

A GENERALISATION OF EBERLEIN'S INTEGRAL OVER FUNCTION SPACE⁽¹⁾

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1. **Introduction.** Let

$$\mathfrak{C} = \{(t_1, t_2, \dots, t_k) = \mathbf{t} : -1 \leq t_j \leq 1, 1 \leq j \leq k\}$$

be a cube in the k -dimensional Euclidean space. Real multiple power series

$$(1.1) \quad \mathbf{x}(t) = \sum x_{n_1 \dots n_k} (t_1)^{n_1} \dots (t_k)^{n_k}$$

where n_1, n_2, \dots, n_k assume all non-negative integral values, and the coefficients satisfy the condition

$$(1.2) \quad \|\mathbf{x}\|_1 = \sum |x_{n_1 \dots n_k}| < \infty,$$

converge absolutely and uniformly for $t \in \mathfrak{C}$. The set of functions defined by (1.1) and (1.2) may be identified with l_1 , regarded as a space of k -fold sequences. That the unit sphere

$$S_\infty = \{\mathbf{x} \in l_1 : \|\mathbf{x}\|_1 \leq 1\}$$

is compact in the weak* topology of l_1 is well known (cf. [1, p. 424]). An elementary integral E can be defined for the weak* continuous real functions on S_∞ . It can then be extended by the Daniell process; and the extended integral induces a countably additive measure on S_∞ . Professor W. F. Eberlein defined E for the case $k = 1$ in [2]. We generalise his results to the case $k \geq 1$. The extended integral and the corresponding measure on S_∞ will be called the *Eberlein integral* and the *Eberlein measure* respectively.

Application of this integral to a discussion of mechanical quadratures will be made in a later paper.

2. **Notation and preliminaries.** We introduce some notation here. More will be explained when the need for it arises.

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Let

$$(2.1) \quad \mathbf{x} = \{x_{n_1 \dots n_k}\}$$

be the sequence of coefficients in (1.1). Setting

$$(2.2) \quad \mathbf{X}_n = \{x_n \dots n_k : n_1 + \dots + n_k = n\}$$

we can write (2.1) as a simple sequence

$$(2.3) \quad \mathbf{x} = (X_0, X_1, \dots, X_n, \dots),$$

the elements of X_n being ordered, say lexicographically. We denote the number of elements in X_n by c_n , so that

$$(2.4) \quad c_n = (k + n - 1)! / ((k - 1)!n!).$$

When the order is relevant, the members of X_n will be enumerated as $\xi_{n1}, \xi_{n2}, \dots, \xi_{nc}$. If \mathbf{y} is a K -fold sequence ordered similarly to \mathbf{x} , the symbols $X_n \pm Y_n$ have unambiguous meanings. If the members of Y are $\eta_{n1}, \eta_{n2}, \dots, \eta_{nc}$ in that order, we write $\langle X_n, Y_n \rangle$ for $\sum_{i=1}^{c_n} \xi_{ni} \eta_{ni}$ and $\langle \mathbf{x}, \mathbf{y} \rangle$ for $\sum_{n=0}^{\infty} \langle X_n, Y_n \rangle$. The following notation will be used to condense long expressions:

$$\sigma_n = \sum_{i=0}^n c_n; \mu_n = \prod_{i=1}^n (c_i!); X_n t^n = \sum x_{n_1 \dots n_k} (t_1)^{n_1} \dots (t_k)^{n_k},$$

the summation being over n_1, \dots, n_k such that $n_1 + \dots + n_k = n$;

$$|X_n| = \sum_{i=1}^{c_n} |\xi_{ni}|;$$

$$dX_n = d\xi_{n1} d\xi_{n2} \dots d\xi_{nc};$$

$$\alpha_n = \sum_{i=0}^n |X_i|;$$

and

$$\phi_n = \prod_{i=1}^n (1 - \alpha_{i-1})^{c_i}.$$

We set down for easy reference, Liouville's generalisation of a theorem of Dirichlet's on multiple integrals [3, p. 160]:

$$\begin{aligned} & \int \dots \int (u_1)^{a_1} \dots (u_n)^{a_n} f(u_1 + \dots + u_n) du_1 \dots du_n \\ &= \frac{\Gamma(a_1 + 1) \dots \Gamma(a_n + 1)}{\Gamma(a_1 + \dots + a_n + n)} \int_0^c h^{a+n-1} f(h) dh \end{aligned}$$

where $a = a_1 + \dots + a_n$, and the multiple integral is over all the non-negative values of the variables u_i such that their sum is not greater than a prescribed constant c . We shall refer to this result as Dirichlet's theorem.

