

DEFORMATIONS, RELATED DEFORMATIONS AND A UNIVERSAL SUBFAMILY

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Introduction. Let $n \geq 2$, $1 = d_1, d_2, \dots, d_n$ be integers such that $d_i \mid d_{i+1}$ and

$$\delta = \begin{pmatrix} d_1 & 0 \\ 0 & d_n \end{pmatrix}.$$

Then, for each element z of the Siegel's generalized upper-half plane H_n , $(n \times 2n)$ -matrix (z, δ) is a Riemann matrix of an abelian torus $T(z)$ which admits

$$P = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}$$

as a principal matrix. By means of theta functions, P gives a polarization on $T(z)$ and we denote by $(T(z), P)$ the resulting polarized abelian torus.

For z and z' in H_n , $(T(z), P)$ and $(T(z'), P)$ are isomorphic if and only if there are complex $(n \times n)$ -matrix C and $(2n \times 2n)$ -integer matrix M such that $(z, \delta)M = C(z', \delta)$ with ${}^tMPM = P$. On the other hand, $\text{Sp}(n, \mathbf{R})$ operates on H_n by $z \rightarrow (Az + B)(Cz + D)^{-1}$, where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is an element of $\text{Sp}(n, \mathbf{R})$. And for a fixed δ , there is a discrete subgroup $\Gamma(\delta)$ of $\text{Sp}(n, \mathbf{R})$ such that the $\Gamma(\delta)$ -orbit of z in H_n coincides with the polarized isomorphism class of $(T(z), P)$ when we identify $z \in H_n$ with $(T(z), P)$. We may regard $H_n/\Gamma(\delta)$ as a variety of moduli for the set of polarized abelian tori arising from H_n .

By means of theta series, which depend on an integer $\mu \geq 3$, we can embed (as complex analytic manifolds) all $T(z)$, $z \in H_n$, simultaneously into a projective space \mathbf{P}^N of dimension $\mu^n |\delta| - 1$. Let $\mathcal{W}(\delta, \mu)$ be the set of abelian varieties in \mathbf{P}^N with the natural polarizations (*i.e.*, the polarization determined by hyperplane

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sections) which are images of the above embedding or their projective transforms. Then, the set of Chow-points of the members of $\mathcal{U}(\delta, \mu)$ form a \mathcal{Q} -open variety and the polarized isomorphism classes of the members of $\mathcal{U}(\delta, \mu)$ coincide with projective families.

If we regard a construction of variety of moduli as finding a locally closed set whose points represent isomorphism classes, then we may enlarge the set $\mathcal{U}(\delta, \mu)$ which depended on μ , to the set $\mathcal{A}(\delta)$ of all polarized abelian varieties which are isomorphic to some member of $\mathcal{U}(\delta, \mu)$. Then we can give another characterization of $\mathcal{A}(\delta)$: let us call a polarized abelian variety a "deformation" of another if there is a finite chain of polarized abelian varieties connecting the two so that each consecutive one is either isomorphic or one of the pair is a specialization of the other over the field \mathcal{Q} . Then, for $A \in \mathcal{A}(\delta)$, let $\Sigma(A)$ be the set of deformations of A . We have $\mathcal{A}(\delta) = \Sigma(A)$ [10], [4].

By rearranging the above sequence of observations backwards, we have the following set which becomes susceptible of generalization to other polarized varieties. Let A_0 be a polarized abelian variety. Then, there is a subset $\mathcal{U} \subset \Sigma(A_0)$ such that (1) for any $A \in \Sigma(A_0)$ there is $U \in \mathcal{U}$ with $A \cong U$, (2) underlying varieties of the members of \mathcal{U} are in a fixed projective space, and (3) the set of Chow-points of the underlying varieties of the members of \mathcal{U} is a \mathcal{Q} -open variety. We may call such \mathcal{U} a universal subfamily of $\Sigma(A_0)$ and there are many such subfamilies for $\Sigma(A_0)$. The special one we have considered, $\mathcal{U}(\delta, \mu)$, renders fibering by projective families and leads to a construction of a variety of moduli for $\Sigma(A_0)$. In this sense, \mathcal{U} may be considered as a geometric counterpart to H_n which was an intermediate stage toward $\dots H_n/\Gamma(\delta)$.

In the case of nonsingular curves of fixed genus n , Baily constructed a variety of moduli by means of their Jacobians and $H_n/\Gamma(I_n)$, where I_n is the identity $(n \times n)$ -matrix. From our point of view, the following aspects of the curves are of main interest:

- (1) the canonical map $\phi: C \rightarrow J$ of a nonsingular curve C into its Jacobian J is biregular (birational) and the image $\phi(C)$ determines a polar divisor Θ with $l(\Theta) = 1$, i.e., $(J, \tilde{\Theta}) \in \mathcal{A}(I_n)$ if we denote the polarization determined by Θ by $\tilde{\Theta}$,
- (2) the canonical map $\phi: C \rightarrow J$ is compatible with the specialization of the curve C (we consider only nonsingular specializations),
- (3) Hoyt's result [3] that the set of Jacobians of curves of fixed genus n in a family of polarized abelian varieties (here we may take $\mathcal{U}(I_n, \mu)$) forms a locally closed family.

