# MAXIMAL SUBGROUPS OF SYMMETRIC GROUPS(1)

BY
RALPH W. BALL

**Introduction.** Let $M$ be any nonempty set and $S(M)$ the group of all permutations of $M$. If $X$ is any cardinal number, then $X^+$ shall denote the successor of $X$. The smallest infinite cardinal number shall be denoted by the symbol $d$. The cardinality of $M$ shall be denoted by $|M|$. The support of $s \in S(M)$ is defined by $\operatorname{spt} s = \{x \in M : s(x) \neq x\}$. The fixed set of $s = fs$ $s = \{x \in M : s(x) = x\} = M - \operatorname{spt} s$. If $|M| = X \geqq d, d \leqq Y \leqq X^+$, define $S(X,Y) = \{s \in S(M) : |\operatorname{spt} s| < Y\}$. The alternating subgroup shall be denoted $A(X)$. It is well known that the normal subgroups of $X(S,Y)$ are $A(X)$ and the set of subgroups $S(X,Z)$ with $d \leqq Z \leqq Y$ [1]. The set of all groups $S(X,Y)$, $d \leqq Y \leqq X^+$, together with $A(X)$ is called the set of symmetric groups on the set $M$. Since $S(X,Y) \cong S(X',Y')$ and $A(X) \cong A(X')$ if and only if $X = X'$ and $Y = Y'$ [1], the notation is unambiguous up to isomorphism and depends only on the cardinalities of the sets. If $|M| = n < d$, the standard notation of $S_n$ for $S(M)$ and $A_n$ for the alternating subgroup will be used.

This paper determines the structure of certain classes of maximal subgroups of symmetric groups. Maximal shall always mean proper maximal. It happens that classification of maximal subgroups is convenient in terms of concepts of transitivity. A subset $H$ of $S(M)$ is transitive if for each $x$ and $y \in M$, there is $s \in H$ such that $s(x) = y$. Otherwise $H$ is intransitive. If $H$ is a subgroup, the relation $R$ on $M$ defined by $xRy$ if and only if there is $s \in H$ such that $s(x) = y$ is an equivalence relation. The equivalence classes are called the sets of transitivity of $H$. Let $n < d$; $H$ is $n$-set transitive if for each $M_1 \subseteq M$ with $|M_1| = n$ and for each $s \in S(M)$, there is $r \in H$ such that $r(M_1) = s(M_1)$. $H$ is $n$-ply transitive if for each $s \in S(M)$, there is $r \in H$ such that $r|M_1 = s|M_1$.

The principal results of this paper are as follows: Partition $M$ into $P$ and $Q$ with $|Q| \leqq |P|$. It will be shown that all intransitive maximal subgroups of $S(X,Y)$ or of $A(X)$ are of the form $S(Q)\cdot S(P) \cap S(X,Y)$ or $S(Q)\cdot S(P) \cap A(X)$ respectively with $|Q|$ finite. Similarly all intransitive maximal subgroups of $S_n$ are of the form $S(Q)\cdot S(P)$ with $|Q| < |P|$. All transitive maximal subgroups of

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$S(X, Y)$ for $Y > d$ will be shown to contain $S(X, d)$. Thus the class of intransitive maximal subgroups of $S(X, Y)$ with $Y > d$ is also the class of maximal subgroups of $S(X, Y)$ containing no subgroups normal in $S(X, Y)$.

If $|M|$ and $|Q|$ are both infinite, then $S(Q) \cdot S(P) \cap S(X, Y)$ is not maximal in $S(X, Y)$, but can be used to construct a transitive maximal subgroup. The class of such transitive maximal subgroups is probably not the class of all transitive maximal subgroups of $S(X, Y)$, although this question is still open.

It will be shown that the transitive maximal subgroups of $S(X, d)$ and $A(X)$ not containing $A(X)$ are $S(X, d) \cap N(K)$ and $A(X) \cap N(K)$ respectively where $K = \prod_{\alpha \in A} S(M_\alpha)$ with $\{M_\alpha\}_{\alpha \in A}$ a partition of $M$ such that $|M_\alpha| = |M_\beta| \, \forall \alpha, \beta \in A$ and where $N(K)$ is the normalizer of $K$. Thus all maximal subgroups of $S(X, d)$ and $A(X)$ are determined. Similarly it will be shown that all imprimitive maximal subgroups of $S_n$ are $N(K)$ with $K$ as above. Hence imprimitive maximal subgroups of $S_n$ are completely determined up to isomorphism by the proper divisors of $n$.