3. **The Eberlein integral.** We begin with the observation that the unit sphere S_∞ of l_1 is a compact metric space in the weak* topology of l_1 . The convergence is pointwise, and a metric is

$$\rho(x, y) = \sum_{n=0}^{\infty} 2^{-n} |X_n - Y_n| / (1 + |X_n - Y_n|) \quad (x, y \in S_\infty).$$

The set weak* continuous real functions on S_∞ , which we denote by $\mathcal{C}(S_\infty)$, is a Banach algebra under pointwise algebraic operations and the norm

$$\|f\|_\infty = \sup \{|f(x)| : x \in S_\infty\} \quad (f \in \mathcal{C}(S_\infty));$$

and closed under the lattice operations, "max" and "min". If we define an elementary integral for functions of $\mathcal{C}(S_\infty)$, it can be extended in the standard fashion [4, Chapter III] to obtain the Eberlein integral, which in turn induces the countably additive Eberlein measure on S_∞ . We address ourselves now to the task of constructing this elementary integral, E .

For $n = 0, 1, 2, \dots$, let P_n be the projection operator on l_1 :

$$P_n x = (X_0, X_1, \dots, X_n, 0, 0, \dots)$$

where

$$x = (X_0, X_1, \dots, X_n, X_{n+1}, \dots) \in l_1$$

and the 0's in the definition of $P_n x$ denote sets of zeros. A function f on l_1 is a cylinder function of degree n iff $f(x) = f(P_n x)$. Denoting the set of cylinder functions of degree n by L_n , we see that

$$(3.1) \quad L_0 \subset L_1 \subset \dots \subset L_n \subset L_{n+1} \subset \dots \subset L_\infty = \bigcup_n L_n.$$

We define the integral E on $\mathcal{C}(S_\infty)$ in such a manner that, for $f \in L_0 \cap \mathcal{C}(S_\infty)$, $E(f)$ reduces to the ordinary average

$$(3.2) \quad E(f) = \frac{1}{2} \int_{-1}^1 f dX_0 = \frac{1}{2} \int_{-1}^1 f d\xi_{01}.$$

Generally, for $f \in L_n \cap \mathcal{C}(S_\infty)$, we define

$$(3.3) \quad E(f) = \frac{\mu_n}{2^{\sigma_n}} \int \dots \int_{\alpha_n \leq 1} \frac{f}{\phi_n} dX_0 \dots dX_n.$$

If we use (3.3) for $f \in L_{n-1} \cap \mathcal{C}(S_\infty)$, we have only to use Dirichlet's theorem to perform the integrations with respect to dX_n , to obtain

$$(3.4) \quad E(f) = \frac{\mu_{n-1}}{2^{\sigma_{n-1}}} \int \dots \int_{\alpha_{n-1} \leq 1} \frac{f}{\phi_{n-1}} dX_0 \dots dX_{n-1},$$

which shows that the definition (3.3) is consistent with the inclusions (3.1). If $f \in L_0 \cap \mathcal{C}(S_\infty)$ and we use (3.3), we can perform n reductions like the above

and arrive precisely at (3.2). Now we let $f \in \mathcal{C}(S_\infty)$ be arbitrary, and define the functions f_n for $n = 0, 1, 2, \dots$ by $f_n(x) = f(P_n x)$ for $x \in S_\infty$. Clearly, $\rho(x, P_n x) \leq 2^{-n} \rightarrow 0$ as $n \rightarrow \infty$. Since f is continuous on the compact space S_∞ , and hence uniformly continuous, we have

$$\|f - f_n\|_\infty = \sup \{ |f(x) - f(P_n x)| : x \in S_\infty \} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence $f_n \rightarrow f$ uniformly. As E is obviously a linear functional of unit norm on each subspace $L_n \cap \mathcal{C}(S_\infty)$ of $\mathcal{C}(S_\infty)$,

$$|E(f_m) - E(f_n)| = \|f_m - f_n\|_\infty \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