When we have a variety of moduli as in the case of curves, it seems idle to look for a universal subfamily. But the case of curves suggests a tentative step of generalization to higher dimensions. In this paper, we will consider some additional structure on a nonsingular polarized variety and one restrictive enough

notion of "deformation" so that the set of deformation admits a universal subfamily.

Let V be a polarized variety and suppose that the underlying variety V of V has following properties: (1) the dimension of the image of V by an albanese map of V is the same as that of V , and (2) no curve on V is crushed to a point by the albanese map. If (1) and (2) are satisfied, we will call V an " α -variety" and define α -deformations of V . In §2, we will construct a universal subfamily for the set of α -deformations, while in §1 we will develop terminology in a slightly more general setting, the main guideline being Matsusaka-Mumford [9].

1. Related deformations. NOTATIONAL CONVENTION. While our basic terminology will be that of [11], we will also follow definitions in [12], [8] and [9]. As specific assumptions and conventions of this paper, we take the field of rational numbers, \mathcal{Q} , as our ground field and confine ourselves to projective geometry. For example, by a specialization of a variety, denoted by \rightsquigarrow , we will mean a specialization as a cycle in an ambient projective space defined over \mathcal{Q} . Furthermore, we will consider polarization only on a nonsingular variety (projective, absolutely irreducible). To avoid repetition, we will denote by "variety V " with boldface the following: a nonsingular variety V , called the underlying variety of V , has a specific polarization structure \mathcal{X} on it and the pair of concepts (V, \mathcal{X}) is denoted by V . For example, $V \rightsquigarrow V'$ means "on V and V' polarizations \mathcal{X} and \mathcal{X}' respectively are given and for a polar divisor $X \in \mathcal{X}$, $(V, X) \rightsquigarrow (V', X')$ over $V \rightsquigarrow V'$ (over \mathcal{Q}) with $X' \in \mathcal{X}'$." As in the above example, if not otherwise mentioned, a capital letter U , corresponding to a boldfaced U which represents a polarized variety, will be understood as denoting the underlying variety of U .

DEFINITION 1. A morphism (everywhere defined birational map) $\phi: V \rightarrow V'$ of the underlying variety V of V into that of V' will be called a (polarized) morphism of V into V' , written again by $\phi: V \rightarrow V'$, if there is a polar divisor X' in \mathcal{X}' such that $\phi^{-1}(X') = \text{pr}_1[\Gamma_{\phi} \cdot (V \times X')]$ is defined and is in \mathcal{X} .

This definition does not depend on the choice of a polar divisor X' in \mathcal{X}' . By the result in [7] of a regular maximal algebraic family of positive divisors on a nonsingular variety, it is enough to check for the case $X' \sim Y'$ (linear equivalence on V') with both $\phi^{-1}(X')$ and $\phi^{-1}(Y')$ being defined. Then, by the Corollary to Theorem 14, Chapter VIII of [11], we have $\phi^{-1}(X') \sim \phi^{-1}(Y')$ on V . We also notice⁽²⁾ that ϕ has to satisfy the following: for any point $a \in V'$, the point set $\phi^{-1}(a)$ has to be finite. V' being polarized and ϕ being into, we may consider the case where V' is a projective space with the natural polarization. Let $L(\phi)$ be the linear system defining $\phi: V^n \rightarrow \mathbf{P}^N$ (hence $L(\phi) \subset \mathcal{X}$) and let $m = \dim \phi(V)$. Suppose $a \in \mathbf{P}^N$ and the set $\phi^{-1}(a)$ is not finite. Let W^s be a component with maximal dimension in $\phi^{-1}(a)$ with $s \geq 1$. Then, $m - (n - s) \geq 0$,

(2) This fact was pointed out to me by the referee, and this eliminates many of the vacuous concepts I have originally dealt with.

and we can choose general hyperplanes H_1, H_2, \dots, H_{n-s} in P^N such that $a \notin H_i$, $\phi^{-1}(V) \cdot H_1 \cdots H_{n-s} \neq \emptyset$ is defined and $\phi^{-1}(H_i)$ are also defined. Going to numerical equivalence class, $W^s \cdot X^{(n-s)} = 0$ with $X \in \mathcal{X}$. This contradicts Nakai's criterion of nondegeneracy [6] of X , namely there exists a positive s -cycle W^s such that $W^s \cdot X^{(n-s)}$ is not positive.

DEFINITION 2. Let V and V' be varieties. They are said to be immediate deformations of each other if V is isomorphic to V' or one of them is a specialization of the other. We say that they are deformations of each other if there are a finite number of varieties $V = V_0, V_1, \dots, V_s = V'$ such that each consecutive one is an immediate deformation of the other. By $\Sigma(V)$, we denote the set of deformations of variety V .