## I. Intransitive Maximal Subgroups

Let the set $M$, $|M| = X \geqq d$, be partitioned into $Q$ and $P$ such that $|P| = X$, $|Q| = Z$, $0 < Z \leqq X$, $d \leqq Y \leqq X^+$. The following notations will be used:

$Q_s = \{x \in Q\colon s(x) \in P\}$.
$Q'_s = \{x \in Q\colon s(x) \in Q\} = Q - Q_s$.
$P_s = \{y \in P\colon s(y) \in Q\}$.
$P'_s = \{y \in P\colon s(y) \in P\} = P - P_s$.
$S(Q, Y) = S(Q) \cap S(X, Y)$.
$S(P, Y) = S(P) \cap S(X, Y)$.
$S(Q, A) = S(Q) \cap A(X)$.
$S(P, A) = S(P) \cap A(X)$.
$J(Z) = S(Q) \cdot S(P)$.
$J(Y, Z) = J(Z) \cap S(X, Y) = S(Q, Y) \cdot S(P, Y)$.
$J(A, Z) = J(Z) \cap A(X)$.

If $\{Q, P\}$ and $\{Q', P'\}$ are two partitions of $M$ such that $|Q| = |Q'|$, then $S(Q) \cdot S(P) \cong S(Q') \cdot S(P')$ in a natural way. Thus $J(Z)$ is completely determined up to isomorphism by $Z$, and the notation is reasonable. The class of all subgroups of $S(X, X^+)$ constructed in the above manner will be called the class of $J$-subgroups.

The following theorem lists some facts about $J(Y, Z)$-subgroups for which the proofs are obvious, hence omitted. The notation $H \leqq G$ means that $H$ is a subgroup of $G$, and $H < G$ means $H$ is a proper subgroup of $G$.

1.1. Theorem. *Each of the following statements is true*:
(1) $J(Y, Z) < S(X, Y)$, $J(A, Z) < A(X)$.

(2) *If* $Z>1$, *then* $fs\ J(Y,Z)=\varnothing$ *and* $fs\ J(A,Z)=\varnothing$.
(3) *If* $Z<d$, *then* $|Q_s|=|P_s|\leqq Z$ *for each* $s\in S(X,X^+)$.
(4) *For each* $s\in S(X,d)$ *or* $A(X)$, $|Q_s|=|P_s|<d$.
(5) *If* $Z\geqq d$, $Y>d$, *then for each* $s\in S(X,Y)$, $|Q_s|\leqq Z$ *and* $|P_s|\leqq Z$, *but* $|Q_s|$ *is not necessarily equal to* $|P_s|$.
(6) *If* $Z<d$, *then* $s\in S(X,Y)-J(Y,Z)$ *or* $A(X)-J(A,Z)$ *if and only if* $s\in S(X,Y)$, *respectively* $A(X)$, *and* $Q_s\neq\varnothing$.
(7) *The permutation* $s\in S(X,d)-J(d,Z)$ *or* $A(X)-J(A,Z)$ *if and only if* $s\in S(X,d)$, *respectively* $A(X)$, *and* $Q_s\neq\varnothing$.
(8) *If* $Z\leqq d$, $Y>d$, *then* $s\in S(X,Y)-J(Y,Z)$ *if and only if* $s\in S(X,Y)$ *and* $Q_s\neq\varnothing$ *or* $P_s\neq\varnothing$.

1.2. Definition. For each $s\in S(X,X^+)$, define the *transfer index of* $s=T(s)=\max\{|Q_s|,|P_s|\}$.

It is clear that $s\in S(X,Y)-J(Y,Z)$ or $A(X)-J(A,Z)$ if and only if $s\in S(X,Y)$ respectively $A(X)$, and $T(s)\neq 0$.

A set of lemmas which are useful in the investigation of the classes of $J(Y,Z)$-subgroups will now be stated and proved. Some of the lemmas are stated in a more general form than actually needed here. For $S\subseteq S(M)$ the notation $\langle S\rangle$ means the subgroup of $S(M)$ generated by $S$.

1.3. Lemma. *Let* $Z<X$ *or* $Y<X^+$, $s\in S(X,Y)-J(Y,Z)$. *Then there is* $r\in\langle J(Y,Z),s\rangle$ *such that*:
(1) $|P_0(r)|=X$ *where* $P_0(r)=fs\ r\cap P$.
(2) $Q_r=Q_s$.
(3) $P_r=P_s$.
(4) $r(Q_r)\cap P_r=\varnothing$.

