Lemmas 4.1 and 4.2 are needed to prove the upper row exact, as well as the fact that $\tilde{\Omega}_*(Y)$ is an odd torsion groupsince $\tilde{H}_*(Y)$ is. According to Corollary 6.4, both $\tilde{\alpha}_A$ and $\tilde{\alpha}_B$ are genuine isomorphisms. The reduced Künneth formula now follows from Lemma 6.1, and can be shown natural as above for complex bordism.

We now establish a $\mathscr{C}_2$ Künneth formula, after which the proof of Theorem A will be resumed.

THEOREM C. *Let X and Y be CW-complexes, X of finite type. Assume further that*

(a) *$\mu: \Omega_*(X) \to H_*(X)$ is an epimorphism;*

(b) *$H_*(X)$ has no two-torsion.*

Then there is a natural $\mathscr{C}_2$*-exact sequence*

$$0 \to \Omega_*(X) \otimes_\Omega \Omega_*(Y) \xrightarrow{\alpha} \Omega_*(X \times Y) \xrightarrow{\beta} \Omega_*(X) *_\Omega \Omega_*(Y) \to 0.$$

Moreover, $\beta \cdot \alpha = 0$ *and* β *is an epimorphism.*

Proof. Proceed just as in the previous proof, noting that Lemma 2.2 is now needed in order that we may take X finite. We now apply Lemma 5.3, which provides an inclusion $i: A \to S^m X$ so that

$$0 \to \tilde{\Omega}_*(B) \xrightarrow{\partial} \tilde{\Omega}_*(A) \xrightarrow{i_*} \tilde{\Omega}_*(S^m X) \to 0$$

is exact. There results a fundamental diagram of the form (7), with all occurrences of U changed to Ω. Both rows are exact and the diagram is commutative. To show the top row exact, we must show that the derived map

$$\tilde{\Omega}_*(A) *_\Omega \tilde{\Omega}_*(Y) \xrightarrow{i_* * 1} \tilde{\Omega}_*(S^m X) *_\Omega \tilde{\Omega}_*(Y)$$

is zero; since $\tilde{\Omega}_*(A)$ is $\mathscr{C}_2$-free, Lemma 3.1 implies that $\tilde{\Omega}_*(A) *_\Omega \tilde{\Omega}_*(Y) \in \mathscr{C}_2$; since $\tilde{H}_*(X)$ has no two-torsion, the cokernel of $\tilde{\Omega}_*^f(X) \to \tilde{\Omega}_*(X)$ is an odd torsion group by Lemma 4.10, and therefore $\tilde{\Omega}_*(S^m X) *_\Omega \tilde{\Omega}_*(Y)$ is an odd torsion group by Lemma 3.2; so $i_* * 1$ is zero, since its image is zero. Furthermore, $\tilde{\alpha}_A$ and $\tilde{\alpha}_B$ are $\mathscr{C}_2$-isomorphisms by Lemma 6.3. Thus we may apply the delicate part of Lemma 6.1 to obtain a reduced $\mathscr{C}_2$ Künneth formula with $\tilde{\beta} \cdot \tilde{\alpha} = 0$ and $\tilde{\beta}$ an epimorphism. All this carries over to the absolute case. Naturality is proved as before.

Proof of Theorem A: $\tilde{H}_*(X)$ an odd torsion group. We apply Theorem C, noting that all the groups appearing in the $\mathscr{C}_2$ Künneth formula are odd torsion groups if $\tilde{H}_*(X)$ is. But a $\mathscr{C}_2$-exact sequence of maps between odd torsion groups is clearly an exact sequence, so we obtain a natural Künneth formula for $\Omega_*(X \times Y)$.