DEFINITION 3. When $\Sigma(U_0)$ is given, a variety V is said to be $\Sigma(U_0)$ -related if there is a morphism $\phi; V \rightarrow U$ for some U in $\Sigma(U_0)$. When this is so, we denote by (V, ϕ) the $\Sigma(U_0)$ -related variety V with relation ϕ .

DEFINITION 4. Let $\Sigma = \Sigma(U_0)$ be given, and (V, ϕ) and (V', ϕ') be Σ -related varieties. They are said to be isomorphic to each other (with Σ -relation), written $(V, \phi) \cong (V', \phi')$, if (a) there is an isomorphism $\alpha; V \rightarrow V'$ and (b) there is an isomorphism $\beta: U \rightarrow U'$ such that $\beta \circ \phi = \phi' \circ \alpha$, where $\phi: V \rightarrow U$ and $\phi': V' \rightarrow U'$. Similarly, (V', ϕ') is said to be a Σ -related specialization of (V, ϕ) , written $(V, \phi) \rightsquigarrow (V', \phi')$, if (a) $V \rightsquigarrow V'$ and (b) there are Σ -related $(\overline{V}, \overline{\phi})$ and $(\overline{V}', \overline{\phi}')$ such that $(V, \phi) \cong (\overline{V}, \overline{\phi})$, $(V', \phi') \cong (\overline{V}', \overline{\phi}')$ and $(\overline{V}, \overline{\phi}) \rightsquigarrow (\overline{V}', \overline{\phi}')$, where Γ_ϕ denotes the graph of ϕ .

In Definition 2, we can replace "isomorphism", "specialization" and "variety V_i " by Σ -related ones respectively and obtain a notion of Σ -related deformation. We denote by $\Sigma(V, \phi)$ the set of Σ -related deformations of (V, ϕ) , when $\Sigma = \Sigma(U_0)$ is fixed.

In the following, we will restrict ourselves in the type of Σ , so that the resulting Σ -related deformation will be seen to be a slight generalization of " α -deformation" mentioned in the Introduction.

DEFINITION 5. The set $\Sigma(U_0)$ of deformations of a variety U_0 will be called "of type c " if there is a positive integer c , such that:

- (1) for every U in Σ and for any basic polar divisor B of U , there is a positive U -divisor Z which is algebraically equivalent to cB , and every such Z is ample;
- (2) for every U in Σ , the set of positive U -divisors Z , which are algebraically equivalent to cB for some basic polar divisor B of U , form an irreducible algebraic family $\mathcal{F}(c, U)$, and $l_u(Z)$ depends only on Σ . ($l_u(Z) = \dim |Z| + 1$, where $|Z|$ denotes the complete linear system on U determined by Z .)

From now on, we fix a set of deformations of a variety U_0 , Σ , of type c , and a Σ -related variety (V_0, ϕ_0) with $\dim V_0 \geq 2$. We may rearrange the notations so that $\phi_0: V_0 \rightarrow U_0$.

For any $(V, \phi) \in \Sigma(V_0, \phi_0)$, and for any $Z \in \mathcal{F}(c, U)$, we can apply Bertini's theorems on the linear system $\phi^{-1}(|Z|)$ (on V) and assert that there is a non-

singular variety W on V as a member of $\phi^{-1}(|Z|)$. ϕ induces the morphism $\phi_w: W \rightarrow U$ and $\text{Tr}_w(\phi^{-1}(|Z|)) = \phi_w^{-1}(|Z|)$ on W . By the second remark after Definition 1, members of $\phi_w^{-1}(|Z|)$ are nondegenerate W -divisors and give an induced polarization on W . Then, $\phi_w: W \rightarrow U$ is a (polarized) morphism so that (W, ϕ_w) is Σ -related.

With these notations, let's consider the following two sets:

- (1) fix a divisor $Z_0 \in \mathcal{F}(c, U_0)$ and also a nonsingular variety $W_0 \in \phi^{-1}(|Z|)$, and then we have $\Sigma(W_0, \phi_{0w_0})$;
- (2) Σ' be the set of all possible pairs (W, ϕ_w) with nonsingular variety $W \subset V$, $W \in \phi^{-1}(|Z|)$ while $Z \in \mathcal{F}(c, U)$ and $(\phi, \phi) \in \Sigma(V_0, \phi_0)$ vary.

LEMMA 1. $\Sigma' \subset \Sigma(W_0, \phi_{0w_0})$.

As $(W_0, \phi_{0w_0}) \in \Sigma'$, it is enough to show that for any $(W, \phi_w) \in \Sigma'$ there are a finite number of (W_i, ϕ_{iw_i}) ($i = 0, \dots, s$) such that each consecutive one is an immediate deformation of the other in the sense of Σ -relation, with $(W, \phi_w) = (W_s, \phi_{sw_s})$.

