ON REE'S SERIES OF SIMPLE GROUPS

BY
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Introduction. The purpose of this paper is to obtain the character tables of the finite simple groups of Ree related to the Lie algebra G_2 (presented in [16], [17]) from certain basic properties of these groups. In the process we shall derive a number of additional properties of the Ree groups. We incorporate the basic properties as conditions in the following definition:

DEFINITION. A finite group G will be said to be of *Ree type* if it satisfies the following five conditions:

I. The 2-Sylow subgroups of G are elementary Abelian of order 8.

II. G has no normal subgroup of index 2.

III. For some element J of order 2 (an "involution") in G, the centralizer $C_G(J)$ of J in G is the direct product of $\langle J \rangle$ and L where L is isomorphic to the linear fractional group $LF(2,q)$.

Condition I implies that $q \equiv 4 + e \pmod 8$ where $e = \pm 1$.

IV. If $\langle R \rangle$ denotes a cyclic subgroup of order $(q+e)/2$ in L, then the normalizer $N_G(\langle R_0 \rangle)$ of any subgroup $\langle R_0 \rangle \neq \langle 1 \rangle$ of $\langle R \rangle$ is contained in $C_G(J)$.

V. Let J' be an involution of L and S an element of L of order $(q-e)/4$ which centralizes J'. Then an element of G of order 3 which normalizes $\langle J, J' \rangle$ does not centralize S.

We call q the *characteristic* of G. The verification of these conditions is straightforward from the description of Ree's groups in [17].

The existence of elements R, J', and S of IV and V is a consequence of the known structure of $LF(2,q)$ which will be summarized in paragraph I–1. Moreover, there is an involution J'' of L commuting with J' for which $J''SJ'' = S^{-1}$; the centralizer of J' in L is $\langle J', J'', S \rangle$. Thus the centralizer $C_G(\langle J, J' \rangle)$ is $\langle J, J', J'', S \rangle$. Also, there is an element of order 3 in L normalizing $\langle J', J'' \rangle$ but not centralizing it.

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Let P_2 be a 2-Sylow subgroup of G. Since P_2 is Abelian, the transfer of G into P_2 has as its image the intersection of P_2 with the center of $N_G(P_2)$ [2, p. 173]. The automorphism group of an elmentary Abelian group of order 8 has order 168, so that $[N_G(P_2): C_G(P_2)]$ divides 21. From the above, 3 divides this index. But if the index is just 3, $N_G(P_2)$ has a 2-element in its center and the kernel of the transfer is proper, violating condition II. Hence the index is 21 and we conclude that all the members of $P_2 - \langle 1 \rangle$ are conjugate, and that any 4-group of P_2 is normalized but not centralized by an element of order 3. Thus all involutions in G are conjugate and an element of order 3 in condition V really exists.

The element S of condition V cannot be the identity, so that the values 3 and 5 for q are excluded. For $5 < q < 27$, the possible orders of $\langle S \rangle$ are (in light of III) 3 and 5; but cyclic groups of these orders do not possess automorphisms of order 3. Thus in fact $q \geqq 27$.

The object of this paper is to prove the following theorem:

THEOREM. *Let G be a group of Ree type of characteristic q. Then the following hold:*

(1) *G is simple of order $q^3(q-1)(q^3+1)$ where $q = 3^{2k+1}$, $k \geqq 1$.*

(2) *If P is a 3-Sylow subgroup of G, P has order q^3 and is disjoint from its conjugates. Its center $Z(P)$ is elementary Abelian of order q, P is of class 3, and P contains a normal elementary Abelian subgroup P_1 of order q^2 containing $Z(P)$ which is both the derived group and the Frattini subgroup of P. The members of $P - P_1$ have order 9, their cubes forming $Z(P) - \langle 1 \rangle$.*

(3) *The normalizer $N(P) = PW$ where W is cyclic of order $q-1$. If J is the involution of W, $C_P(J) = C_{P_1}(J)$ is elementary Abelian of order q and $C_P(J) \cap Z(P) = \langle 1 \rangle$. If R is an element of W of (odd) order $(q-1)/2$, then $C_P(R^a) = \langle 1 \rangle$ for all $R^a \neq 1$.*

(4) *$q^2 - q + 1 = (q+1+3m)(q+1-3m)$, where $m = 3^k$. G possesses Abelian Hall subgroups M^+ and M^- of orders $q+1+3m$ and $q+1-3m$, respectively. M^+ and M^- are respectively the centralizers of their nonidentity elements and are each disjoint from their conjugates. Furthermore, $N(M^+) = M^+W^+$ and $N(M^-) = M^-W^-$ where W^+ and W^- are cyclic of order 6. W^+ and W^- induce regular groups of automorphisms of M^+ and M^-, respectively.*

(5) *The permutation representation of G on the right cosets of $N(P)$ represents G faithfully as a doubly transitive permutation group in such a way that the subgroup fixing three letters has order 2.*

(6) *The character table of G is uniquely determined up to some values of certain "exceptional characters;" the table appears at the end of Chapter V.*

(7) *The decompositions of the various 2-blocks of G are uniquely determined (and appear at the end of Chapter I).*

The present paper is meant to fit into the general program of the characterization of the known simple groups. Since from the result of Feit and Thompson noncyclic simple groups are of even order, the most natural approach to this problem is by a study of the structure of the group relating to its 2-Sylow subgroup and elements of even order.

More particularly, the material presented here bears on the general problem of characterizing groups with Abelian 2-Sylow subgroups. J. H. Walter has worked on showing that such simple groups, assuming their proper subgroups are known, must satisfy conditions I–III for groups of Ree type. Thompson and Janko in turn have shown that conditions IV and V follow from I–III provided $q > 5$. Thus in the program of classifying groups with Abelian 2-Sylow subgroups, it must be shown that a group of Ree type really is one of Ree's groups. (The exceptional case $q = 5$ (with condition V waived) corresponds to Janko's new simple group.)

One approach is by the study of permutation groups. Doubly transitive groups in which only the identity fixes three letters have been classified by Zassenhaus, Feit, Suzuki, and Ito. In addition, Suzuki has classified doubly transitive groups which are similar in nature to the Ree groups but act on an odd number of letters [20]. As an indication of how close the connection to permutation groups is, Ree [18] has shown that a doubly transitive group on an even number ($\geqq 4$) of letters such that the subgroup fixing two letters has exactly one nonidentity element fixing at least three letters and such that each involution fixes at least three letters is indeed of Ree type. (These properties of groups of Ree type follow readily from the discussion of Chapter III.)

The two papers of Brauer, [2] and [3], are referred to as BI and BII, and the numbering of these papers is followed.

Chapter I. The Principal 2-Block of G and Two Families of Exceptional Characters

The main result of this chapter is that the principal 2-block of G has eight ordinary characters. Formulas of Brauer then lead to the degrees of these characters and the order of the group. These formulas are the main tool of the derivation, for they give number-theoretic requirements on G which are very stringent. In preparation for these considerations, two families of exceptional characters are obtained from condition IV on G. Likewise, information about the 2-blocks of defects 1 and 2 is produced.

1. The following remarks describe the nonidentity conjugate classes of $LF(2, q)$: There are two elements R and S_0 of orders $\frac{1}{2}(q + e)$ and $\frac{1}{2}(q - e)$, respectively, each conjugate to its inverse but to no other power. There is a subgroup $\{T(x) \mid x \in F_q\}$, F_q the field of q elements, isomorphic to the additive group of F_q by $x \to T(x)$. $T(x)$ and $T(y)$ are conjugate iff xy is a square in F_q, for $x, y \neq 0$. Let T and T' be members of this subgroup representing the two (nonidentity) classes so obtained. Let $S_0^2 = S$ and $S_0^{(q-e)/4} = J'$ (an involution). The orders

of the centralizers in $LF(2,q)$ are $\frac{1}{2}(q+e)$ for $R^a \neq 1$; $\frac{1}{2}(q-e)$ for $S_0^b \neq 1, J'$; $q-e$ for J'; q for $T(x) \neq 1$. The order of $LF(2,q)$ is $\frac{1}{2}q(q^2-1)$. (For all this see [11].)

2. The character table for $LF(2,q)$ is in Table 1 (from [14]). Here r ($\neq 1$) is any $\frac{1}{2}(q+e)$th root of unity and s ($\neq \pm 1$) any $\frac{1}{2}(q-e)$th root of unity; r and r^{-1} give the same character, as do s and s^{-1}. There are $\frac{1}{4}(q+e-2)$ θ_r's and $\frac{1}{4}(q-e-4)$ θ_s's.

TABLE 1

Value on	1	T	T'	R^a	S_0^b
Character					
θ_1	1	1	1	1	1
θ_2	$\frac{1}{2}(q+e)$	$\frac{1}{2}(e+(eq)^{1/2})$	$\frac{1}{2}(e-(eq)^{1/2})$	0	$e(-1)^b$
θ_3	$\frac{1}{2}(q+e)$	$\frac{1}{2}(e-(eq)^{1/2})$	$\frac{1}{2}(e+(eq)^{1/2})$	0	$e(-1)^b$
θ_4	q	0	0	$-e$	e
θ_r	$q-e$	$-e$	$-e$	$(-e)(r^a+r^{-a})$	0
θ_s	$q+e$	e	e	0	$e(s^b+s^{-b})$

3. The modular characters for the prime 2 of $LF(2,q)$ are as follows (facts used but not cited are in BI): the number of modular characters is the number of 2-regular classes, and that is $2+(3q+e)/8$. The θ_r are of defect 0 and thus are the sole members of their blocks and provide modular characters when restricted to 2-regular elements. On 2-regular elements, θ_s and θ_{-s} coincide and thus belong to the same block. Since in the present situation all characters have height 0 and any block of defect d has at most 2^d (ordinary) characters [8], θ_s and θ_{-s} are the ordinary characters of a block of defect 1. Since there is only one modular character in their block (cf. [10, p. 617]), the Cartan matrix must be [2] and the decomposition matrix

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus θ_s restricted to 2-regular elements provides a modular character. Finally, since the only 2-regular class of defect 2 is that of 1, the principal 2-block is the only block of defect 2. Its ordinary characters are θ_1, θ_2, θ_3, θ_4; there are three modular characters not counted so far, and they must be in this block.

4. We consider the principal 2-block more closely. There are two 2-sections (see BII) in $LF(2,q)$, those of 1 and J'. By BII 7D there are four columns of (nonzero) generalized decomposition numbers for the principal 2-block for these two sections, three for that of 1 and one for J'. The modular character of the

centralizer of J' in $LF(2,q)$ involved must be the principal (identity) character. That centralizer is $\langle J_0, S_0 \rangle$ where $J_0^2 = 1$ and $J_0 S_0 J_0 = S_0^{-1}$. If S_0^b is 2-regular, $J' S_0^b$ is an odd power of S_0. The column of decomposition numbers for $\theta_1, \theta_2, \theta_3, \theta_4$, and J' is then $(1, -e, -e, e)$. The other three columns form a basis for the set of columns with rational integral entries orthogonal to this one (see [4]); by replacing the three modular characters by integral combinations, we may assume these columns are

$$\begin{matrix} 1 & 0 & 0 \\ 0 & -e & 0 \\ 0 & 0 & -e \\ -e & -e & -e \end{matrix} \quad .$$

The functions so obtained are $\theta_1, -e\theta_2, -e\theta_3$ restricted to 2-regular elements. They will be denoted by $\phi_1^J, \phi_2^J, \phi_3^J$. ϕ_2^J and ϕ_3^J are algebraically conjugate (complex conjugate if $e = -1$). The Cartan matrix is then

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} .$$

5. The characters of the entire centralizer $C(J)$ are obtained from the fact that $C(J) = \langle J \rangle L$, where L is isomorphic to $LF(2,q)$ (and we now use the letters used for $LF(2,q)$ for elements of L). The modular characters of $C(J)$ are those of L. To each ordinary character χ of L correspond two of $C(J)$. One (also labelled χ) is χ on L and has value $\chi(A)$ on JA, $A \in L$. The other, labelled χ', is χ on L but $-\chi(A)$ on JA. χ and χ' are in the same 2-block, but if χ_1 and χ_2 (of L) are not, neither are χ_1' and χ_2'. χ and χ' have the same (ordinary) decomposition numbers (see BII).