Then $\{E(f_n) : n = 0, 1, 2, \dots\}$ is a Cauchy sequence of real numbers, and we define $\lim_n E(f_n)$ as $E(f)$. Explicitly, for $f \in \mathcal{C}(S_\infty)$,

$$(3.5) \quad E(f) = \lim_{\alpha_n \leq 1} \frac{\mu_n}{2^{\sigma_n}} \int \dots \int \frac{f(P_n x)}{\phi_n} dX_0 \dots dX_n.$$

We remark that this same formula serves to define $E(f)$ for any bounded Baire function f . From here the standard Daniell extension process leads to the Eberlein integral and the Eberlein measure on S_∞ . We shall denote the extended integral also by the symbol $\int_{S_\infty} f d_E x$.

THEOREM. *The Eberlein measure on S_∞ is concentrated on $\{x \in S_\infty : \|x\|_1 = 1\}$, the strong boundary of S_∞ .*

Proof. As the function $(1 - \|x\|_1)$ is positive in the interior and zero on the boundary of S_∞ , it is clearly enough to show that

$$\int_{S_\infty} (1 - \|x\|_1) d_E x = 0.$$

Noting that $\|x\|_1 = \lim_n \alpha_n$ and that $\alpha_n \in \mathcal{C}(S_\infty)$ for $n = 0, 1, 2, \dots$, and using (3.5) we have, for $K > -1$,

$$\int_{S_\infty} (1 - \|x\|_1)^K d_E x = \lim_n Q_n^K$$

where

$$Q_n^K = (\mu_n / 2^{\sigma_n}) \int \dots \int [(1 - \alpha_n)^K / \phi_n] dX_0 \dots dX_n.$$

On performing the indicated integrations with respect to dX_n by the aid of Dirichlet's theorem, we see that

$$Q_n^K = \frac{c_n! \Gamma(K + 1)}{\Gamma(c_n + K + 1)} Q_{n-1} \leq \frac{1}{K + 1} Q_{n-1}^K.$$

Hence

$$0 < Q_n^K \leq Q_0^K / (K + 1)^n = 1 / (K + 1)^{n+1};$$

and when $K > 0$, $Q_n^K \rightarrow 0$ as $n \rightarrow \infty$. Q.E.D.

COROLLARY. When $K > -1$,

$$Q_n^K = \frac{\mu_n [\Gamma(K + 1)]^n}{(K + 1) \prod_{i=1}^n \Gamma(c_i + K + 1)}.$$

4. Some special integrals. We evaluate a number of special integrals leading up to (4.19).

(i) We begin with the obvious observation that, when K is an odd positive integer,

$$(4.1) \quad \int_{S_\infty} (\xi_{ni})^K d_E \mathbf{x} = 0.$$

(ii) For any real $K > -1$,

$$(4.2) \quad \int_{S_\infty} |\xi_{ni}|^K d_E \mathbf{x} = \frac{\mu_n [\Gamma(K + 1)]^n}{(K + 1) \prod_{i=1}^n \Gamma(c_i + K + 1)}.$$

By definition, this integral equals

$$(4.3) \quad \frac{\mu_n}{2\sigma_n} \int \dots \int_{\alpha \leq 1} \frac{1}{\phi_n} \{ \} dX_0 \dots dX_{n-1}$$

where

$$\{ \} = \int_{|\mathbf{x}| \leq 1 - \alpha_{n-1}} |\xi_{ni}|^K dX_n.$$

Evaluating this last integral by the use of Dirichlet's theorem and substituting in (4.3), we find the value of the integral in (4.2) to be

$$\frac{c_n! \Gamma(K + 1)}{\Gamma(c_n + K + 1)} Q_{n-1}^K.$$

Substituting for Q_{n-1}^K from (3.6), we obtain the equality (4.2). When K is an integer, (4.2) may be written as

$$(4.4) \quad \int_{S_\infty} |\xi_{ni}|^K d_E \mathbf{x} = \frac{(K!)}{(K + 1) \prod_{i=1}^n \prod_{l=1}^K (c_i + l)}.$$

(iii) If n and r are non-negative integers, then

$$(4.5) \quad \int_{S_\infty} \langle X_n, Y_n \rangle d_E \mathbf{x}$$

equals zero or

$$(4.6) \quad \frac{(r!)^n}{(r+1) \prod_{i=1}^n \prod_{l=1}^r (c_i + l)} \sum' (\eta_{n1})^{K_1} \dots (\eta_{nc})^{K_c},$$

according as r is odd or even. Here \sum' denotes that the sum is taken over all non-negative *even* values of the K 's such that their sum is equal to r .