Case 1. $\phi = \phi_0$ and $Z = Z_0$. Taking a sufficiently generic member \bar{W} of $\phi_0^{-1}(|Z_0|)$, we have $(W_0, \phi_{0w_0}) \leftarrow \sim (\bar{W}, \phi_{0\bar{w}}) \rightsquigarrow (W, \phi_w)$.

Case 2. $\phi = \phi_0$ and $Z \in \mathcal{F}(c, U_0)$. By the Case 1 above and property (2) of Definition 5, it is enough to consider the case where $W_0 = \phi_0^{-1}(S_0)$, $S_0 \in |Z_0|$ and $\bar{W} = \phi_0^{-1}(\bar{S})$, $\bar{S} \in |Z|$ so that \bar{S} is a sufficiently generic member of the family $\mathcal{F}(c, U_0)$. Then, $(\bar{W}, \phi_{0\bar{w}}) \rightsquigarrow (W_0, \phi_{0w_0})$.

General case. Over the deformation U_0 to U , members of $\mathcal{F}(c, U_0)$ deform to those of $\mathcal{F}(c, U)$. Hence, in finite steps, this case reduces to Case 1 or 2 above.

With the same notation as above, we obtain a set $\Sigma_1(V_0, W_0)$ of pairs of varieties (V, W) such that $W \subset V$, $(V, \phi) \in \Sigma(V_0, \phi_0)$ and $(W, \phi_w) \in \Sigma'$.

PROPOSITION 1⁽³⁾. There is a constant c_r of the set Σ_1 such that for any integer $m \geq c_r$,

(r, A). $H^i(V, \mathcal{L}_v(mW')) = 0$ for $i \geq 1$, where $\mathcal{L}_v(mW')$ is the sheaf on V defined by $|mW'|$ [12],

(r, B). The minimal sum of $\phi^{-1}(|Z|)$ and $|mW'|$ on V is complete; where $r = \dim V$, $(V, W) \in \Sigma_1$, and $W' \in \phi^{-1}(|Z|)$ such that $W \cdot W'$ is defined.

Proof. We will use induction on r . Suppose $r > 2$ and assume that $(r - 1, A)$ and $(r - 1, B)$ hold with constant c_{r-1} . By Lemma 1, these assumptions hold for the pairs (W, T) in $\Sigma_1(W_0^{r-1}, T_0^{r-2})$, which we get from Σ' .

For a given pair $(V, W) \in \Sigma_1(V_0, W_0)$, and an integer $m > 1$, consider the following exact sequence of sheaves determined by respective divisors;

$$0 \rightarrow \mathcal{L}_V((m - 1)W') \rightarrow \mathcal{L}_V(mW') \rightarrow \mathcal{L}_V(mW \cdot W') \rightarrow 0.$$

(3) I am grateful to the referee who has pointed out that this proposition is proved in a much more general context in the forthcoming thesis of S. Kleiman (Ph. D., Harvard, 1965).

For $m \geq c_{r-1}$, by $(r - 1, A)$, we have an exact sequence

$$\begin{aligned} 0 \rightarrow H^0(V, (m-1)W') &\rightarrow H^0(V, mW') \rightarrow H^0(W, mW \cdot W') \rightarrow H^1(V, (m-1)W') \\ &\rightarrow H^1(V, mW') \rightarrow 0 \rightarrow H^2(V, (m-1)W') \rightarrow H^2(V, mW') \rightarrow 0 \rightarrow \dots \rightarrow 0 \\ &\rightarrow H^r(V, (m-1)W') \rightarrow H^r(V, mW') \rightarrow 0 \end{aligned}$$

where we abbreviated $H^i(V, \mathcal{L}_V(D))$ into $H^i(V, D)$.

W' being a nondegenerate V -divisor, a high multiple of W' is ample and hence equivalent to a hyperplane section of a variety which is the embedded image of V by the ample linear system. By the semi-fineness of the sheaf determined by a hyperplane section, [12], we get

$$H^i(V, mW') = 0 \text{ for } i \geq 2 \text{ and large } m.$$

But, for $m \geq c_{r-1}$, $H^i(V, (m-1)W') = H^i(V, mW')$ ($i \geq 2$) by the exact sequence above. That is, m is large enough if $m \geq c_{r-1}$ to obtain the above equality.

Let us denote $e_{r,m} = \chi(V, mW') = \sum_{i=0}^r (-1)^i h^i(V, mW')$, where $h^i(V, mW') = \dim H^i(V, mW')$. We have $e_{r,m} = h^0(V, mW') - h^1(V, mW')$ for $m \geq c_{r-1}$. V being nonsingular, $e_{r,m}$ is a specialization invariant of the pair (V, mW') , [1], and hence a constant of the set $\Sigma_1(V_0, W_0)$ when m is fixed. We also have the result in [9] that $h^0(V, mW') \leq d_{r,m}$ for a fixed m , where $d_{r,m}$ is a constant of the set $\Sigma_1(V_0, W_0)$. Put $e_r = e_{r,c_{r-1}}$, $d_r = d_{r,c_{r-1}}$ and $b_r = d_r - e_r$. Then, for any pair (V, W) in Σ_1 , we have $h^1(V, c_{r-1}W') \leq b_r$.