6. We now plan to use the theory of exceptional characters (see [19] and the expositions in [6] and [10]). If H is a subgroup of G, a collection of conjugate classes $K_1, \cdots, K_m$ of H is a collection of *special classes* if whenever $X \in K_i$ and $A^{-1}XA \in H$ for some $A \in G$, then in fact $A \in H$.

Suppose $\theta_1, \cdots, \theta_n$ is a family of irreducible characters of H, $n > 1$, maximal with the property that $\theta_1(X) = \cdots = \theta_n(X)$ for all X in H not members of special classes. Then:

a. There exist n irreducible characters $\chi_1, \cdots, \chi_n$ of G (the *exceptional characters*) such that $\chi_i | H = \varepsilon\theta_i + \theta$, where $\varepsilon = \pm 1$ (independent of i) and θ is a character of H containing the θ_i with equal multiplicity. If χ is an irreducible character of G not one of the χ_i, $\chi | H$ contains the θ_i with equal multiplicity.

b. If $A \in G$ is not conjugate to any member of any K_i, then $\chi_1, \cdots, \chi_n$ all have the same value on A. (In particular, if $G \neq H$, the class of 1 cannot be special and all χ_i have the same degree.)

Moreover, two different maximal families lead to disjoint families of exceptional characters.

Property IV of G makes the classes of the $R^a \neq 1$ in $C(J)$ special; the classes of the JR^a also become special. The families of the θ_r and the θ'_r provide then two families of exceptional characters for G, the η_r and the η'_r: $\eta_r \mid C(J) = \varepsilon\theta_r + \eta$, $\eta'_r \mid C(J) = \varepsilon'\theta'_r + \eta'$, $\varepsilon = \pm 1$, $\varepsilon' = \pm 1$, and the multiplicity of θ_r in η and θ'_r in η' is independent of r. Each family has $(q-2+e)/4$ members.

7. The characters η and η' actually vanish on the special classes of $C(J)$. For, the orthogonality relations in L, applied to $R^a \neq 1$ and T' give $\sum\theta_r(R^a) = e$, summed over the θ_r. Then also $\sum\theta'_r(R^a) = e$. From the constant multiplicity property above and the fact that the other characters of $C(J)$ have rational integral values on R^a independent of a, $\chi(R^a)$ is a rational integer independent of a for any nonexceptional irreducible character χ of G. The same is true for η and η'.

Condition IV implies that $C(R^a)$, for $R^a \neq 1$, is the centralizer in $C(J)$ of R^a. Thus $c(R^a) = q + e$. Now the order of the centralizer of any element of G is the sum of the squares of the absolute values of the irreducible characters of G on that element. Let ξ_1 be the identity character of G and let ξ_j in general denote a non-exceptional character of G. Then this relation for R^a becomes

$$q + e = \sum_r |\eta_r(R^a)|^2 + \sum_r |\eta'_r(R^a)|^2 + \sum_j |\xi_j(R^a)|^2.$$

Let $\eta(R^a) = x$, $\eta'(R^a) = x'$. Then

$$\begin{aligned}\sum_r |\eta_r(R^a)|^2 &= \sum_r |\varepsilon\theta_r(R^a) + x|^2 \\ &= \sum_r |\theta_r(R^a)|^2 + \varepsilon x \sum_r (\theta_r(R^a) + \overline{\theta_r(R^a)}) + (q-2+e)x^2/4.\end{aligned}$$

The first sum may be evaluated by using this same centralizer relation in L, yielding $\sum_r|\theta_r(R^a)|^2 = \frac{1}{2}(q+e) - 2$. The middle term is $2e\varepsilon x$. Using the same procedure on the η'_r terms, and the fact that θ'_r and θ_r agree on R^a, one finally obtains

$$(1)\quad 4 = (\tfrac{1}{4}(q-2+e)x^2 + 2e\varepsilon x) + (\tfrac{1}{4}(q-2+e)x'^2 + 2e\varepsilon' x') + \sum_j |\xi_j(R^a)|^2.$$

Because the terms in the last sum are integers (one of which is 1) and because $q \geqq 27$, it can only be that $x = x' = 0$. Moreover, four of the ξ_j must have values ± 1 on $R^a(\neq 1)$ independent of a; call them $\xi_1, \xi_2, \xi_3, \xi_4$.

Quite similarly $\eta(JR^a) = \eta'(JR^a) = 0$, and these same four characters ξ_1, ξ_2, ξ_3, and ξ_4 have values ± 1 on $JR^a(\neq J)$ because $\xi_j(JR^a) \equiv \xi_j(R)\ (\mathrm{mod}\ 2)$.

Note that $\sum_r\eta_r(R^a) = \sum_r\eta_r'(R^a) = \varepsilon e$.

8. We wish now to show that there are two possibilities for the number of characters for the principal 2-block, namely seven and eight.

If B is a 2-block of positive defect d of G, a defect group may be taken with J in its center. Then (see [1]) there is a block b of defect d of $C(J)$ such that $B = b^G$ (cf. BII) and for every ordinary character χ of B there is a modular character ϕ^J of b such that the decomposition numbers of χ for the section of J and ϕ^J is not 0.

From BII 6A the decomposition numbers for a character in the principal 2-block B_0 of G for the section of J can be nonzero only for blocks b of $C(J)$ for which $b^G = B_0$. Here, as $\langle J \rangle$ is in the defect group of each block of $C(J)$ (BI 9F), the centralizer of such a defect group is in $C(J)$. Then $b^G = B_0$ iff b is the principal 2-block of $C(J)$ (see [4] or [7]). The Cartan matrix of the principal 2-block of $C(J)$ (relative to ϕ_1^J, ϕ_2^J, ϕ_3^J of paragraph 4) is

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix},$$

double that of L.

There are thus three columns of decomposition numbers for the principal 2-block B_0 of G for the section of J and they correspond to ϕ_1^J, ϕ_2^J, ϕ_3^J. The inner product of the ith and jth columns is the ij entry in this Cartan matrix. Because J is of order 2, the columns consist of rational integers, and the inner product condition forces the entries to be 0, ± 1, or ± 2. But no ± 2 can appear, for it would then be the only entry in its column. However, as these columns are orthogonal to the columns of degrees of the characters in B_0, a single-entry column is out.

From the first part of the discussion, no row can be entirely 0's. On the basis of the preceding remarks there are two possible patterns:

ϕ_1^J	ϕ_2^J	ϕ_3^J
1	0	0
δ_2	0	0
δ_3	δ_3	δ_3
δ_4	δ_4	δ_4
0	δ_5	0
0	δ_6	0
0	0	δ_7
0	0	δ_8

ϕ_1^J	ϕ_2^J	ϕ_3^J
1	0	0
δ_2	δ_2	0
δ_3	0	δ_3
δ_4	δ_4	δ_4
0	δ_5	0
0	0	δ_6
0	δ_7	δ_7 .

Each δ_i is ± 1. The first four characters in either case are ξ_1, ξ_2, ξ_3, and ξ_4, of paragraph 7. The 2-Sylow subgroup of G is its own centralizer (from the structure of L) so that by BI 6D this block B_0 is the only one of 2-defect 3.

9. Before analyzing the principal 2-block in detail, we derive information on the other 2-blocks of G of positive defect. First we consider blocks of defect 1.

Let ϕ_r^J be the modular character of $C(J)$ given by θ_r (paragraph 3). The defect of the block b_r of ϕ_r^J is 1 and the Cartan matrix is [2]. As $C(J)$ is the normalizer of $\langle J\rangle$ in G, BI 10B says the map $b_r \to b_r^G$ produces a 1-1 correspondence between the blocks b_r of $C(G)$ of defect 1 and those of G having defect group $\langle J\rangle$. As the involutions of G are all conjugate, this means all blocks of G of defect 1. The column of decomposition numbers for the section of J and ϕ_r^J has then two nonzero entries, ± 1. By the orthogonality to the degrees, one is 1, the other -1. From BII 6A the corresponding characters are in b_r^G. No other block b of $C(J)$ has $b^G = b_r^G$, for either b is another block of defect 1 or it, and therefore b^G, has defect greater than 1. Then all the other columns (for the section of J) have 0's for these two characters. From paragraph 8, no character in b_r^G has a row of 0's for its decomposition numbers for J. b_r^G thus contains exactly two characters. From $\eta_r(JA) = \varepsilon\phi_r^J(A) + \eta(JA)$ for $A \in L$, and the equal multiplicity result for η, η_r decomposed for the section of J involves ϕ_r^J with a multiplicity different from that of the other modular characters of defect 1 of $C(J)$. Because there are at least three such characters, η_r must be in b_r^G. Similarly η_r' is in this block, and $\varepsilon = \varepsilon'$: the column for ϕ_r^J is $(\varepsilon, -\varepsilon)$.

10. Consider now 2-blocks of G of defect 2. Because the involutions of L are all conjugate in L, all groups of order 4 of G are conjugate. Any group of order 4 in L is normalized but not centralized by an element of order 3 (cf. [11]). By a conjugation, we may take $\langle J, J'\rangle$ to be the defect group of any 2-block of defect 2, with an involution J'' such that $\langle J, J', J''\rangle$ is of order 8. Then $C(J, J') = \langle J, J', J'', S\rangle$ and $J''SJ'' = S^{-1}$. $\langle S\rangle$ is the commutator subgroup of $C(J, J')$.

From BI 12A the blocks in G with defect group $\langle J, J'\rangle$ are in 1-1 correspondence with families of associated characters of defect 0 of $C(J, J')/\langle J, J'\rangle$. (The relatively prime condition of BI 12A is automatic here.) Let θ be a character of one of these families, belonging to the block b when regarded as a character of $C(J, J')$. If θ's family corresponds to the block B of G, then $b^G = B$ (BII 2D). The 2-blocks of $C(J)$ of defect 2 also have defect group $\langle J, J'\rangle$. By BI 10B there is a 1-1 correspondence between blocks of $C(J)$ with defect group $\langle J, J'\rangle$ and blocks of defect 2 of the normalizer in $C(J)$ of $\langle J, J'\rangle$, i.e., $C(J, J')$. If the block b_1 of $C(J, J')$ corresponds to the block B_1 of $C(J)$, then $b_1^{C(J)} = B_1$. From BII 2G the blocks of $C(J, J')$ and $C(J, J')/\langle J, J'\rangle$ are in 1-1 correspondence: a character of $C(J, J')/\langle J, J'\rangle$ belonging to a given block belongs to the corresponding block when regarded as a character of $C(J, J')$. As the characters of defect 0 of $C(J, J')/\langle J, J'\rangle$ are in 1-1 correspondence with blocks of defect 0, each block of defect 2 of $C(J,J')$ contains exactly one such character (as a character of $C(J,J')$).