**Proof.** If $X>d$, consider the unique decomposition of $s$ into disjoint cycles. Since $|Q_s|\leqq W$ and $|P_s|\leqq W$, where $W\leqq Z$ and $W<Y$, then $|s(Q_s)\cup P_s|\leqq 2W<X$. Thus there are at most $2W$ cycles whose intersection with $s(Q_s)\cup P_s$ is nonvoid. Let $C_y$ denote the cycle in $s$ in which the letter $y$ occurs, and let $P_1=\{y\in P\colon C_y\cap(s(Q_s)\cup P_s)=\varnothing\}$. Clearly $P_1\subset P$ and $|P_1|=X$. Define $s_1\in S(P,Y)$ by $s_1(y)=y$ if $y\in P-P_1$ and $s_1(y)=s^{-1}(y)$ if $y\in P_1$. Then $s_2=s_1s\in\langle J(Y,Z),s\rangle$ and satisfies conditions (1), (2), (3).

If $X=d$, then $Z<d$ or $Y=d$, so $|Q_s|=|P_s|=n<d$ by Theorem 1.1. Denote $Q_s=\{x_i\}_{i=1}^n$ and $P_s=\{y_i\}_{i=1}^n$. Define $s_1\in S(P,Y)$ by $s_1(y)=y_i$ if $y=s(x_i)$ for some $i=1,\cdots,n$, and $s_1(y)=s^{-1}(y)$ if $y\notin s(Q_s)$. Then $s_2=s_1s\in\langle J(Y,Z),s\rangle$ and satisfies conditions (1), (2), (3). Let $P_1=P-P_s$.

Let $s_2$ and $P_1$ be defined as above for the pertinent case and denote $Q_s=\{x_\alpha\}_{\alpha\in A}$. Choose $\{y_\alpha\}_{\alpha\in A}\subseteq P_1$, and define $s_3\in S(P,Y)$ by $s_3=\prod_{\alpha\in A}(s_2(x_\alpha),y_\alpha)$. Then $r=s_3s_2\in\langle J(Y,Z),s\rangle$ and satisfies conditions (2), (3), (4). Also $r$ satisfies condition (1) since $|P_1|=X$, $P_0(r)=P_1-\{y_\alpha\}_{\alpha\in A}$, and $|A|<X$.

1.4. LEMMA. *Let* $Z < X$ *or* $Y < X^+$, $s \in S(X, Y) - J(Y, Z)$, $\{x_i\}_{i=1}^n \subseteq Q$, $\{y_i\}_{i=1}^n \subset P$, $1 \leqq n < d$, $n \leqq Z$. *Then* $\prod_{i=1}^n (x_i, y_i) \in \langle J(Y,Z), s\rangle$.

**Proof.** It may be assumed that $|Q_s| \geqq 1$ since $\langle J(Y,Z), s\rangle = \langle J(Y,Z), s^{-1}\rangle$. It may also be assumed that $s$ has the properties implied by Lemma 1.3. Let $x \in Q_s$ and $y \in P_0(s)$. Define $s_1 = (s(x), y) \in S(P, Y)$. Then $s^{-1}s_1 s = (x, y) \in \langle J(Y,Z), s\rangle$. Now $(x, x_1) \in S(Q, Y)$ and $(y, y_1) \in S(P, Y)$. Thus

$$(x_1, y_1) = (y, y_1)(x, x_1)(x, y)(x, x_1)(y, y_1) \in \langle J(Y,Z), s\rangle .$$

The lemma is now obvious.

1.5. LEMMA. *Let* $s, r \in S(X, Y) - J(Y,Z)$ *such that* $Q_s = Q_r$, $P_s = P_r$. *Then* $r \in \langle J(Y,Z), s\rangle$.

**Proof.** Now $s^{-1}(Q) = r^{-1}(Q)$ and $s^{-1}(P) = r^{-1}(P)$, hence $rs^{-1}(Q) = Q$ and $rs^{-1}(P) = P$, so $rs^{-1} \in J(Y,Z)$. Then $r = rs^{-1}s \in \langle J(Y,Z), s\rangle$.

1.6. LEMMA. *Let* $Z < X$ *or* $Y < X^+$, $s \in S(X, Y) - J(Y,Z)$ *with* $|Q_s| = W \geqq d$. *Let* $0 \leqq V \leqq W$. *Then there is* $r \in \langle J(Y,Z), s\rangle$ *such that* $|Q_r| = W$, $|P_r| = V$.