From the exactness of $H^1(V, (m-1)W') \rightarrow H^1(V, mW') \rightarrow 0$ for $m \geq c_{r-1}$, we get a decreasing sequence of nonnegative integers

$$h^1(V, c_{r-1}W') \geq h^1(V, (c_{r-1} + 1)W') \geq \dots$$

Hence, for any pair (V, W) in Σ_1 , there are only two possibilities;

- (1) $h^1(V, (c_{r-1} + b_r + j)W') = 0$ for all $j \geq 1$, or
- (2) $h^1(V, (c_{r-1} + i)W') = h^1(V, (c_{r-1} + i + 1)W')$ for some i , $0 \leq i \leq b_r$.

Now we claim that (2) implies (1), i.e., (1) is always the case.

Suppose (2) is the case. Then, the following sequence is exact,

$$0 \rightarrow H^0(V, (c_{r-1} + i)W') \rightarrow H^0(V, (c_{r-1} + i + 1)W') \rightarrow H^0(W, (c_{r-1} + i + 1)W' \cdot W) \rightarrow 0.$$

This is equivalent to saying that $\text{Tr}_W |(c_{r-1} + i + 1)W'| = |(c_{r-1} + i + 1)W' \cdot W|$.

But,

$$\begin{aligned} \text{Tr}_W |(c_{r-1} + i + 2)W'| &\supset \min \text{sum} [\text{Tr}_W(\phi^{-1}(|Z|)), \text{Tr}_W |(c_{r-1} + i + 1)W'|] \\ &= \min \text{sum} [\phi_W^{-1}(|Z|), |(c_{r-1} + i + 1)W' \cdot W|], \end{aligned}$$

and by $(r - 1, B)$, the last linear system is complete and hence

$$|(c_{r-1} + i + 2)W' \cdot W| = \text{Tr}_W |(c_{r-1} + i + 2)W'|.$$

Back in terms of exact sequences and h^1 , we have

$$h^1(V, (c_{r-1} + i + 1)W') = h^1(V, (c_{r-1} + i + 2)W'),$$

and repeating the same argument and using the semifineness argument, we get Case (1) as claimed.

Let $c_r = c_{r-1} + b_r$. Then, (r, A) is true with the constant c_r . Now we claim that (r, B) is also true with the same c_r . By (r, A) , we have $\text{Tr}_W |mW'| = |mW' \cdot W|$ for $m \geq c_r$, and from $c_r \geq c_{r-1}$ and $(r - 1, B)$ we have

$$\min \text{sum} [\text{Tr}_W(\phi^{-1}(|Z|)), \text{Tr}_W |mW'|] = |(m + 1)W' \cdot W| \text{ when } m \geq c_r.$$

We have to show

$$\min \text{sum} [\phi^{-1}(|Z|), |mW'|] = |(m + 1)W'| \text{ on } V \text{ when } m \geq c_r.$$

Let the function module determined by $\phi^{-1}(|Z|)$ and a member W' be denoted by L and the function module determined by $\min \text{sum} [\phi^{-1}(|Z|), |mW'|]$ and a member $(m + 1)W'$ be M and the one determined by $|mW'|$ and mW' be $L_V(mW')$, and so forth. Then, we have the following exact sequence of modules;

$$\begin{array}{ccccccc} 0 & \rightarrow & K & \rightarrow & M & & \rightarrow L_W((m + 1)W' \cdot W) \rightarrow 0 \\ & & \cap & & \cap & & \parallel \\ 0 & \rightarrow & K' & \rightarrow & L_V((m + 1)W') & \rightarrow & L_W((m + 1)W' \cdot W) \rightarrow 0 \end{array}$$

where K and K' are respective kernels of the map which sends functions on V to their traces on W (see [13] for terminology). It suffices to show $K' \subset M$, i.e., $K = K'$ and hence $\dim M = \dim L_V((m + 1)W')$. Suppose $f \in K'$, i.e., $(f)_0 > W$. Then, $(f) = W + P - (m + 1)W'$ with some $P > 0$, and hence $(f) = (W - W') + (P - mW')$ gives $f = g \cdot h$ with $g \in L$ and $h \in L_V(mW')$, and hence $f \in M$.

To complete our proof, we need the following $(1, A^*)$, $(1, B^*)$ and c_1^* to replace $(r - 1, A)$, $(r - 1, B)$ and c_{r-1} for the case $r = 2$.

$(1, A^*)$. Let $(V, C) \in \Sigma_1(V_0^2, C_0)$. On the nonsingular curve C , we use the Riemann-Roch theorem for the C -divisor $mC' \cdot C$, and get $H^1(C, mC' \cdot C) = 0$ for $m \geq 2g - 2$, where the genus g of the curve C is a constant of the set Σ_1 .