Thus if b is a 2-block of $C(J,J')$ of defect 2, b^G and $b^{C(J)}$ are blocks of defect 2; from BII 2C, $(b^{C(J)})^G = b^G$. Now $C(J,J')/\langle J,J'\rangle$ is isomorphic to $\langle J'',S\rangle$. If A is an element of order 3 normalizing but not centralizing $\langle J,J'\rangle$, condition V on G yields $A^{-1}SA \neq S^{-1}$. $\langle J'',S\rangle$ is a dihedral group. The two-dimensional irreducible characters are induced from the nontrivial one-dimensional characters of $\langle S\rangle$. If λ generates the character group of $\langle S\rangle$ and if $A^{-1}SA = S^n$, then λ^A (the character associated to λ by A) is λ^n. As λ^m and λ^{-m} induce the same character, we may let the class $\{S^m, S^{-m}\}$ of $\langle J'',S\rangle$ correspond to the character induced by λ^m; this correspondence is preserved under the action of A. Since A moves at least one class of $\langle S\rangle$, it moves at least one two-dimensional character; that is, there is a character θ of degree 2 generating a family of *three* associate characters (A generates the normalizer of $\langle J,J'\rangle$ over $C(J,J')$). Thus there is at least one set of three blocks b_4, b_5, b_6 of $C(J)$ of defect 2 for which $b_4^G = b_5^G = b_6^G = B_2$, a block of G of defect 2. These are the only blocks b of $C(J)$ with $b^G = B_2$. Let $\phi_4^J, \phi_5^J, \phi_6^J$ be the modular characters for b_4, b_5, b_6, respectively (cf. paragraph 3). The Cartan matrix for each of these blocks is [4] (the ordinary characters of any one are $\theta_s, \theta'_s, \theta_{-s}, \theta'_{-s}$ for suitable s).

If χ is an ordinary character of B_2, the Cartan matrix [4] means no decomposition number for the section of J and χ is ± 2, otherwise that is the only entry in its column (cf. paragraph 8). The entries are 0 or ± 1, then. Because the defect is 2, B_2 contains at most four ordinary characters [8]. There must be then exactly four characters, and the orthogonality to the column of degrees means that they all have the same degree and the following decomposition ($\delta = \pm 1$):

	ϕ_4^J	ϕ_5^J	ϕ_6^J
ξ_{11}	δ	δ	δ
ξ_{12}	$-\delta$	$-\delta$	$-\delta$
ξ_{13}	$-\delta$	δ	$-\delta$
ξ_{14}	δ	$-\delta$	$-\delta$.

11. A formula of Brauer [5] will be applied several times to the principal block and the block of defect 2 above. The formula is given not in its most general form, but in the form it takes in the present case. Let b be a 2-block of $C(J)$ and ϕ_j^J any member of a basis of the modular characters of b, i.e., any member of the set of transforms of the modular characters by a unimodular transformation (integral coefficients). Let $B = b^G$. Let d_{ij}^J and $d_{ij}'^J$ be the decomposition numbers for the section of J relative to ϕ_j^J for the characters $\chi_i \in B$ and $\chi'_i \in b$, respectively. Let $h_i = \chi'_i(J)/q(q^2-1) + \chi'_i(J')/2(q-e) + \chi'_i(JJ')/2(q-e)$. Then

(2) $$g \sum_{\chi_i \in B} (\chi_i(J))^2 d_{ij}^J / \chi_i(1) = (q(q^2-1))^3 \sum_{\chi'_i \in b} (h_i)^2 d_{ij}'^J / \chi_i'(1).$$

12. If (2) is applied to the block B_2 above and the modular character ϕ_4^J, there results $\delta g = q^3(q-e)\xi_{11}(1)$, forcing $\delta = 1$. Generally, if H is a subgroup of G and χ is a character of G, the multiplicity of the principal character of H in $\chi | H$ is $|H|^{-1} \sum_{A \in H} \chi(A)$. As this must be a rational integer, one obtains $\sum_{A \in H} \chi(A) \equiv 0 \pmod{|H|}$. For ξ_{11} and $H = \langle R \rangle$ this yields $\xi_{11}(1) \equiv 0 \pmod{\frac{1}{2}(q+e)}$, or, as $\xi_{11}(1)$ is even, $\xi_{11}(1) = (q+e)a$, a an integer. Thus $g = q^3(q^2-1)a$.

13. One further numerical function is useful in discussing the principal block. For $A \in G$, let $\sigma(A)$ be the number of $X \in G$ with $X^2 = A$. Then σ is a class function, so that $\sigma(A) = \sum \sigma_\chi \chi(A)$ summed over the characters of G. From [12] σ_χ is given as follows: if χ comes from a representation in the real numbers, $\sigma_\chi = 1$. If χ is real but the representation cannot be put in the real numbers, $\sigma_\chi = -1$. Otherwise, $\sigma_\chi = 0$. Moreover, if χ has an odd degree, $\sigma_\chi \neq -1$.

If $A \in C(J)$ and A is 2-regular, $\sigma(JA) = 0$, or $\sum \sigma_\chi \chi(JA) = 0$. Expressing the χ's in terms of the decomposition numbers and modular characters for the section of J and using the independence of the modular characters of $C(J)$, one finds that the column of σ_χ's is orthogonal to each column of decomposition numbers for the section of J. Let $\sigma_i = \sigma_{\xi_i}$.

14. Certain relations involving the characters of the principal block are valid regardless of which decomposition is correct. For $1 \leqq i \leqq 4$ let $\xi_i(R^b) = e_i$, $R^b \neq 1$, so that $e_i (= \pm 1)$ is independent of b (paragraph 7). The orthogonality relations for the sections of 1 and J imply $1 + \delta_2 e_2 + \delta_3 e_3 + \delta_4 e_4 = 0$.

The characters of defect 1 all have the same degree d, by paragraph 9 and their exceptional nature. The results of paragraph 7 and the orthogonality relations for R^b and 1 give $2\varepsilon e d + 1 + e_2 d_2 + e_3 d_3 + e_4 d_4 = 0$, where $d_i = \xi_i(1)$. Finally, the orthogonality of degrees and decomposition numbers (in the first column) gives $1 + \delta_2 d_2 + \delta_3 d_3 + \delta_4 d_4 = 0$.

15. Suppose now there were only seven characters in the principal block. As the algebraic conjugate of any character in this block is also in it [7], ξ_2 and ξ_3 are algebraic conjugates (by paragraph 4) and $\delta_2 = \delta_3$, $d_2 = d_3$, $e_2 = e_3$, $\sigma_2 = \sigma_3$. Since $\xi_4(J) = -\delta_4 e q$, d_4 is odd; as ξ_4 must be its own complex conjugate (regardless of e), $\sigma_4 = 1$. By paragraph 13, $1 + 2\delta_2\sigma_2 + \delta_4 = 0$. If $e = -1$, ξ_2 and ξ_3 are complex conjugate, $\sigma_2 = 0$, and $\delta_4 = -1$. If $e = 1$, each ξ_i is real (as the ϕ_j^J are), $\sigma_2 \neq 0$, and $\delta_4 = 1$. Thus $\delta_4 = e$.

From the orthogonality of sections using the second column, $\delta_2 e_2 + \delta_4 e_4 = 0$. The relations of paragraph 14 give $\delta_3 e_3 = -1$, so $\delta_4 e_4 = 1$. As $1 + 2\delta_2 d_2 + \delta_4 d_4 = 0$ and degrees are positive, $\delta_2 = -\delta_4$. Thus $\delta_2 = \delta_3 = -\delta_4 = -e$, $e_2 = e_3 = e_4 = e$, and $d_4 = 2d_2 - e$. Also, from paragraph 14, $d\varepsilon e = -2d_2$, forcing $e\varepsilon = -1$.

16. Formula (2) applied to the principal block and ϕ_1^J, simplified with the information above (and with $\xi_2(J) = \xi_3(J) = -\frac{1}{2}\delta_3 e(q-e)$) becomes

$$g(d+(q-e))^2 = q^2(q^2-1)^2(q+e)d(d-e). \tag{3}$$

Using the result of paragraph 12 we obtain

$$qa(d+(q-e))^2 = (q^2-1)(q+e)\,d(d-e). \tag{4}$$

Now q is a prime power, so $q\,|\,d$ or $q\,|\,(d-e)$. However, if $q\,|\,(d-e)$, the left side of (4) is divisible by q^3, so in fact $q^3\,|\,(d-e)$. Since $(g/c(J))(\xi_2(J)/d_2)$ is an algebraic integer (cf. BI), $q^2a(q-e)/d$ is a rational integer. As q^2 is prime to d, this means $d\,|\,a(q-e)$. Since $d \geqq q^3+e$, we have $a \geqq (q^3+e)(q-e)^{-1}$. But (4) implies $a < q^{-1}(q^2-1)(q+e)$, and the combination of these two inequalities yields the contradiction $qe < -2q^2+1$.

Thus $q\,|\,d$. Then $d-e$ is prime to q and to $d+q-e$, so that $(d-e)\,|\,a$, from (4). Now $\xi_4(R^b) = e$ $(R^b \neq 1)$; thus (cf. paragraph 12) $d_4 \equiv e \pmod{\tfrac{1}{2}(q+e)}$ and $d \equiv 2e \pmod{\tfrac{1}{2}(q+e)}$. As d is even, then $(q+e)\,|\,(d-2e)$. Since q and $q+e$ are relatively prime, we have $d = bq(q+e)-2q$ for some $b \geqq 1$. Then the conditions $d-e \leqq a < q^{-1}(q^2-1)(q+e)$ force $b=1$, $a=d-e$. Substitution in (4) leads to $e=1$. Thus, potentially, $g = q^3(q^2-1)(q^2-q-1)$, and $d = q(q-1)$.

17. To rule out this case, we use (2) applied to ϕ_2^J. The right-hand side in general is $-eq^2(q^2-1)(q+e)(3q+e)$. The orthogonality of sections (with the second column) gives $\delta_2d_2+\delta_4d_4+\delta_5d_5+\delta_7d_7=0$. Let $f=\delta_5d_5$. Then in the present case, $\delta_7d_7 = 1-\tfrac{1}{2}q(q-1)-f$. Substitution into (2) of these values and the values found above (along with $\xi_5(J) = -\tfrac{1}{2}\delta_5e(q+e)$, $\xi_7(J) = -\delta_7e(q+e)$) yields the quadratic equation for f:

$$\begin{aligned} &4(q^2+7q+1)f^2 - 4(q+1)(q^3-4q^2+5q+1)f \\ &\qquad + q(q+1)^2(q-2)(q^2-q-1) = 0. \end{aligned} \tag{5}$$

However, the discriminant of this quadratic is $16(q+1)^2(4q+1)(-3q^2+6q+1)$, which is negative, whereas f is real.

18. Thus the correct number of characters in the principal block is eight. From $\xi_2(J)=\delta_2$, $\xi_3(J) = -e\delta_3q$, $\xi_4(J) = -e\delta_4q$, (2) for the character ϕ_1^J becomes

$$g(1+\delta_2/d_2+\delta_3q^2/d_3+\delta_4q^2/d_4) = q^2(q^2-1)^2\ (q+e). \tag{6}$$

Now in general if A, B, C are in G, then the number of times C is a product $A'B'$ with A' conjugate to A, B' to B, is given by

$$(A,B;C) = (g/c(A)c(B)) \sum \chi(A)\chi(B)\overline{\chi(C)}/\chi(1) \tag{7}$$

summed over the characters of G [9]. (This relation is the source of Brauer's formulas.) In the present case, $(J,J;C)$ is the number of involutions J^* with $J^*CJ^* = C^{-1}$. When $C = R^b \neq 1$ this number is $q+e$. Since $\sum \eta_r(R^b) = \sum \eta_r'(R^b) = \varepsilon e$ (paragraph 7), summed over each family, and since $(\eta_r(J))^2 = (\eta_r'(J))^2 = (q-e)^2$ (paragraph 9), (7) for $(J,J;R^b)$ becomes

(8) $\quad (q+e) = (g/q^2(q^2-1)^2)((q-e)^2 2\varepsilon e/d + 1 + e_2 d_2 + e_3 q^2/d_3 + e_4 q^2/d_4)$

($d = \eta_r(1) = \eta_r'(1)$.) Elimination of g from (7) and (8) yields

(9) $\quad (q-e)^2\ 2\varepsilon e/d + e_2/d_2 + e_3 q^2/d_3 + e_4 q^2/d_4 = \delta_2/d_2 + \delta_3 q^2/d_3 + \delta_4 q^2/d_4.$

Orthogonality of the second column with the e_i's gives $\delta_3 e_3 + \delta_4 e_4 = 0$. As these numbers are ± 1's, we may assume $\delta_3 = e_3$, $\delta_4 = -e_4$. From $1 + \delta_2 e_2 + \delta_3 e_3 + \delta_4 e_4 = 0$ (paragraph 14), $\delta_2 = -e_2$. Then the relations of paragraph 14 give $e\varepsilon d = -1 - \delta_3 d_3$. If now in (9) we substitute these results, we obtain $1 + \delta_3 d_3 = e(q-e)\delta_2 d_2$, and $\delta_4 d_4 = -eq\delta_2 d_2$. Because the degrees are positive, $\delta_3 = e\delta_2$ and $\delta_4 = -e\delta_2$.