Expanding $\langle X_n, Y_n \rangle^r$ by the multinomial theorem and using the linearity of the Eberlein integral, we can write (4.5) as

$$(4.7) \quad \sum' \frac{r!}{K_1! \dots K_c!} \int_{S_\infty} (\xi_{n1} \eta_{n1})^{K_1} \dots (\xi_{nc} \eta_{nc})^{K_c} d_E \mathbf{x}$$

where the summation is as explained above. It is obvious that if, in the sum (4.7), any of the K 's were odd in a term, the term vanishes. It follows that (4.5) itself vanishes when r is odd. The typical integral appearing in (4.7) equals

$$(4.8) \quad \frac{\mu_n}{2^{\sigma_n}} \int \dots \int_{\alpha_n \leq 1} \frac{1}{\phi_n} \{ \} dX_0 \dots dX_n$$

where

$$\{ \} = \int_{|\mathbf{x}_n| \leq 1 - \alpha_n} (\xi_{n1})^{K_1} \dots (\xi_{nc})^{K_c} dX_n.$$

Using Dirichlet's theorem to evaluate the last integral and using the evaluation of Q_{n-1}^r from (3.6), we reduce (4.8) to

$$\frac{(r!)^{n-1} K_1! K_2! \dots K_c!}{(r+1) \prod_{i=1}^n \prod_{l=1}^r (c_i + l)}.$$

Using this in the general term of (4.7), we see the equality of (4.5) and (4.6).

(iv) For a fixed non-negative integer n , let r_0, r_1, \dots, r_n be any non-negative integers. We shall evaluate

$$(4.9) \quad I = \int_{S_\infty} \langle X_0, Y_0 \rangle^{r_0} \dots \langle X_n, Y_n \rangle^{r_n} d_E \mathbf{x}.$$

As this integral vanishes whenever one or more of the r 's is odd, we assume them all to be even. The following condensed notation will be useful in evaluating the above integral:

$$R_{-1} = 0, \text{ and } R_j = \sum_{i=0}^j r_{n-i} \quad (0 \leq j \leq n);$$

$$N(n, r) = \sum' (\eta_{n1})^{K_1} \dots (\eta_{nc})^{K_c},$$

where \sum' means the same as in (4.6); and

$$I_{n-j} = \frac{\mu_{n-j}}{2^{\sigma_{n-j}}} \int \cdots \int_{\alpha_{n-j} \leq 1} \frac{1}{\phi_{n-j}} \left[\prod_{i=0}^{n-j} \langle X_i, Y_i \rangle^{r_i} \right] (1 - \alpha_{n-j})^{R_{j-1}} dX_0 \cdots dX_{n-j}.$$

We rewrite

$$(4.10) \quad I_{n-j} = \frac{\mu_{n-j}}{2^{\sigma_{n-j}}} \int \cdots \int_{\alpha_{n-j} \leq 1} \frac{\prod_{i=0}^{n-j-i} \langle X_i, Y_i \rangle^{r_i}}{\phi_{n-j}} \cdot \{ \} dX_0 dX_1 \cdots dX_{n-j-i}$$

where

$$\{ \} = \int_{|X_{n-j}| \leq 1 - \alpha_{n-j-1}} \langle X_{n-j}, Y_{n-j} \rangle^{r_{n-j}} \cdot (1 - \alpha_{n-j-1} - |X_{n-j}|)^{R_{j-1}} dX_{n-j}.$$