$(1, B^*)$. $\text{deg}(C' \cdot C) = d > 0$ is a constant of Σ_1 and hence we can find c_1^* such that $c_1^* \geq 2g - 2$ and for $m \geq c_1^*$, $(m - 1)d \geq 2g - 2$. Then, $|mC' \cdot C| - C' \cdot C$ is nonspecial and the Castelnuovo's lemma (Lemma 6 in [9]) becomes available. That is, $\min \text{sum} [\phi_c^{-1}(|Z|), |mC' \cdot C|] = |(m + 1)C' \cdot C|$ on C for $m \geq c_1^*$.

COROLLARY. *Notations and assumptions being the same as above, there is a projective space P such that, for any pair (V, W) in Σ_1 , V can be embedded into P by means of the complete linear system $|c_r W|$, and furthermore, the degree of the image variety is a constant of Σ_1 .*

Proof For any (V, W) in Σ_1 , $c_r W$ is an ample V -divisor because a very high multiple of W is ample and then we can step downwards to $c_r W$ by (r, B) and

Lemma 5 of [9]. On the other hand, by (r, A) and the constancy of $\chi(V, c_r W)$, $l(c_r W)$ is a constant of the set Σ_1 . Hence, we can embed V into \mathbf{P}^N with $N + 1 = l(c_r W)$ by means of $|c_r W|$ for any $(V, W) \in \Sigma_1$. As the divisor $c_r W$ goes to a hyperplane section by the above embedding, and the degree of a zero cycle on V obtained by intersecting r members of $|W|$ being a constant of the set Σ_1 , we get the second part of the corollary.

The above corollary comes very close to the notion of a universal subfamily mentioned in the introduction, and we put the definition down here formally.

DEFINITION 7. Let Σ be a subset of the set of deformations of a variety V_0 . A subset \mathcal{U} of Σ is called a universal subfamily of Σ if the following conditions are satisfied:

- (1) for any $V \in \Sigma$, there is a $V' \in \mathcal{U}$ such that $V \cong V'$,
- (2) there is a fixed projective space \mathbf{P} such that $V' \subset \mathbf{P}$ for any $V' \in \mathcal{U}$,
- (3) the set $\{c(V') \mid V' \in \mathcal{U}\}$ can be expressed as a finite union of open varieties, where $c(V')$ denotes the Chow-point of V' .

2. α -deformations and a universal subfamily. Let V be a variety with $\dim V \geq 2$ such that an albanese (ϕ, A) of V satisfies (1) $\dim V = \dim \phi(V)$ and (2) the morphism ϕ does not crush any curve on V to a point. These properties do not depend on the choice of (ϕ, A) of V , and we will say that V has property " α " if the above conditions are satisfied. For an α -variety V and a given albanese (ϕ, A) of V , we will construct a polarization \mathcal{Y} on A uniquely (and denote $A = (A, \mathcal{Y})$) and a polarization $\tilde{\mathcal{X}}$ on V (denote $\tilde{V} = (V, \tilde{\mathcal{X}})$) such that (\tilde{V}, ϕ) is properly $\Sigma(A)$ -related.

Let us fix an α -variety V_0 and an albanese (ϕ_0, A_0) of V_0 . We define α -deformation of V_0 by requiring each V_i appearing in the chain of immediate deformations (in the Definition 2 of §1.) to be α -varieties. And $\Sigma_\alpha(V_0)$ will denote the set of α -deformations of V_0 . Then, comparing the two sets $\Sigma_\alpha(V_0)$ and $\Sigma(V_0, \phi_0)$, we will construct a universal subfamily of $\Sigma_\alpha(V_0)$ in this section.

Let V be an α -variety and X be an ample polar divisor of V . Let C be a 1-cycle of the form $X_1 \cdot X_2 \cdots X_{r-1}$ where $X_i \in |X|$ such that the intersection is defined. Let (ϕ, A^q) be an albanese of V . First, we know that, by suitable choice of C , $\phi(C)$ generates A and the Pontrjagin sum of $\phi(C)$, $q - 1$ times in A , gives a non-degenerate A -divisor Y [5].

LEMMA 1. Let V and (ϕ, A) be as before, and D_i ($i = 1, 2$) be 1-cycles on V whose supports are irreducible curves C_i respectively. Suppose $D_1 \equiv D_2$ (alg. eq.). Then, $\text{pr}_2[(D_1 \times A) \cdot \Gamma_\phi]$ and $\text{pr}_2[(D_2 \times A) \cdot \Gamma_\phi]$ are numerically equivalent to each other on A .