19. The order of G may now be determined up to a determination of e. First, if ξ_3 and ξ_4 were complex conjugate, then $\delta_3 = \delta_4$, $e_3 = e_4$. But this would violate $\delta_3 e_3 + \delta_4 e_4 = 0$. Thus each character is real, and as the degrees are odd (from the values on J), $\sigma_3 = \sigma_4 = 1$. In addition, ξ_2 is self-conjugate and of odd degree; thus $\sigma_2 = 1$. Then from paragraph 13, $1 + \delta_2 + \delta_3 + \delta_4 = 0$. From paragraph 18, this is $1 + \delta_2 = 0$. Hence $\delta_2 = -1$, $\delta_3 = -e$, $\delta_4 = e$ (and $\varepsilon = 1$). With these values and the fact that $d_4 = qd_2$, (6) becomes

(10) $$g(d_2-1)^2 = q^2(q+e)^3(q-e)d_2 d_3.$$

Because $e_2 = 1$ $(= -\delta_2)$, restriction of ξ_2 to $\langle R \rangle$ shows that $(q+e)\,|\,(d_2-1)$ (since d_2 is odd). As $d_3 = (q-e)d_2 + e$, d_3 is then prime to both $q+e$ and $q-e$. Writing $g = q^3(q^2-1)a$ (paragraph 12), we have $d_3\,|\,q^3 a$. Thus (10) in the form $(q^3 a/d_3)(d_2-1)^2 = q^2(q+e)^2 d_2$ yields $(d_2-1)\,|\,q(q+e)$. Since $(q+e)\,|\,(d_2-1)$, $(d_2-1) = b(q+e)$ where $b\,|\,q$.

On the other hand, as $q^3\,|\,g$, $q\,|\,d_2$ or $q\,|\,d_3$. As $d_2 \equiv be + 1 \pmod q$ and $d_3 \equiv -b \pmod q$, it must be that either $b = 1$ (and $e = -1$) or $b = q$. But from ξ_2 restricted to a 2-Sylow subgroup we have $d_2 \equiv \delta_2 \equiv -1 \pmod 8$, or $(2+e)b \equiv 3 \pmod 4$. If then $b = 1$, e would have to be 1. Thus $b = q$.

In summary, then: $d_2 = q(q+e)+1$, $d_3 = q^3$, $d_4 = q^2(q+e)+q$, and $g = q^3(q^2-1)(q(q+e)+1)$.

20. To determine e (and obtain other degrees) we apply (2) to ϕ_2^J (the right side is given in paragraph 17). Let $f_5 = \delta_5 d_5$, $f_6 = \delta_6 d_6$. Then from the orthogonality of the second column with the column of degrees, $\delta_3 d_3 + \delta_4 d_4 + f_5 + f_6 = 0$, giving $f_5 + f_6 = -q(q+e)$. Substituting all known so far in (2) yields

(11) $$12ef_5^2 + 12eq(q+e)f_5 + q(q+e)^2(q(q+e)+1) = 0.$$

The discriminant of this quadratic is $16(q^2-1)^2(-3eq)$, and as a rational root is required, we must have $e = -1$ and q an odd power of 3. Set $q = 3m^2$, m a power of 3. Then $g = q^3(q^2-1)(q+1-3m)(q+1+3m)$.

ξ_7 may be taken to be the complex conjugate of ξ_5, and ξ_8 that of ξ_6. f_5 and

f_6 have opposite signs and are the roots of (11); we choose $\delta_5 = -1$, $\delta_6 = 1$ (so $\delta_7 = -1$, $\delta_8 = 1$).

The decompositions and degrees so far are given below:

Principal 2-Block

	ϕ_1^J	ϕ_2^J	ϕ_3^J	Degree
ξ_1	1	0	0	1
ξ_2	-1	0	0	$q^2 - q + 1$
ξ_3	1	1	1	q^3
ξ_4	-1	-1	-1	$q(q^2 - q + 1)$
ξ_5	0	-1	0	$\frac{1}{2}(q-1)m(q+1+3m)$
ξ_6	0	1	0	$\frac{1}{2}(q-1)m(q+1-3m)$
ξ_7	0	0	-1	$\frac{1}{2}(q-1)m(q+1+3m)$
ξ_8	0	0	1	$\frac{1}{2}(q-1)m(q+1-3m)$

2-Block of Defect 1

	ϕ_r^J	Degree
η_r	1	$q^3 + 1$
η_r'	-1	$q^3 + 1$

2-Block of Defect 2 (B_2)

	ϕ_4^J	ϕ_5^J	ϕ_6^J	Degree
ξ_{11}	1	1	1	$(q-1)(q^2-q+1)$
ξ_{12}	-1	-1	1	$(q-1)(q^2-q+1)$
ξ_{13}	-1	1	-1	$(q-1)(q^2-q+1)$
ξ_{14}	1	-1	-1	$(q-1)(q^2-q+1)$

Chapter II. 2-Blocks of Defect 2

In this chapter the characters of defect 2 for the prime 2 are studied. It is shown that all the blocks of defect 2 have a structure like that of the block B_2 of paragraph I–10. Moreover, the ordinary characters of these blocks make up the members of two more families of exceptional characters associated with the elements of $\langle S\rangle$.

1. The goal of the first few paragraphs is the determination of the centralizers of elements $S^a \neq 1$, and this is done by ruling certain primes out of the orders of these centralizers.

Let $A \in G$ be a 2-regular element whose order is divisible by 3. Then we shall show that $c(A) \mid 2q^3$. First, $\eta_r(A) - \eta_r'(A) = 0$ and $1 - \xi_2(A) + \xi_3(A) - \xi_4(A) = 0$, from orthogonality with decomposition numbers. As $d_3 = q^3$, ξ_3 has defect 0 for 3 and $\xi_3(A) = 0$. In addition, $\eta_r(A)$ and $\eta_r'(A)$ do not depend on r. The orthogonality relations for A and R then give $\eta_r(A) = \eta_r'(A) = 1$. Since $(g/c(A))(\eta_r(A)/\eta_r(1))$ is an algebraic integer, $c(A) \mid q^3(q-1)$. From property IV, $c(R^a) = q-1$ for

$R^a \neq 1$. Thus no element of order dividing $\frac{1}{2}(q-1)$ centralizes A, because $\langle R \rangle$ contains a p-Sylow subgroup for any prime p dividing $\frac{1}{2}(q-1)$. Hence $c(A) \mid 2q^3$. It follows that no element of odd order dividing q^3+1 (except 1) is centralized by such an A.

2. At this point the structure of the 2-blocks of defect 2 may be established. Let $T_0 \in N(\langle J, J' \rangle) \cap C(J'')$ of order 3 give $T_0^{-1} J T_0 = J'$ (cf. paragraph I–10). T_0 normalizes $\langle S \rangle$, but, by assumption V, does not centralize S. In fact, the result above implies T_0 centralizes no member of $\langle S \rangle$ but 1. Because T_0 conjugates no member of $\langle S \rangle$ (but 1) to its inverse, the discussion in paragraph I–10 implies that each character of degree 2 of $\langle J'', S \rangle$ belongs to a family of three associate characters, and each family leads to a 2-block of defect 2 of G. Thus all these blocks have the decomposition given for B_2 at the end of Chapter I. Each has four ordinary characters of degree $(q-1)(q^2-q+1)$.

3. Consider now elements whose orders contain primes dividing q^2-q+1. (Such elements will be discussed in detail in Chapter IV.) The characters of 2-defect 2 of the previous paragraph and those of defect 1 (paragraph I–9) all have orders divisible by q^2-q+1. Thus they are of defect 0 for any prime dividing q^2-q+1 and vanish on the elements under consideration. The same holds for ξ_2 and ξ_4. The degrees of ξ_5 and ξ_7 being divisible by $q+1+3m$, these characters have defect 0 for primes dividing $q+1+3m$. Similarly, ξ_6 and ξ_8 have defect 0 for primes dividing $q+1-3m$.

Let V be an element whose order contains a prime dividing $q+1-3m$, and W one whose order contains a prime dividing $q+1+3m$. Both elements are 2-regular, and from the orthogonalities to the decomposition numbers in the principal block along with the results on vanishings, we have $\xi_3(V) = \xi_3(W) = -1$, $\xi_5(V) = \xi_7(V) = -1$, and $\xi_6(W) = \xi_8(W) = 1$. In the computation of $(J, V; W)$ and $(J, W; V)$ (cf. paragraph I–18) only characters of positive 2-defect need be considered, as the others vanish on J. Thus one obtains

(1) $(J, V; W) = (q^2+1)(q^2-q+1)/c(V)$ and $(J, W; V) = (q^2+1)(q^2-q+1)/c(W)$.

Now $c(V)$ is odd and $\frac{1}{2}(q^2+1)$ is prime to g; thus as the left sides are integers, $c(V) \mid (q^2-q+1)$. Similarly, $c(W) \mid (q^2-q+1)$.

4. The centralizer of S^a $(\neq 1)$ may now be determined. If $A \in C(S^a)$ has prime order p, then $p \neq 3$ by paragraph 1 and $p \nmid (q^2-q+1)$ by paragraph 3. As in paragraph 1, $p \nmid \frac{1}{2}(q-1)$. Thus the prime factors of $c(S^a)$ are among those of $q+1$. Thus $C(S^a) = \langle J, J', S \rangle$ of order $q+1$.

5. The theory of exceptional characters will now be applied. $\langle S \rangle$ is normal in $N(\langle J, J' \rangle) = \langle J, J', J'', S, T_0 \rangle$. If $A^{-1} S^a A \in N(\langle J, J' \rangle)$ for $A \in G, S^a \neq 1$, then $A^{-1} S^a A = S^b$. Since $C(S^a) = C(S^b)$, A normalizes $\langle J, J', S \rangle$ and thence $\langle J, J' \rangle$, i.e., $A \in N(\langle J, J' \rangle)$. As S^a is a power of JS^a (S^a is of odd order), the same holds if $A^{-1} J S^a A \in N(\langle J, J' \rangle)$. The members of $\langle JS \rangle$ other than 1 and J then represent special classes of $N(\langle J, J' \rangle)$.

The group $N(\langle J,J'\rangle)$ is the semi-direct product of the normal subgroup $\langle J,J'\rangle$ and the subgroup $\langle J'',T_0,S\rangle$. The characters may be found following Mackey [15]. Table 2 gives the part of the character table needed. Here ψ_t is the character of degree 2 of $\langle J'',S\rangle$ whose value on S^a is t^a+t^{-a}, where $t\,(\neq 1)$ is a $(q+1)/4$th root of unity. If $T_0^{-1}ST_0=S^n$, set $t'=t^n$ and $t''=t^{n^2}$. $\zeta_u=\zeta_t$ whenever $u=t^{\pm n^j}$, and $\zeta'_t=\zeta'_{t^{-1}}$. There are $(q-3)/24$ of the ζ_t and $(q-3)/8$ of the ζ'_t.