Using the multinomial theorem to expand $\langle X_{n-j}, Y_{n-j} \rangle^{r_{n-j}}$ and integrating termwise using Dirichlet's theorem, we obtain

$$\{ \} = 2^{c_{n-j}} \cdot \frac{r_{n-j}! R_{j-1}!}{(R_j + c_{n-j})!} \cdot (1 - \alpha_{n-j-1})^{R_j + c_{n-j}} \cdot N(n-j, r_{n-j}).$$

Substituting this in (4.10), we obtain

$$(4.11) \quad I_{n-j} = I_{n-j-1} \cdot \frac{r_{n-j}! R_{j-1}! N(n-j, r_{n-j})}{\prod_{i=1}^{R_j} (c_{n-j} + l)},$$

a reduction formula for I_{n-j} . Repeated application of this formula yields

$$(4.12) \quad I = I_n = \prod_{i=0}^n \frac{r_i! R_{n-i-1}! N(i, r_i)}{\prod_{t=1}^{R_{n-i}} (c_i + l)}.$$

(v) If y is a bounded k -fold sequence of real numbers, i.e. if $y \in m = (l_1)^*$, then the absolutely convergent series

$$\langle x, y \rangle = \sum_{n=0}^{\infty} \langle X_n, Y_n \rangle$$

defines a bounded Baire function of class 1 on the compact set S_{∞} to the real numbers. Hence

$$(4.13) \quad \int_{S_{\infty}} \langle x, y \rangle^p d_E x$$

exists for every non-negative p . It is clear that the integral reduces to 1 when $p = 0$, and that it vanishes when p is an odd positive integer. First we note that

$$(4.14) \quad \begin{aligned} \langle x, y \rangle^p &= \left[\sum_{n=0}^{\infty} \langle X_n, Y_n \rangle \right]^p \\ &= \sum_1 \sum_2 \sum_3 \frac{p!}{r_{n_1}! r_{n_2}! \cdots r_{n_j}!} \langle X_{n_1}, Y_{n_1} \rangle^{r_{n_1}} \cdots \langle X_{n_j}, Y_{n_j} \rangle^{r_{n_j}} \end{aligned}$$

and that term-by-term integration is permissible. Here \sum_1 is a sum over j from 1 to p ; \sum_2 is a sum over all integers n_1, n_2, \dots, n_j such that $0 \leq n_1 < n_2 < \dots < n_j < \infty$; and \sum_3 , a sum over all strictly positive integers $r_{n_1}, r_{n_2}, \dots, r_{n_j}$ whose sum equals p . As we are going to integrate over S_∞ with respect to $d_E \mathbf{x}$, it is obvious that we can also stipulate that each of $r_{n_1}, r_{n_2}, \dots, r_{n_j}$ be even. To integrate the typical term of (4.14), we can exploit the result (4.12) which gives the value of the integral (4.9). First we have to re-express this term (apart from the constant factor) in the same form as the integrand of (4.9). To this end, we define

$$(4.15) \quad r_m = 0 \quad (0 \leq m \leq n_j; m \neq n_1, n_2, \dots, \text{ or } n_j)$$

and rewrite the summand of (4.14) as

$$\frac{p!}{r_0! r_1! \dots r_{n_j}!} \langle X_0, Y_0 \rangle^{r_0} \langle X_1, Y_1 \rangle^{r_1} \dots \langle X_{n_j}, Y_{n_j} \rangle^{r_{n_j}} .$$

Using the result of (4.12), we conclude that the integral of the summand in (4.14) is

$$(4.16) \quad \frac{p! \prod_{i=0}^{n_j} (R_{n_j-i-1}!) \cdot \prod_{i=0}^{n_j} N(i, r_i)}{\prod_{i=0}^{n_j} \prod_{l=1}^{R_{n_j-1}} (c_i + l)} .$$

This expression, however, involves some extraneous r 's introduced through (4.15). Since it is not always expedient to keep track of these vanishing r 's, we have to purge the expression (4.16) of its dependence on them. First we note that $N(i, r_i) = 1$ whenever $r_i = 0$, so that we can rewrite the product $\prod_{i=0}^{n_j} N(i, r_i)$ as $\prod_{i=1}^{n_j} N(n_i, r_{n_i})$. Then we define the quantities

$$P_{j-m} = \sum_{i=m}^j (r_{n_i}) \quad (1 \leq m \leq j)$$

so that

$$P_{j-1} = R_{n_j-n_1} = R_{n_j-n_1+1} = \dots = R_{n_j}$$

and

$$P_i = R_{n_j-n_{j-i}} = R_{n_j-n_{j-i}+1} = \dots = R_{n_j-n_{j-i-1}-1} \quad (0 \leq i \leq j-2).$$