Proof. For any A -divisor W such that

$$\{\text{pr}_2[(D_1 \times A) \cdot \Gamma_\phi] - \text{pr}_2[(D_2 \times A) \cdot \Gamma_\phi]\} \cdot W$$

is defined, we want $\text{deg}\{\text{pr}_2[(D_1 \times A) \cdot \Gamma_\phi] - \text{pr}_2[(D_2 \times A) \cdot \Gamma_\phi]\} \cdot W = 0$. By our

assumption and changing W within the algebraic equivalence class (hence within the numerical equivalence class) if necessary, we may assume $(\text{pr}_2(D_i \times A) \cdot \Gamma_\phi) \cdot W$ to be defined for each $i = 1, 2$. Then by the property α of ϕ , $E_i \cdot (V \times W)$ are defined on Γ_ϕ , where E_1, E_2 are transforms of D_1, D_2 respectively by the isomorphism between V and Γ_ϕ . As the algebraic projection preserves the degree of 0-cycles, we get the lemma.

LEMMA 2. *Let V and (ϕ, A) be as before. Then, there is a polarization \mathcal{Y} on A which is uniquely determined by V and ϕ .*

Proof. By the remark before Lemma 1, there is a polarization on A determined by Y . Suppose now that X_1 is another ample polar divisor of V . Then, there are integers a and a_1 such that $aX \equiv a_1X_1$ (alg. eq.) and if C_1 is a curve obtained from X_1 in the same way as we got C from X , then $a^{-1}C \equiv a_1^{-1}C_1$ (alg. eq.). Then, by Lemma 1 and by linearly extending the Pontrjagin sum, we see that some multiple of Y is numerically equivalent to some multiple of Y_1 and we get the same polarization on A .

By this lemma, we call (ϕ, A) an albanese of V , where $A = (A, \mathcal{Y})$, \mathcal{Y} being the polarization determined by Y in the above notation.

LEMMA 3. *With the notations as above, A gives a new polarization $\mathcal{X}(A)$ on the underlying variety V of V . If we denote $V(A) = (V, \mathcal{X}(A))$, then $\phi: V(A) \rightarrow A$ is a morphism. Furthermore, if (ψ, B) were another albanese of V , then $\mathcal{X}(A) = \mathcal{X}(B)$. (Hence, we denote $\tilde{V} = (V, \mathcal{X}(A))$.)*

Proof. Let Y be an ample polar divisor of A . Then, we can find Y_1 , linearly equivalent to Y , such that $\text{pr}_1[\Gamma_\phi \cdot (V \times Y_1)] = \phi^{-1}(Y_1)$ is defined. Then, by the property α , $\phi^{-1}(Y_1)$ is nondegenerate on V and hence gives $\mathcal{X}(A)$. The last part of the lemma comes from the universal mapping property of an albanese map.

With the notations of Lemma 3, (\tilde{V}, ϕ) is properly $\Sigma(A)$ -related and for any other albanese (ϕ', A') of V , we have $(\tilde{V}, \phi) \cong (\tilde{V}, \phi')$ in the sense of Definition 4, §1.

Now let us consider specializations. Let V be an α -variety and fix an albanese (ϕ, A) of V . Then, we have A and \tilde{V} uniquely by Lemma 2 and Lemma 3. Let V_1 be a nonsingular variety such that $V \rightsquigarrow V_1$. Then, for an albanese (ϕ_1, A_1) of V_1 , $\dim A = \dim A_1 = q$. On the other hand, there is an albanese (ϕ', A') of V such that $(V, \Gamma_{\phi'}, A') \rightsquigarrow (V_1, \Gamma_{\psi}, B)$ with the property that B is abelian and (ψ, V_1) generates B . By the behavior of an abelian subvariety over the specialization of an abelian variety, and the fact that we are in characteristic 0, (ψ, B) is actually an albanese of V_1 [3], [4].

LEMMA 4. *Let V be an α -variety and fix an albanese (ϕ, A) of V . V_1 be another α -variety such that $V \rightsquigarrow V_1$. Then, for any albanese (ϕ_1, A_1) of V_1 , we have $(\tilde{V}, \phi) \rightsquigarrow (\tilde{V}_1, \phi_1)$.*

Proof. From $V \rightsquigarrow V_1$, we have $(V, \Gamma_{\phi'}, A') \rightsquigarrow (V_1, \Gamma_{\psi}, B)$ with

$$(\tilde{V}, \phi) \cong (\tilde{V}, \phi') \text{ and } (\tilde{V}_1, \phi_1) \cong (\tilde{V}_1, \psi).$$

We can choose ample polar divisors X and X_1 respectively of V and V_1 such that $(V, \Gamma_{\phi'}, A', X) \rightsquigarrow (V_1, \Gamma_{\psi}, B, X_1)$. In the construction of the nondegenerate divisors Y and Y_1 respectively of A' and B (from X and X_1 , in Lemma 2), we used operations which are compatible with specialization. Hence, $(V, \Gamma_{\phi'}, A') \rightsquigarrow (V_1, \Gamma_{\psi}, B)$. Now, we can choose ample polar divisors Y' and Y'_1 of A' and B respectively so that $Y' \rightsquigarrow Y'_1$ over the already constructed specialization. Then, the operations involved in obtaining \tilde{X} and \tilde{X}_1 from Y' and Y'_1 respectively (in Lemma 3) are again compatible with specialization. i.e., $(\tilde{V}, \Gamma_{\phi'}, A') \rightsquigarrow (\tilde{V}_1, \Gamma_{\psi}, B)$ over $V \rightsquigarrow V_1$.