TABLE 2

Value on	1	J	S^a	JS^a
Character				
ζ_t	6	6	$\psi_t(S^a)+\psi_{t'}(S^a)+\psi_{t''}(S^a)$	$\zeta_t(S^a)$
ζ'_t	6	-2	$\zeta_t(S^a)$	$\psi_{t''}(S^a)-\psi_t(S^a)-\psi_{t'}(S^a)$
six of degree one	1	1	1	1
two of degree three	3	-1	3	-1

6. The families $\{\zeta_t\}$ and $\{\zeta'_t\}$ serve as the source of exceptional characters (paragraph I–6). As the case $q=27$ is different, let $q>27$ for the moment. The two families are maximal for the special classes of paragraph 5. Let $\{\eta_t\}$ and $\{\eta'_t\}$ be the resulting families of exceptional characters. Here $\eta_t \,|\, N(\langle J,J'\rangle) = \varepsilon_1\zeta_t+\eta_1$ and $\eta'_t \,|\, N(\langle J,J'\rangle)=\varepsilon'_1\zeta'_t+\eta'_1$, with $\varepsilon_1=\pm 1$, $\varepsilon'_1=\pm 1$. The ζ_t appear in η_1 and in η'_1 with equal multiplicity, the ζ'_t in η_1 and in η'_1 with equal multiplicity. The orthogonality relations on $N(\langle J,J'\rangle)$ yield $\sum\zeta_t(S^a)=-1$, $\sum\zeta'_t(S^a)=-3$, $\sum\zeta_t(JS^a)=-1$, and $\sum\zeta'_t(JS^a)=1$, for $S^a\neq 1$; the sums are over the indicated families. As in paragraph I–7, one finds that the following are all rational integers independent of $a(S^a\neq 1)$: $\eta_1(S^a)$, $\eta_1(JS^a)$, $\eta'_1(S^a)$, $\eta'_1(JS^a)$, and $\chi(S^a)$ and $\chi(JS^a)$ for nonexceptional characters χ.

7. These exceptional characters are exactly the characters of 2-defect 2 as will now be shown. The characters ψ_t introduced above are 0 on J'', and $\langle J'',S\rangle$ has besides the 1-character a linear character which is -1 on J'' and 1 on $\langle S\rangle$. The orthogonality relations in $\langle J'',S\rangle$ give $\sum\psi_t(S^a)=-1$, $S^a\neq 1$. These equations used in the matrix of character values of $\langle J'',S\rangle$ show that the ψ_t are a basis for functions on the set K of $(q-3)/8$ classes of $\langle J'',S\rangle$ represented by the $S^a\neq 1$. Moreover, if ϕ_s^J is the modular character of $C(J)$ corresponding to θ_s (paragraph I–3), $-\phi_s^J(S^a)=\psi_t(S^a)$ where $t=s^2$. If χ is a character of G, consider the function on K given by $\chi(JS^a)$, expressed as a linear combination of the ψ_t. From paragraph 6 any nonexceptional character gives equal coefficients, while the exceptional ones do not because $(q-3)/8>3$. But the characters of 2-defect 2 do not give equal coefficients, either. These characters then coincide with the exceptional ones. Since (when $q>27$) $(q-3)/8>6$, a comparison of

coefficients shows that the ordinary characters of a 2-block of defect 2 are η_t, η'_t, $\eta'_{t'}$, and $\eta'_{t''}$, and that the modular characters of $C(J)$ involved are ϕ^J_s, $\phi^J_{s'}$, and $\phi^J_{s''}$ where $s^2 = t$, $(s')^2 = t'$, $(s'')^2 = t''$. It follows also that $\varepsilon_1 = \varepsilon'_1 = -1$. Finally, since the ϕ^J_s are real, the decomposition for the section of J of a conjugate character is the same as that of the original. But as the decompositions for the characters of 2-defect 2 specify the character, these characters are all real.

8. The values of the characters of G other than these exceptional characters on $S^a \neq 1$ may now be determinant by a technique like that of paragraph I–7.

Restriction of a nonexceptional character χ to $\langle S\rangle$ gives $\chi(1) \equiv \chi(S^a)$ $(\mathrm{mod}\,(q+1)/4)$, $S^a \neq 1$ (recall $\chi(S^a)$ is independent of a). Thus modulo $(q+1)/4$, $\xi_2(S^a) \equiv -\xi_4(S^a) \equiv 3$, $\xi_3(S^a) \equiv \xi_6(S^a) \equiv \xi_8(S^a) \equiv -\xi_5(S^a) \equiv -\xi_7(S^a) \equiv -1$. Now $c(S) = q + 1 = \sum|\chi(S)|^2$, summed over the characters of G. Let $\eta'_1(S^a) = x$, $\eta'_1(S^a) = y$. The contribution to the sum from the principal 2-block is at least 24, which is what comes if all the congruences are equalities. If we delete all nonexceptional characters but those of the principal 2-block, then

$$q + 1 \geqq 24 + \sum|\eta_t(S)|^2 + \sum|\eta'_t(S)|^2. \tag{2}$$

This simplifies (by use of character relations in $N(\langle J, J'\rangle)$) to

$$((q-3)x^2/24 + 2x) + ((q-3)y^2/8 + 6y) \leqq 0. \tag{3}$$

Since still $q > 27$, so that in fact $q \geqq 243$, this forces $x = y = 0$. Then not only are all the congruences equalities (because (2) is now an equation) but all nonexceptional characters other than those of the principal 2-block vanish on S and therefore on $S^a \neq 1$. Further, $\eta_t(S^a) = -\zeta_t(S^a)$, $\eta'_t(S^a) = -\zeta'_t(S^a)$.

9. Suppose now that $q = 27$. Then only the ζ'_t furnish exceptional characters. All powers of S are conjugate (except 1) and $\zeta_t(S) = -1$. The argument of paragraph 7 still implies that the exceptional characters are the last three characters of the single block of 2-defect 2. The first character of that block, now being nonexceptional, has a rational integral value on S. The rest then have the same value, from the orthogonality to the decomposition numbers for J. The same sort of argument as in paragraph 8, but without the quadratic discussion, yields again that those congruences are equalities and that the common value on S for the characters of 2-defect 2 is ± 1. The congruence obtained from the restriction to $\langle S\rangle$ shows it to be 1. (The first character will still be labelled η_t.)

CHAPTER III. THE 3-SYLOW SUBGROUP

In this chapter the results of the main theorem related to the 3-Sylow subgroup are obtained, along with values of characters on the various 3-elements. One of the main tools is the fact that $\langle JR\rangle$ can be included in the normalizer of 3-groups used in a step-by-step build-up of the 3-Sylow subgroup; the rough effect is that q itself behaves like a prime (this kind of behavior is common in families of simple groups).

1. First, some character values: let $X \neq 1$ be any member of G of order a (positive) power of 3. Since X is 2-regular, all characters of a 2-block of defect 2 have the same value on X; because these characters are exceptional, *all* characters of 2-defect 2 have the same value on X. Let $b(X)$ be this value. From the principal 2-block, $\xi_2(X) = 1 - \xi_4(X)$, $\xi_5(X) - \xi_6(X) = \xi_7(X) - \xi_8(X) = -\xi_4(X)$ (recall $\xi_3(X) = 0$). These relations and the orthogonality of X and S give $b(X) = 2\xi_4(X) - 1$.

Let F be the group of all $T(x)$, $x \in F_q$ (paragraph I–1) and let $T = T(1)$. Restriction of the real character ξ_4 to F gives $\xi_4(T) = aq$, where a is a rational integer. As $\xi_4(JT) = 0$, a is even. From above, $b(T) = 2aq - 1$. The contribution to $c(T) = \sum |\chi(T)|^2$ from characters of 2-defect 2 is then $(q-3)(2aq-1)^2/6$. As $c(T) \leqq 2q^3$ (paragraph II–1) and a is even, a must be 0. Then $\xi_2(T) = 1$, $\xi_4(T) = 0$, and $b(T) = -1$. Then for $(T, T; R)$ one obtains the integer $(q-1)(q^2+1)q^4/c(T)^2$, implying $c(T) \mid 2q^2$. In particular, T cannot be a central element of a 3-Sylow subgroup of G.

2. We now start "building" a 3-Sylow subgroup. Since $C(T) \supseteq C(F)$, $C(F)$ (which contains J) has order twice a power of 3, and its 3-Sylow subgroup P_1 is normal. $\langle R \rangle$ normalizes F (see [11]) and therefore $C(F)$. Consequently $\langle R \rangle$ normalizes P_1. In general, if $\langle R \rangle$ normalizes the 3-group H, no nonidentity element of $\langle R \rangle$ centralizes a nonidentity element of H. If $|H| = h$, then $(q-1)/2$ divides $h - 1$, or, as h is a power of 3 and odd, $h \equiv 1 \pmod{q-1}$. Since $h \mid q^3$, h can only be 1, q, q^2, or q^3.

If X is in the center of some 3-Sylow subgroup of G containing F, then X is also in P_1. Moreover, if $X \neq 1$, then $X \notin F$, for the nonidentity elements of F are conjugate to T or T^{-1} (-1 is not a square in F_q). Thus there are central elements of order 3 of P_1 not in F. Thus $P_1 \neq F$ and so $|P_1| = q^2$. As $\langle R \rangle$ normalizes the center of P_1, P_1 must be Abelian. Since F is elementary Abelian, so is P_1. Finally, $c(T)$ must now be $2q^2$.

3. Certain character values may now be made more precise. Let $X \neq 1$ be an element in the center of a 3-Sylow subgroup of G. Because the central elements of any 3-Sylow subgroup of G containing F are in P_1, it follows that X is of order 3 and $c(X) = q^3$. (X cannot centralize an involution.) Then one finds

$$(X, T; R) = (q-1)(q^3 + q - (q+1)\xi_4(X))/2q^2. \tag{1}$$

As this is an integer, $\xi_4(X)$ is a rational integer and $\xi_4(X) = q + aq^2$ for some rational integer a. Since $|\xi_4(X)|^2 < q^3$, $a = 0$. Thus $\xi_4(X) = q$, $\xi_2(X) = 1 - q$, and $b(X) = 2q - 1$ (paragraph 1).

More generally, let Y be any element of order 3 for which $q^2 \mid c(Y)$ (e.g., $Y \in P_1$). Then one finds

$$(Y, X; R) = (q-1)(q^3 - q^2 + q(q+1)\xi_4(Y))/c(Y). \tag{2}$$

If the power of 3 in $c(Y)$ is kq^2, $k \mid q$, it must be that $\xi_4(Y) = q(1 + ak)$, a a rational integer. As $k \geqq 1$ and $(Y,X;R) \geqq 0$, $a \geqq -1$. But if $a = -1$, k must be 1 and $\xi_4(Y) = 0$. On the other hand, since $|\xi_4(Y)|^2 < c(Y) \leqq 2kq^2$, $a \leqq 0$. If $a = 0$, then Y is not conjugate to T or T^{-1}. The contribution to $c(Y) = kq^2$ from the characters of 2-defect 2 then being $(q-3)(2q-1)^2/6$, which is more than $q^3/3$, k must be q and Y must be a central element.

In short, either $\xi_4(Y) = 0$ and $c(Y) \leqq 2q^2$ or $\xi_4(Y) = q$ and $c(Y) = q^3$.