7. **An application.** If G is a finite group and $B(G)$ a classifying space of G, We denote the bordism groups $\Omega_n(B(G))$ by $\Omega_n(G)$. Recall that $B(G \times H) = B(G) \times B(H)$. In the case $G = Z_p$, a cyclic group of odd prime order, Conner and Floyd have made in [3] a thorough study of $\Omega_*(Z_p)$, both as a graded abelian group and as a graded Ω-module. Much less is known concerning

$$\Omega_*(Z_p \times Z_p) \cong \Omega_*(B(Z_p) \times B(Z_p)).$$

According to [3, (34.1)], the bordism spectral sequence of $B(Z_p)$ collapses; but this is not the case for $B(Z_p) \times B(Z_p)$.

As was mentioned in the introduction, we shall apply Theorem A to obtain the following result.

THEOREM 7.1. *The Künneth formula for* $\Omega_*(Z_p \times Z_p)$ *splits into a direct sum decomposition*

$$\Omega_*(Z_p \times Z_p) = [\Omega_*(Z_p) \otimes_\Omega \Omega_*(Z_p)] \oplus [\Omega_*(Z_p) *_\Omega \Omega_*(Z_p)].$$

Proof. As was mentioned in the introduction, Theorem A yields a Künneth formula

$$(8) \qquad 0 \to \Omega_*(Z_p) \otimes_\Omega \Omega_*(Z_p) \xrightarrow{\alpha} \Omega_*(Z_p \times Z_p) \xrightarrow{\beta} \Omega_*(Z_p) *_\Omega \Omega_*(Z_p) \to 0$$

for $\Omega_*(Z_p \times Z_p)$. Here α is the bordism product and β is of degree -1. It is convenient to strip this down to a reduced Künneth formula

$$(9) \qquad 0 \to \tilde{\Omega}_*(Z_p) \otimes_\Omega \tilde{\Omega}_*(Z_p) \xrightarrow{\tilde{\alpha}} \tilde{\Omega}_*(Z_p \wedge Z_p) \xrightarrow{\tilde{\beta}} \tilde{\Omega}_*(Z_p) *_\Omega \tilde{\Omega}_*(Z_p) \to 0,$$

in which $\tilde{\Omega}_*(Z_p \wedge Z_p)$ denotes $\tilde{\Omega}_*(B(Z_p) \wedge B(Z_p))$ by abuse of notation.

This latter exact sequence will be shown to split canonically, for the elementary reason that both graded groups $\tilde{\Omega}_*(Z_p) \otimes_\Omega \tilde{\Omega}_*(Z_p)$ and $\tilde{\Omega}_*(Z_p) *_\Omega \tilde{\Omega}_*(Z_p)$ are zero in odd degrees and $\tilde{\beta}$ has degree -1. It then follows that the former sequence also splits.

Generators and relations for the Ω-module $\tilde{\Omega}_*(Z_p)$ have been described by Conner and Floyd in [3, §46], from which the following statement results. There is an exact sequence of Ω-modules

$$(10) \qquad 0 \to B_* \to A_* \to \tilde{\Omega}_*(Z_p) \to 0,$$

for which A_* and B_* are free on generators in odd dimensions. Thus (10) is a free resolution of the graded Ω-module $\tilde{\Omega}_*(Z_p)$, and so there results an exact sequence

$$(11)\ 0 \to \tilde{\Omega}_*(Z_p) *_\Omega \tilde{\Omega}_*(Z_p) \to B_* \otimes_\Omega \tilde{\Omega}_*(Z_p) \to A_* \otimes_\Omega \tilde{\Omega}_*(Z_p) \to \tilde{\Omega}_*(Z_p) \otimes_\Omega \tilde{\Omega}_*(Z_p) \to 0$$

by elementary homological algebra, since A_* is a free Ω-module.