Using this notation, we can write

$$\prod_{i=0}^{n_j} (R_{n_j-i-1}!) = \frac{\prod_{i=1}^j (P_{j-i}!)^{n_i}}{\prod_{i=2}^j (P_{j-i}!)^{n_{i-1}}} .$$

In considering

$$\prod_{i=0}^{n_j} \prod_{l=1}^{R_{j-1}} (c_i + l),$$

we take the product over i in different stretches:

$$i = 0; 1 \leq i \leq n_1; n_1 + 1 \leq i \leq n_2; \dots; n_{j-1} + 1 \leq i \leq n_j;$$

and obtain

$$\left[\prod_{i=1}^{P_{j-1}} (c_0 - l) \right] \left[\prod_{i=1}^j \prod_{l=1}^{P_{j-1}} (c_{n_i} + l)(c_{n_{i-1}} + l) \dots (c_{n_{i-1}+1} + l) \right].$$

Now we can write (4.16) in the form

(4.17)

$$p! \prod_{i=1}^j (P_{j-i}!)^{n_i} \cdot \prod_{i=1}^j (N(n_i, r_{n_i}))$$

$$\left[\prod_{i=2}^j (P_{j-i}!)^{n_i-1} \right] \left[\prod_{i=1}^{P_{j-1}} (c_0 + l) \right] \left[\prod_{i=1}^j \prod_{l=1}^{P_{j-1}} (c_{n_i} + l)(c_{n_{i-1}} + l) \dots (c_{n_{i-1}+1} + l) \right]$$

Denoting this expression by the symbol, $\{(4.17)\}$, we get, for the value of the integral (4.13), the expression

(4.18)

$$\sum_1 \sum_2 \sum_3 \{(4.17)\}$$

where \sum_1, \sum_2, \sum_3 , mean the same as in (4.14), and we understand that $r_{n_1}, r_{n_2}, \dots, r_{n_j}$ are all strictly positive and even.

(vi) Of considerable importance to the discussion of mechanical quadrature formulae is the special case, $p = 2$, of the general result (4.18). When $p = 2$, the index j in the sum \sum_1 assumes the two values, 1 and 2. When $j = 2$, the corresponding sum \sum_3 is empty. Collecting the terms for $j = 1$, we get

$$(4.19) \quad \int_{S_\infty} \langle x, y \rangle^2 d_E x = \frac{1}{3} \sum_{n=0}^\infty \frac{2^n \sum_{i=1}^c (\eta_{ni})^2}{\prod_{i=1}^n (c_i + 1)(c_i + 2)}.$$

Similarly, when $p = 4$,

$$\int_{S_\infty} \langle x, y \rangle^4 d_E x$$

is the sum of

$$\frac{1}{5} \sum_{0 \leq m \leq n = 1}^\infty \frac{12^m 2^n \left[\sum_{i=1}^c (\eta_{mi})^2 \right] \left[\sum_{i=1}^c (\eta_{ni})^2 \right]}{\left[\prod_{i=1}^m \prod_{l=1}^4 (c_i + l) \right] \left[\prod_{i=m+1}^n (c_i + 1)(c_i + 2) \right]}$$

and

$$\frac{1}{5} \sum_{n=0}^\infty \frac{24^n \left[\sum_{i=1}^c (\eta_{ni})^2 \right]^2}{\prod_{i=1}^n \prod_{l=1}^4 (c_i + l)}.$$

It is clear now that the value of the integral (4.13) can, in principle, be explicitly evaluated for any non-negative integral value of p , though the work can become prohibitively unwieldy for large even values of p .

BIBLIOGRAPHY

1. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.
2. W. F. Eberlein, *An integral over function space*, *Canad. J. Math.* **14** (1962), 379–384.
3. J. A. Edwards, *A treatise on integral calculus*, Vol. II, Macmillan, London, 1922.
4. L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953.

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