4. With this information we can establish the center of a 3-Sylow subgroup. In P_1, consider the set of elements $X \neq 1$ such that X is in the center of some 3-Sylow subgroup of G containing F. Although these elements and 1 do not form a subgroup a priori, nevertheless they are permuted under conjugation by the group $\langle JR \rangle$ and no such element is fixed by a member of $\langle JR \rangle - \langle 1 \rangle$. There are then $a(q-1)$ such elements, $a > 0$ an integer. The multiplicity of the identity in $\xi_2 | P_1$ is then, from paragraph 3, $[(q^2 - q + 1) + a(q-1)(1-q) + ((q^2-1) - a(q-1))]/q^2 = 2 - a + (a-1)/q$. As this must be a non-negative integer, a can only be 1.

If X is one of these $q-1$ central elements, so is X^{-1}, and for some $A \in \langle JR \rangle$, $A^{-1}XA = X^{-1}$. Since A^2 then centralizes X, $A = J$. In general, if $\langle J \rangle$ normalizes the odd subgroup H and H' is the set of elements conjugated to their inverses by J, then $|C(J) \cap H| \, |H'| = |H|$. With $H = P_1$ there are then $q-1$ nonidentity elements of P_1 inverted by J. Thus the $q-1$ central elements are exactly these elements. Since they all commute, they and 1 form a group C of order q, which will turn out to be the center of a 3-Sylow subgroup.

The set $C - \langle 1 \rangle$ can be characterized as the set of $X \in P_1$, $X \neq 1$, such that X is in the center of some 3-Sylow subgroup of G containing P_1, for that set is contained in $C - \langle 1 \rangle$ and is also permuted by conjugation by $\langle JR \rangle$. Consequently, anything normalizing P_1 normalizes C. P_1 being properly normal in some 3-group of G, P_1 is not a 3-Sylow subgroup of $N(P_1)$. An odd prime factor p of q^3+1 cannot divide $N(P_1)$, for an element of order p would normalize C and thence, as $p \nmid (q-1)$, centralize a member of $C - \langle 1 \rangle$. Quite similarly, there is no 4-group in $N(P_1)$. Thus $|N(P_1)| = kq^2(q-1)$ where $k | q$, $k > 1$. Now for any prime p dividing $q-1$, the p-Sylow subgroup of $N(P_1)$ is in the center of its normalizer in $N(P_1)$: for $p=2$, this Sylow subgroup is $\langle J \rangle$, and for other p it is $\langle R^\tau \rangle$ for some a, with normalizer $\langle JR \rangle$ (no involution conjugates R^a to R^{-a} without centralizing J). From Burnside's Theorem [13, p. 203], each such Sylow subgroup has a normal complement. The intersection of these complements is a 3-Sylow subgroup P of $N(P_1)$. Since $\langle R \rangle$ then normalizes P, $k = q$ and P is a 3-Sylow subgroup of G.

Any central element of P lies in C, as $P_1 \subset P$. But $\langle R \rangle$ normalizes the center of P; consequently C must be that center. $P_1 = CF$ (for $C \cap F = \langle 1 \rangle$). Just as for P_1, since anything normalizing P normalizes C, $|N(P)| = q^3(q-1)$ and $N(P) = N(P_1)$. Then P_1 is normal in $N(P)$.

5. Consider the permutation representation of G given by the action of right

multiplication of cosets of $N(P)$. The degree of the character θ of this representation is q^3+1 and it contains ξ_1 with multiplicity 1 (cf.[13]). None of the η_r or η_r' can be in θ because their degrees are too large. If J_1 is an involution with $J_1RJ_1=R^{-1}$, then R and JR fix $N(P)J_1 \neq N(P)$, in addition to $N(P)$. If then ξ_i has multiplicity m_i in θ, $2 \leqq 1+m_2+m_3+m_4$ and $2 \leqq 1-m_2+m_3-m_4$. Thus $1 \leqq m_3$; as ξ_3 has degree q^3, $\theta=\xi_1+\xi_3$. Thus [13] this action is doubly transitive. In particular, if $A \in P-\langle 1\rangle$, $\theta(A)=1$ and P is the only 3-Sylow subgroup containing A. There are thus q^6-1 elements with order a positive power of 3. Finally, if $A \in P-\langle 1\rangle$ and $B^{-1}AB \in P$, then $B \in N(P)$.

6. The group $\langle R\rangle$ normalizes P and no member of $\langle R\rangle - 1$ centralizes any member of $P-\langle 1\rangle$. If H is any normal subgroup of P normalized by $\langle R\rangle$, the same conclusion holds for the action of $\langle R\rangle$ on P/H. Moreover, if H is a subgroup of P of index q that is normalized by $\langle R\rangle$, the normalizer of H in P is also normalized by $\langle R\rangle$ and must be P itself. Then P/H must be elementary Abelian. Thus P/CF is elementary Abelian. Furthermore, CF/C is in the center of P/C. For if not, CF/C meets that center only in 1; but then P/C would be Abelian because CF/C is, and CF/C would be in the center after all.

7. We may obtain further character values for central elements of P. Because $c(T)=2q^2$, T has $q(q-1)/2$ conjugates in $N(P)$, all of which lie in the normal subgroup P_1 (cf. the last remark of paragraph 5). T and T^{-1} are not conjugate, for if so, $\xi_5(T)=\xi_5(T^{-1})$ would imply that $\xi_5(JT)-\xi_5(JT^{-1})$ belongs to a prime ideal divisor of 2 in the field of gth roots of unity, which is not so. T^{-1} also has $q(q-1)/2$ conjugates in P_1, and P_1-C thus consists entirely of conjugates of T and T^{-1}.

If then χ is a character of G, because the members of $C-\langle 1\rangle$ are all conjugate (paragraph 4), $\chi \mid P_1$ gives

$$\chi(1)+(q-1)\chi(X)+q(q-1)(\chi(T)+\chi(T^{-1}))/2 \equiv 0 \pmod{q^2}$$

where X stands for a member of $C-\langle 1\rangle$. Since from $\chi \mid F$ one has

$$(q-1)(\chi(T)+\chi(T^{-1}))/2 \equiv -1 \pmod{q},$$

one obtains

$$\chi(X) \equiv \chi(1) \pmod{q^2}. \tag{3}$$

Because $|\chi(X)|^2 \leqq c(X)=q^3$, this congruence can usually be replaced by an equality. Thus $\xi_5(X)=\xi_7(X)=-(q+m)/2$ and $\xi_6(X)=\xi_8(X)=(q-m)/2$.

8. The rest of this chapter concerns the elements of P outside $P_1=CF$. If Y is such an element, $Y^3 \in P_1$ (paragraph 6). However, if $Y^3=AB$, $A \in C$, with $B \in F$ and $B \neq 1$, then Y centralizes B, an impossibility. Thus $Y^3 \in C$. Moreover, if $B \in F$, then $B^{-1}YB \in YC$ (paragraph 6). Two different B's give two different members of YC, for otherwise again Y centralizes a member of $F-\langle 1\rangle$. Thus

YC consists of conjugates of Y. $\langle Y\rangle$ is of order 3 or 9; in either case $\xi_2 | \langle Y\rangle$ shows $\xi_2(Y)$ to be a rational integer congruent to 1 (mod 3) (ξ_2 is real).

That this congruence is an equality comes as follows: there is enough information on orders of centralizers to compute the contribution to $\sum|\xi_2(A)|^2$, summed over $A \in G$, from elements not conjugate to members of $P - P_1$. (The section of J has $3g/8$ members and ξ_2 is 0 on any element whose order is not prime to q^2-q+1.) The contribution to the sum from elements conjugate to members of $P - P_1$ is then g less the first contribution; explicitly, it is $q^2(q-1)(q^3+1)$. But this is exactly the number of such elements, and the congruence implies $\xi_2(Y) = 1$ for all such Y.

9. Consider now those $Y \in P - P_1$ for which $JYJ = Y^{-1}$. Since $|C(J) \cap P| = q$, there are $q^2 - 1$ members A of $P - \langle 1\rangle$ with $JAJ = A^{-1}$ (cf. paragraph 4). Because only $q-1$ of them are in P_1 (viz., the members of $C - \langle 1\rangle$), there are $q^2 - q$ such Y. From $\xi_2(Y) = 1$ one obtains $\xi_4(Y) = 0$, and ξ_5, ξ_6, ξ_7, and ξ_8 all have the same value on Y. In the computation of $(Y,X;J)$ only the characters of the principal 2-block enter, for in 2-blocks of defect 1 or 2 all characters have the same value on Y and on X and the sum of the values on J is 0. Using the values known, one has

$$(Y,X;J) = q(q^2-1)(q+6m\xi_5(Y))/c(Y). \tag{4}$$

Then first of all $\xi_5(Y)$ is a rational integer. The centralizer of J permutes (by conjugation) the pairs (Y_1,X_1) for which $Y_1X_1 = J$, Y_1 conjugate Y, X_1 to X; only the identity can fix a pair. Consequently, $c(Y) | (q+6m\xi_5(Y))$. If $c(Y) = kq$, k a power of 3, then $k \geqq 3$, as $\langle Y\rangle C \subseteq C(Y)$. Then $\xi_5(Y) = am$, a an integer, and $1 + 2a = kb$, b an integer. From $kq = c(Y) > 4|\xi_5(Y)|^2$ and the fact that b must be odd, one finds $k = 3$ and $\xi_5(Y) = m$. Thus $C(Y) = \langle Y\rangle C$.

10. The conjugates by members of P of an element Y of paragraph 9 may be determined as follows: as P/P_1 is Abelian (paragraph 6) the conjugates of Y lie in $YP_1 = YCF$. If $Q_i^{-1}YQ_i = YX_iT_i$, $i = 1,2$, are two such conjugates, where $Q_i \in P$, $X_i \in C$, and $T_i \in F$, then if $Q_2^{-1}T_1Q_2 = X_3T_1$, $X_3 \in C$, we have $(Q_1Q_2)^{-1}Y(Q_1Q_2) = Y(X_1T_1)(X_2T_2)X_3$. But for some $T_3 \in F$, $T_3^{-1}YT_3 = YX_3^{-1}$ (paragraph 8), so that $Y(X_1T_1)(X_2T_2)$ is a conjugate of Y. Thus the elements A for which YA is a conjugate of Y by members of P form a subgroup of CF containing C and therefore of the form $CF(Y)$, $F(Y)$ a subgroup of F of index 3 (since Y has $q^2/3$ conjugates by members of P). Replacing Y by a conjugate by an element of $\langle R\rangle$, if necessary, we may assume $T \notin F(Y)$.

11. The other elements of $P - P_1$ may now be described. Let $Q \in P$ and $Q^{-1}YQ = YA$, $A \in CF(Y)$. If $Q^{-1}TQ = TX_1$, $X_1 \in C$, and if $T_1 \in F$ gives $T_1^{-1}YT_1 = YX_1^{-1}$, then $(QT_1)^{-1}(YT)(QT_1) = YTA$. Thus $(YT)CF(Y)$ consists of conjugates of YT (by members of P). Since $c(YT) \geqq 3q$, this set is all the conjugates of YT by members of P and $c(YT) = 3q$. Similarly $c(YT^{-1}) = 3q$ and $(YT^{-1})CF(Y)$ are all conjugates of YT^{-1} by members of P. Note that

$$YT^{-1} = (JT^{-1})^{-1}(YT)^{-1}(JT^{-1}).$$

The conjugates of Y by $\langle JR\rangle$ are coset representatives of the cosets of P_1 in P other than P_1 (cf. paragraph 6). If $Y_1 \in P - P_1$ some conjugate of Y_1 by a member of $\langle JR\rangle$ is in YP_1. As $YP_1 = YCF(Y) \cup (YT)CF(Y) \cup (YT^{-1})CF(Y)$, Y_1 is conjugate to Y, YT, or YT^{-1}. The same sort of discussion shows Y, YT, and YT^{-1} to be mutually nonconjugate, as an element effecting the conjugation would be in P.