Let us fix an α -variety V_0 and an albanese (ϕ_0, A_0) of V_0 . And consider the following two sets: $\Sigma_{\alpha}(V_0)$ and $\Sigma(\tilde{V}_0, \phi_0)$. For any number V of Σ_{α} and any albanese (ϕ, A) of V , $(\tilde{V}, \phi) \in \Sigma(\tilde{V}_0, \phi_0)$. Applying the Corollary of Proposition 1, §1, to the subset of pairs (V, W) in $\Sigma_1(V_0, W_0)$ which arise from $(\tilde{V}, \phi) \in \Sigma(\tilde{V}_0, \phi_0)$ such that $V \in \Sigma_{\alpha}$ and ϕ is an albanese map of V , we have:

PROPOSITION 2. *Underlying varieties of members of $\Sigma_{\alpha}(V_0)$ can be embedded into a fixed projective space so that the degree of the image variety is a constant of the set Σ_{α} .*

THEOREM 1. *Let V_0 be an α -variety. The set of α -deformations of V_0 , $\Sigma_{\alpha}(V_0)$, considered as a subset of $\Sigma(V_0)$, has a universal subfamily.*

Proof. Let P be the projective space appearing in Proposition 2 above and \mathcal{U} be the subset of Σ_{α} , consisting of embedded images in P with the transferred (by embedding) polarizations. For any $U \in \mathcal{U}$, a pair (U, B) , where B is a basic polar divisor of U , has the following property: $|B + C_m|$ on U is not empty where m is a suitable constant of the set \mathcal{U} and C_m denotes an m -ple of hyperplane section of U . ($|B + C_m|$ becomes ample if $m \geq m_0$ where m_0 depends on the degree of B which is a constant of the set \mathcal{U} .) Hence, the underlying varieties of members of \mathcal{U} from a subset of the set of nonsingular varieties in P of a fixed degree d ($= \deg(U)$) which carry at least a positive divisor of a fixed degree e ($= \deg(Z)$ with $Z \in |B + C_m|$).

Let the Chow-bunch of positive cycles in P of dimension r , degree d , be \mathcal{A} and that of dimension $r - 1$, degree e , be \mathcal{B} , and fix a pair (U_0, Z_0) where U_0 is an underlying variety of a member of \mathcal{U} and Z_0 is a positive U_0 -cycle in $|B_0 + C_m|$. Choose an irreducible component \mathcal{A}_0 of \mathcal{A} containing $c(U_0)$, and an irreducible component \mathcal{B}_0 of \mathcal{B} containing $c(Z_0)$. The possible choices of the pair $(\mathcal{A}_0, \mathcal{B}_0)$ are finite in number. From the graph of the algebraic correspondence between \mathcal{A}_0 and \mathcal{B}_0 , which associates $c(U)$ to $c(Z)$ so that the support of U contains the support of Z , choose an irreducible component \mathcal{C}_0 which contains the point $(c(U_0), c(Z_0))$. (The set of such components is nonempty, finite.)

Let (\bar{u}, \bar{z}) be a generic point of \mathcal{C}_0 and \bar{U} and \bar{Z} be P -cycles so that $c(\bar{U}) = \bar{u}$, $c(\bar{Z}) = \bar{z}$, respectively. Then, from $(\bar{U}, \bar{Z}) \rightsquigarrow (U_0, Z_0)$ we can deduce the following:

- (a) \bar{U} is a nonsingular variety which has the property α , and
- (b) let $\bar{B} = \bar{Z} - \bar{C}_m$ where \bar{C}_m is an m -ple of a hyperplane section of \bar{U} . Then, \bar{B} is a nondegenerate \bar{U} -divisor and the polarization determined by \bar{B} on \bar{U} gives a variety \bar{U} so that $\bar{U} \rightsquigarrow U_0$ ($U_0 \in \mathcal{U}$).

Let the geometric projection of \mathcal{C}_0 into \mathcal{A}_0 be \mathcal{C}'_0 , and denote by \mathcal{C}''_0 the subset of points $c(U)$ so that U is nonsingular and has property α . Then, \mathcal{C}''_0 is an open subvariety of \mathcal{C}'_0 . In the process of obtaining \mathcal{C}''_0 , we were given only finite choices in each step. Let \mathcal{C}'' be the set union of all possible \mathcal{C}''_0 . Then, \mathcal{C}'' has the following property: (a) if $U \in \mathcal{U}$, then $c(U) \in \mathcal{C}''$, and (b) for any $c(U) \in \mathcal{C}''$, we can give a polarization on U , in a uniform manner, so that $U \in \Sigma_\alpha$. From (b) we obtain a subset \mathcal{U}'' of Σ_α so that $\mathcal{U} \subset \mathcal{U}''$ by (a), and hence \mathcal{U}'' is a universal subfamily of Σ_α .

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