Now all the Ω-modules $\tilde{\Omega}_*(Z_p)$, A_* and B_* have only two-torsion in even degrees; for their generators are in odd degrees, and Ω has only two-torsion in odd degrees. Thus $B_* \otimes_\Omega \tilde{\Omega}_*(Z_p)$ and $A_* \otimes_\Omega \tilde{\Omega}_*(Z_p)$ have only two-torsion in odd degrees. It follows from (11) that this is also true for $\tilde{\Omega}_*(Z_p) \otimes_\Omega \tilde{\Omega}_*(Z_p)$ and $\tilde{\Omega}_*(Z_p) *_\Omega \tilde{\Omega}_*(Z_p)$. But $\tilde{\Omega}_*(Z_p)$ consists entirely of odd torsion, since $\tilde{H}_*(B(Z_p))$ does. So by Lemma 4.2, both $\tilde{\Omega}_*(Z_p) \otimes_\Omega \tilde{\Omega}_*(Z_p)$ and $\tilde{\Omega}_*(Z_p) *_\Omega \tilde{\Omega}_*(Z_p)$ are odd torsion groups with all elements of odd degree zero. Since the map $\tilde{\beta}$ of (9) has degree -1, it follows that (9) splits canonically, with $\tilde{\Omega}_*(Z_p) \otimes_\Omega \tilde{\Omega}_*(Z_p)$ isomorphic to the even part of $\tilde{\Omega}_*(Z_p \wedge Z_p)$ and $\tilde{\Omega}_*(Z_p) *_\Omega \tilde{\Omega}_*(Z_p)$ isomorphic to the odd part. Thus also (8) splits, and the theorem is proved.

We remark that these methods also apply to complex bordism, in which case we may allow $p = 2$ as well. See [4], [5] and use Theorem B.

8. **A generalization.** The Künneth formula for complex bordism may be easily carried over to generalized homology theories defined by spectra $\boldsymbol{M}$ which share certain of the properties of the Milnor spectrum MU (and do not share the complications of the Thom spectrum MSO). There are also Künneth formulas for the resulting generalized cohomology theories, in which case only finite complexes are allowed.

Thus let M be a multiplicative spectrum with an augmentation $\mu: M \to K(Z)$ of M into the integral Eilenberg-MacLane spectrum (see [8]) subject to several additional conditions. Here $M = \{M_k, f_k, g_{k,1}\}$, $k, 1 = 0$, with the M_k CW-complexes of finite type with base point; the suspension maps $f_k: SM_k \to M_{k+1}$ and the product maps $g_{k,1}: M_k \wedge M_1 \to M_{k+1}$ are subject to a condition of homotopy commutativity. μ is a map of multiplicative spectra, in the usual sense. The additional conditions are:

(a) M_k is $(k-1)$-connected;

(b) f_k induces isomorphisms $\pi_i(SM_k) \to \pi_i(M_{k+1})$ for $i < 2k$;

(c) μ induces isomorphisms $\pi_k(M_k) \to \pi_k(K(Z, k)) = Z$;

(d) the stable homotopy groups $\pi_i(M)$ are free abelian;

(e) $H^*(M_k)$ is free abelian, for all k.

For such an augmented spectrum, $H_*(\ \ ; M)$ is a multiplicative homology theory, and there is a natural multiplicative transformation $\mu: H_*(\ ;M) \to H_*(\ ;Z)$. By (c) and (d) ,for any CW-pair (X, A), the spectral sequence of (X, A) collapses if and only if $\mu: H_*(X, A; M) \to H_*(X, A; Z)$ is onto. Furthermore, it is easy to conclude, from (d) and a result of Dold [6], that if $H_*(X, A; Z)$ is free abelian, then the spectral sequence of (X, A) collapses. From this and (a), Lemma 2.1 carries over easily. Lemma 3.2 carries over by virtue of (e). The results on generators of bordism modules have analogues for generators of $H_*(X, A; M)$ as a module over the graded coefficient ring $H_*(pt; M) \cong \pi_*(M)$. Thus the results of §§5,6 on resolutions and Künneth formulas may be translated to the present situation. In particular, if X is a CW-complex of finite type for which

$$\mu: H_*(X;)M \to H_*(X; Z)$$

is an epimorphism, then $H_*(X; M)$ is of homological dimension at most one as a (right or left) $\pi_*(M)$-module.

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