Y has order 3 or 9, from paragraph 8. However, the order must be 9: for if $Q \in P_1$, $Q^{-1}YQ = YX_1$ with $X_1 \in C$, so that $QYQ^{-1} = YX_1^{-1}$. Then $(YQ)^3 = Y^3$. Thus if $Y^3 = 1$, P would consist of elements of order 3 or 1. But in such a group any element commutes with all its conjugates; yet Y has more than $c(Y) = 3q$ conjugates (by members of P). Thus Y has order 9; the same is true for YT and YT^{-1}.

Note that the members of $CF(Y)$ are commutator elements $Y^{-1}Q^{-1}YQ$, $Q \in P$. Thus the derived group, P', contains C properly. As P' is characteristic, $|P'| = q^2$, and, in fact, $P' = CF$. Since P/CF is actually elementary Abelian, CF is also the Frattini subgroup of P.

12. In this paragraph we show that the roles of YT and YT^{-1} may be exchanged. Explicitly, there is an element R^a such that if $Y_1 = R^{-a}YR^a$, then $T_e \notin (Y_1)$, just as for Y, and Y_1T is conjugate in G to YT^{-1}.

For a given R^a, $F(Y_1) = R^{-a}F(Y)R^a$ $(Y_1 = R^{-a}YR^a)$. Y_1T is conjugate to YR^aTR^{-a} and this is to be conjugate to YT^{-1}. Thus TR^aTR^{-a} must be in $F(Y)$. If this is so, $T \notin F(Y_1)$, for otherwise $R^aTR^{-a} \in F(Y)$, forcing $T \in F(Y)$.

The conjugates of T by the members of $\langle R\rangle$ are the $T(x)$ for which $x \in F_q$ is a (nonzero) square. If $R^aTR^{-a} = T(x)$, $TR^aTR^{-a} = T(1+x)$. Let

$$H = \{x \mid T(x) \in F(Y)\},$$

a subgroup of index 3 of the additive group of F_q. Needed is a nonzero square x with $1 + x \in H$. Suppose no such x exists.

$0, 1$, and -1 represent the cosets of H in F_q. Then all squares of F_q are in H and $1 + H$. As -1 is not a square, there are $(q-3)/4$ squares $b^2 \neq 0$ with $1 + b^2$ a square [11, p. 48]. If $b^2 \in 1 + H$, $1 + b^2$ is a square in $-1 + H$. So $b^2 \in H$. Since there are $(q-1)/2$ nonzero squares in F_q, this means the squares (including 0) are evenly distributed between H and $1 + H$ (and each square in $1 + H$ is 1 plus one in H). Now $-1 = a^2 + b^2$ has $q + 1$ pairs of solutions in F_q [11, p. 46]. As $(q+1)/4 > 1$, there is a pair with $a^2 \neq 1$, $b^2 \neq 1$. If a^2 or b^2 is in H, the other is in $-1 + H$. So both are in $1 + H$. Then there are c^2, d^2 in H with $a^2 = 1 + c^2$, $b^2 = 1 + d^2$. Then $c^2 + d^2 = 0$. As -1 is not a square, $c = d = 0$, violating $a^2 \neq 1$, $b^2 \neq 1$. Thus x exists and with it Y_1.

Chapter IV. Characters of 2-defect 0

In this chapter the rest of the characters of G are obtained. All but two of them are exceptional characters associated with elements in G whose orders contain primes dividing $q^2 - q + 1$.

1. First of all, consider elements of G whose order is not prime to $q^2 - q + 1$. From paragraph II–4 the order of the centralizer of such an element divides $q^2 - q + 1$. Let V be any element whose order is not prime to $q + 1 - 3m$. Using the values of the characters of the principal 2-block on V determined in paragraph II–4, one finds $(J, J; V) = q + 1 - 3m$. The $q + 1 - 3m$ involutions J_1 with $J_1 V J_1 = V^{-1}$ all lie in the same coset of $C(V)$, so that $c(V) \geqq q + 1 - 3m$. But $(g/c(V))(\xi_5(V)/\xi_5(1))$ is an algebraic integer; i.e., $c(V) \mid -6mq^2(q + 1)(q+1-3m)$. Consequently $c(V) = q + 1 - 3m$. $C(V)$ thus has a whole coset of involutions and is therefore Abelian. Quite similarly, if W is an element whose order is not prime to $q + 1 + 3m$, $C(W)$ is Abelian of order $q + 1 + 3m$ and has a coset of involutions.

Thus we obtain two Abelian subgroups M^- and M^+ of orders $q + 1 - 3m$ and $q + 1 + 3m$, respectively. Each is the centralizer of any nonidentity member. As each is in addition the centralizer of a Sylow subgroup of G (namely, any of its own Sylow subgroups), any Abelian subgroup of G of order $q + 1 - 3m$ is conjugate to M^-, any of order $q + 1 + 3m$ to M^+. Any conjugate of M^- is either M^- or meets M^- only in $\langle 1 \rangle$, for otherwise a common element has too large a centralizer. Similarly for M^+.

2. We next determine the orders of the normalizers of M^- and M^+. As in paragraph III–8, the number of members of G with orders prime to $q^2 - q + 1$ can be counted: it is $q^3(q^2 - 1)(2q^2 - q + 3)/3$. From paragraph 1, the normalizers $N(M^-)$ and $N(M^+)$ are normalizers of Sylow subgroups. Thus $[G : N(M^-)] \equiv 1$ (mod p) for any prime p dividing $q + 1 - 3m$. If $|N(M^-)| = (q + 1 - 3m)a$, then as $p \mid q^2 - q + 1$ and a is prime to $q + 1 - 3m$, the congruence becomes $a \equiv 6$ (mod p). Since $p \mid (q^3 + 1)$ and q^3 is an odd power of 3, -3 is a square mod p. Then $p \geqq 7$ and $a \geqq 6$. Similarly, if $|N(M^+)| = (q + 1 + 3m)b$, $b \geqq 6$. Thus the number of elements in all conjugates of M^- and M^+ (other than 1) is the left side of

$$(1)\quad (q - 3m)g/a(q + 1 - 3m) + (q + 3m)g/b(q + 1 + 3m) = q^4(q^2 - 1)(q-2)/3.$$

With the inequalities on a and b, this forces $a = b = 6$.

3. An element of order 3 in $N(M^-)$ permutes (by conjugation) the $q + 1 - 3m$ involutions of paragraph 1 and so centralizes one of them. Taking suitable conjugates we assume $\langle J, T \rangle \subset N(M^-)$. Similarly we assume $\langle J, T \rangle \subset N(M^+)$. The results of paragraph 1 then say that $N(M^-)$ and $N(M^+)$ are Frobenius groups with Frobenius kernels M^- and M^+, respectively.

4. Preparatory to applying the exceptional theory we discuss the characters of $N(M^-)$ and $N(M^+)$. These groups are semi-direct products of $\langle J, T \rangle$ with M^∞ and M^+, respectively, and this gives their characters (following [15]). Each group has six one-dimensional characters. If λ is a (linear) character of M^-, $\lambda \neq 1$,

then the associate character λ^B, $B \in \langle J, T\rangle - \langle 1\rangle$ is not λ. For if $\lambda^B = \lambda$, B normalizes the kernel K of λ and induces an action on M^-/K with no fixed element but 1. But as the map on M^-/K by λ is $1-1$, this cannot be. A similar situation holds for M^+.

Thus for $N(M^-)$ there are $(q-3m)/6$ 6-dimensional irreducible characters θ_i^- which are 0 outside M^-, and for $N(M^+)$, $(q+3m)/6$ 6-dimensional characters θ_i^+, 0 outside M^+. The members of $M^- - \langle 1\rangle$ form special classes of $N(M^-)$, from paragraph 1 and the fact that M^- is actually a characteristic subgroup of $N(M^-)$. The characters θ_i^- form a maximal family for these classes. Similarly the members of $M^+ - \langle 1\rangle$ form special classes of $N(M^+)$ and the θ_i^+ are a maximal family.

5. From the exceptional theory there are then two families (although they must be shown to be distinct, below) of irreducible characters of G, $\{\eta_i^-\}$ and $\{\eta_i^+\}$, for which $\eta_i^- \mid N(M^-) = \varepsilon^-\theta_i^- + \eta^-$ and $\eta_i^+ \mid N(M^+) = \varepsilon^+\theta_i^+ + \eta^+$, $\varepsilon^- = \pm 1$, $\varepsilon^+ = \pm 1$, η^- containing the θ_i^- with equal multiplicity, η^+ the θ_i^+ with equal multiplicity. The orthogonality relations in $N(M^-)$ give $\sum\theta_i^-(A) = -1$ (summed over i), $A \in M^- - \langle 1\rangle$. Similarly $\sum\theta_i^+(A) = -1$, $A \in M^+ - \langle 1\rangle$. Thus if χ is not one of the η_i^-, as $\chi \mid N(M^-)$ contains the θ_i^- with equal multiplicity and the 1-dimensional characters of $N(M^-)$ are 1 on M^-, $\chi(A)$ is a rational integer independent of A. Similarly, if χ is not an η_i^+, $\chi(A)$ is a rational integer independent of $A \in M^+ - \langle 1\rangle$.

6. It must be shown that these families are distinct and that the characters in them are new (that is, of 2-defect 0).

In general, let $\chi_1, \cdots, \chi_n$ be a family of exceptional characters of G (not containing the principal character) and K the set of members of G conjugate to members of the special classes for this family (paragraph I-6). Suppose the character χ of G (irreducible or not) is constant on K. Then if the multiplicity of χ_i in χ is x_i, x_i does not depend on i. For, $gx_i = \sum_{A\in K}\chi(A)\overline{\chi_i(A)} + \sum_{A\in G-K}\chi(A)\overline{\chi_i(A)}$. When $A \in G-K$, $\chi_i(A)$ does not depend on i, while in the first sum $\chi(A)$ is constant and $\sum_{A\in K}\chi_i(A) = -\sum_{A\in G-K}\chi_i(A)$ (from the orthogonality to the principal character) does not depend on i either. In particular, if χ is irreducible, χ is none of the $\chi_i (n>1)$.

Thus, if an irreducible character of G has the same value on all conjugates to members of $M^- - \langle 1\rangle$, that character is not one of the η_i^-; the same type of conclusion holds for M^+. As the members of 2-blocks of positive defect all have this property (for both groups—cf. paragraph II-3), these exceptional characters are indeed new.

If $A \in M^- - \langle 1\rangle$, A is not conjugate to a member of M^+, and the η_i^+ all have the same value on A. No η_j^- is constant on $M^- - \langle 1\rangle$, for if so it is not exceptional. If $\eta_k^+ = \eta_j^-$, then η_k^+ is not constant on $M^- - \langle 1\rangle$, and then none of the η_i^+ are constant there. But then all the η_i^+ are exceptional for M^-. As there are more η_i^+ than η_i^-, this cannot be. Thus the two families are distinct.

7. The characters of these families are real. For, as θ_i^- is 0 off M^-, its multiplicity in a character on $N(M^-)$ depends only on the values on M^-. Since each member of M^- is conjugate to its inverse, θ_i^- is real: η^- has real values on M^-, so that η_i^- and $\overline{\eta_i^-}$ have the same values on M^-. Their restrictions to $N(M^-)$ then contain θ_i^- the same number of times. Since $(q-3m)/6 \geqq 3$, this means that $\eta_i^- = \overline{\eta_i^-}$ or η_i^- is real. Similarly η_i^+ is real.

8. The remaining characters of G can now be given. The members of $(M^- \cup M^+) - \langle 1 \rangle$ represent $(q-3m)/6 + (q+3m)/6 = q/3$ conjugate classes of G. The total number of conjugate classes of G is then $q + 8$. As $q + 6$ characters have been described so far, there are two left, ξ_9 and ξ_{10}. Of the characters of positive 2-defect, only ξ_5, ξ_6, ξ_7, and ξ_8 are nonreal. The number of pairs of nonreal characters is the same as the number of pairs of mutually inverse classes of G [12]. That number is 3 (from T, T^{-1}; YT, YT^{-1}; JT, JT^{-1}). Thus $\overline{\xi_9} = \xi_{10}$.

9. The rest of the chapter is devoted to determining some of the values of these new characters, including their degrees.

First the technique of paragraph I–7 and paragraph II–8 is applied to the exceptional characters. Let V and W stand generically for members of $M^- - \langle 1 \rangle$ and $M^+ - \langle 1 \rangle$, respectively. $\sum |\chi(V)|^2 = c(V) = q + 1 - 3m$, summed over the characters. Those other than the η_i^- have rational integral values on V; if c is the contribution to the sum from these, $c \geqq 4$ (4 from the principal 2-block, from paragraph II–3). Moreover, $\eta_i^-(V) = \varepsilon^- \theta_i^-(V) + x$, $x = \eta^-(V)$ a rational integer. The centralizer equation in $N(M^-)$ used to simplify the corresponding equation in G gives

$$6 = q - 3mx^2/6 - 2\varepsilon^- x + c. \tag{2}$$

With the restrictions given above, for $q > 27$ it can only be that $x = 0$ and $c = 6$. When $q = 27$, x could be 0 or ε^-. If it were ε^-, c would be 5. Thus outside of the principal 2-block and the η_i^-, one character has value ± 1 on V. As $\overline{\xi_9} = \xi_{10}$, it is not one of these; and there are too many η_i^+ (they all have the same value on V). So $x = \varepsilon^-$ is ruled out, and again $x = 0$, $c = 6$. The same argument as for the contradiction gives $\xi_9(V) = \xi_{10}(V) = \pm 1$. All the other characters except the η_i^- and the four in the principal 2-block are 0 on V.

Quite similarly, $\eta^+(W) = 0$, $\xi_9(W) = \xi_{10}(W) = \pm 1$ and all other characters besides the η_i^+ and the four in the principal 2-block are 0 on W.

10. The η_i^-, the η_i^+, ξ_9, and ξ_{10} are of 2-defect 0 and vanish on $\langle R \rangle - \langle 1 \rangle$ and $\langle S \rangle - \langle 1 \rangle$ (paragraph I–7 and paragraph II–8). The congruences obtained from $\langle R \rangle$ and $\langle S \rangle$ imply that their degrees (being divisible by 8) are divisible by $q^2 - 1$. Since η_i^- is 0 on $M^+ - \langle 1 \rangle$ and η_i^+ is 0 on $M^- - \langle 1 \rangle$, the degree of η_i^- is divisible by $q + 1 + 3m$ and that of η_i^+ by $q + 1 - 3m$. Thus let $\xi_9(1) = \xi_{10}(1) = c(q^2 - 1)$, $\eta_i^-(1) = a(q^2 - 1)(q + 1 + 3m)$, $\eta_i^+(1) = b(q^2 - 1)(q + 1 - 3m)$. As the sum of the squares of the degrees of the characters is g, we have (removing $q^2 - 1$)

$$(3) \qquad (q-3m)(q+1+3m)^2(a^2-1)/6 + (q+3m)(q+1-3m)^2(b^2-1)/6 + 2c^2 = 2q/3.$$

Because the coefficients of a^2-1 and b^2-1 exceed $2q/3$, $a=b=1$. Then $c=m$. Thus $\eta_i^-(1)=(q^2-1)(q+1+3m)$, $\eta_i^+(1)=(q^2-1)(q+1-3m)$, and $\xi_9(1)=\xi_{10}(1)=m(q^2-1)$.

Use of the congruences obtained from ξ_9 and ξ_{10} restricted to M^- and M^+ gives $\xi_9(V)=\xi_{10}(V)=-1$ and $\xi_9(W)=\xi_{10}(W)=1$. The orthogonality relations for V and 1 give $\varepsilon^-=-1$; for W and 1, $\varepsilon^+=-1$.

Chapter V. Completion of the Character Table

The gaps in the table are all for elements of the 3-Sylow subgroup, and they are almost all filled from the orthogonality relations. We shall use the phrase "A to B" to mean the orthogonality relations applied to the classes of A and B. A similar phrase is used for characters.

Y to S gives $b(Y)=-1$ and YT to S gives $b(YT)=-1$ (cf. paragraph III–1) so that $b(YT^{-1})=-1$ also. V to Y, W to Y, and the centralizer relation for Y give $\xi_9(Y)=\xi_{10}(Y)=-m, \eta_i^-(Y)=\eta_i^+(Y)=-1$. V to X, W to X, and the centralizer relation on X give $\xi_9(X)=\xi_{10}(X)=-m$, $\eta_i^-(X)=-(q+1+3m)$, $\eta_i^+(X)=-(q+1-3m)$.

V to T, W to T, and Y to T give $-m/2$ for the real part of $\xi_5(T)$ and a linear relation connecting the real part of $\xi_9(T)$, $\eta_i^-(T)$, and $\eta_i^+(T)$. Congruences from restriction to F and the bound provided by $c(T)=2q^2$, along with the fact that from $\eta_i^-(JT)=\eta_i^+(JT)=0$, it follows that $\eta_i^-(T)$ and $\eta_i^+(T)$ must be even, yield $\eta_i^-(T)=-3m-1$, $\eta_i^+(T)=3m-1$. T to T^{-1} gives a quadratic relation between the imaginary parts of $\xi_5(T)$ and $\xi_9(T)$. Finally, $(T,T;J)=0$. For if T_1 and T_2 are conjugate to T, then $T_1T_2=J$ implies $(T_1J)^{-1}T_2(T_1J)=T_1^{-1}$, an impossibility. This gives the imaginary parts up to a sign. The fact that $\xi_5(T)-\xi_5(JT)$ belongs to a prime ideal divisor of 2 in the field of gth roots of unity determines one sign, and as ξ_9 and ξ_{10} have not heretofore been distinguished, the other can be chosen arbitrarily. The remaining values come from conjugation and orthogonality with the decomposition numbers.

Finally the values for YT: η_r to η_i^- gives $\eta_i^-(YT)=\eta_i^-(YT^{-1})=-1$, and η_r to η_i^+ gives $\eta_i^+(YT)=\eta_i^+(YT^{-1})=-1$. ξ_9 to η_r gives $m/2$ as the real part of $\xi_9(YT)$, and Y to YT gives the real part of $\xi_5(YT)$ as $-m/2$. T to YT gives the sum of the imaginary parts of $\xi_5(YT)$ and $\xi_9(YT)$ to be 0. Finally, the centralizer relation on YT gives these imaginary parts up to a sign. From paragraph III–12, we may replace Y by a conjugate Y_1 in such a way that YT^{-1} is conjugate to Y_1T. This option allows the sign of the imaginary part of $\xi_5(YT)$ to be chosen arbitrarily.

The simplicity of G follows from the fact that no character takes on a value equal to its degree on any nonidentity element.

In the character table below, a listing of a paragraph number refers to exceptional values and where they are discussed (or introduced).

Value on Character	1	$R^a \neq 1$	$S^a \neq 1$	V	W	X	Y
ξ_1	1	1	1	1	1	1	1
ξ_2	q^2-q+1	1	3	0	0	$1-q$	1
ξ_3	q^3	1	-1	-1	-1	0	0
ξ_4	$q(q^2-q+1)$	1	-3	0	0	q	0
ξ_5	$(q-1)m(q+1+3m)/2$	0	1	-1	0	$-(q+m)/2$	m
ξ_6	$(q-1)m(q+1-3m)/2$	0	-1	0	1	$(q-m)/2$	m
ξ_7	$(q-1)m(q+1+3m)/2$	0	1	-1	0	$-(q+m)/2$	m
ξ_8	$(q-1)m(q+1-3m)/2$	0	-1	0	1	$(q-m)/2$	m
ξ_9	$m(q^2-1)$	0	0	-1	1	$-m$	$-m$
ξ_{10}	$m(q^2-1)$	0	0	-1	1	$-m$	$-m$
η_r	q^3+1	I–6	0	0	0	1	1
η_r'	q^3+1	I–6	0	0	0	1	1
η_t	$(q-1)(q^2-q+1)$	0	II–6	0	0	$2q-1$	-1
η_t'	$(q-1)(q^2-q+1)$	0	II–6	0	0	$2q-1$	-1
η_i^-	$(q^2-1)(q+1+3m)$	0	0	IV–5	0	$-q-1-3m$	-1
η_i^+	$(q^2-1)(q+1-3m)$	0	0	0	IV–5	$-q-1+3m$	-1

Value on Character	T	T^{-1}	YT	YT^{-1}
ξ_1	1	1	1	1
ξ_2	1	1	1	1
ξ_3	0	0	0	0
ξ_4	0	0	0	0
ξ_5	$(-m+im^2\sqrt{3})/2$	$(-m-im^2\sqrt{3})/2$	$(-m-im\sqrt{3})/2$	$(-m+im\sqrt{3})/2$
ξ_6	$(-m+im^2\sqrt{3})/2$	$(-m-im^2\sqrt{3})/2$	$(-m-im\sqrt{3})/2$	$(-m+im\sqrt{3})/2$
ξ_7	$(-m-im^2\sqrt{3})/2$	$(-m+im^2\sqrt{3})/2$	$(-m+im\sqrt{3})/2$	$(-m-im\sqrt{3})/2$
ξ_8	$(-m-im^2\sqrt{3})/2$	$(-m+im^2\sqrt{3})/2$	$(-m+im\sqrt{3})/2$	$(-m-im\sqrt{3})/2$
ξ_9	$-m+im^2\sqrt{3}$	$-m-im^2\sqrt{3}$	$(m+im\sqrt{3})/2$	$(m-im\sqrt{3})/2$
ξ_{10}	$-m-im^2\sqrt{3}$	$-m+im^2\sqrt{3}$	$(m-im\sqrt{3})/2$	$(m+im\sqrt{3})/2$
η_r	1	1	1	1
η_r'	1	1	1	1
η_t	-1	-1	-1	-1
η_t'	-1	-1	-1	-1
η_i^-	$-3m-1$	$-3m-1$	-1	-1
η_i^+	$3m-1$	$3m-1$	-1	-1

Value on / Character	JT	JT^{-1}	$JR^a \neq J$	$JS^a \neq J$	J
ξ_1	1	1	1	1	1
ξ_2	-1	-1	-1	-1	-1
ξ_3	0	0	1	-1	q
ξ_4	0	0	-1	1	$-q$
ξ_5	$(1-im\sqrt{3})/2$	$(1+im\sqrt{3})/2$	0	1	$-(q-1)/2$
ξ_6	$(-1+im\sqrt{3})/2$	$(-1-im\sqrt{3})/2$	0	-1	$(q-1)/2$
ξ_7	$(1+im\sqrt{3})/2$	$(1-im\sqrt{3})/2$	0	1	$-(q-1)/2$
ξ_8	$(-1-im\sqrt{3})/2$	$(-1+im\sqrt{3})/2$	0	-1	$(q-1)/2$
ξ_9	0	0	0	0	0
ξ_{10}	0	0	0	0	0
η_r	1	1	I–6	0	$q+1$
η_r	-1	-1	I–6	0	$-(q+1)$
η_t	-3	-3	0	II–6	$3(q-1)$
η_t	1	1	0	II–6	$-(q-1)$
η_i^-	0	0	0	0	0
η_i^+	0	0	0	0	0

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