**Proof.** It may be assumed that $s$ has the properties implied by Lemma 1.3. Denote $Q_s = \{x_\alpha\}_{\alpha \in A}$ and partition $A$ into $A_1$, $A_2$, $A_3$ with $|A_1| = |A_2| = W$ and $|A_3| = V$. Choose $\{y_\alpha\}_{\alpha \in A_3} \subset P_0(s)$ and let $s_1 = \prod_{\alpha \in A_3}(s(x_\alpha), y_\alpha) \in S(P, Y)$. Now partition $P_0(s) - \{y_\alpha\}_{\alpha \in A_3}$ into $P_1$, $P_2$, $P_3$ with $|P_1| = |P_2| = W$ and $|P_3| = X$. Now choose $s_2 \in S(P, Y)$ such that $s_2(P_1 \cup P_2) = P_2$, $s_2(\{s(x_\alpha)\}_{\alpha \in A_2}) = P_1$, $s_2(\{s(x_\alpha)\}_{\alpha \in A_1}) = \{s(x_\alpha)\}_{\alpha \in A_1 \cup A_2}$, $s_2$ fixes $P - (P_1 \cup P_2 \cup \{s(x_\alpha)\}_{\alpha \in A_1 \cup A_2})$. That such an $s_2$ exists is a consequence of the cardinalities of the various involved sets. Then $r = s^{-1}s_2 s_1 s \in \langle J(Y, Z), s\rangle$, $Q_r = \{x_\alpha\}_{\alpha \in A_2 \cup A_3}$, $P_r = \{y_\alpha\}_{\alpha \in A_3}$. Thus $|Q_r| = W + V = W$ and $|P_r| = V$.

1.7. LEMMA. *Let* $Z < X$ *or* $Y < X^+$, $s \in S(X, Y) - J(Y,Z)$, $|Q_s| = W$, $0 \leqq V \leqq W$. *Then there* is $r \in \langle J(Y,Z), s\rangle$ *such that* $T(r) = V$.

**Proof.** It may be assumed that $s$ has the properties implied by Lemma 1.3. Denote $Q_s = \{x_\alpha\}_{\alpha \in A}$ and partition $A$ into $A_1, A_2$ with $|A_1| = V$. Choose $\{y_\alpha\}_{\alpha \in A_1} \subset P_0(s)$, and let $s_1 = \prod_{\alpha \in A_1}(s(x_\alpha), y_\alpha) \in S(P, Y)$. Then $r = s^{-1}s_1 s \in \langle J(Y,Z), s\rangle$, $Q_r = \{x_\alpha\}_{\alpha \in A_1}$, and $P_r = \{y_\alpha\}_{\alpha \in A_1}$. Hence $|Q_r| = |P_r| = V$, so $T(r) = V$.

1.8. LEMMA. *Let* $Z < X$ *or* $Y < X^+$, $s, r \in S(X, Y) - J(Y,Z)$, $|Q_s| = |Q_r|$, $|P_s| = |P_r|$. *Then* $r \in \langle J(Y,Z), s\rangle$.

**Proof.** Now $|Q_s| \leqq \min\{Y,Z\}$, $|P_s| \leqq \min\{Y,Z\}$, $|P - P_s| = X = |P - P_r|$.

*Case* 1. $|Q - Q_s| = |Q - Q_r|$. In particular the situations where $|Q_s| < Z$ or $Z < d$ are included. If $|Q - Q_s| < Y$, choose $s_1 \in S(Q, Y)$ such that $s_1(Q_r) = Q_s$ and $s_1(Q - Q_r) = Q - Q_s$. If $|Q - Q_s| \geqq Y = d$, choose $s_1 \in S(Q, Y)$ such that

$s_1(Q_r) = Q_s$. If $|Q - Q_s| \geqq Y > d$, then $|Q - (Q_s \cup Q_r)| \geqq Y$. Partition $Q - (Q_s \cup Q_r)$ into $Q_1, Q_2, Q_3$ such that $|Q_1| = |Q_s - Q_r|$, $d \leqq |Q_2| < Y$, $|Q_2| \geqq |Q_r| \geqq |Q_1|$. Choose $s_1 \in S(Q, Y)$ such that $s_1(Q_r) = Q_s$, $s_1(Q_s - Q_r) = Q_1$, $s_1(Q_1 \cup Q_2) = Q_2 \cup (Q_r - Q_s)$, $s_1$ fixes $Q_3$. If $Y = d$, choose $s_2 \in S(P, Y)$ such that $s_2(P_r) = P_s$. If $Y > d$, partition $P - (P_s \cup P_r)$ into $P_1$, $P_2$, $P_3$ such that $|P_1| = |P_s - P_r|$, $d \leqq |P_2| < Y$, $|P_2| \geqq |P_r| \geqq |P_1|$. Choose $s_2 \in S(P, Y)$ such that $s_2(P_r) = P_s$, $s_2(P_s - P_r) = P_1$, $s_2(P_1 \cup P_2) = P_2 \cup (P_r - P_s)$, $s_2$ fixes $P_3$. For $s_1$ and $s_2$ chosen for the appropriate situation as above, $s_3 = s s_2 s_1 \in \langle J(Y,Z), s\rangle$, $Q_{s_3} = Q_r$, $P_{s_3} = P_r$. By Lemma 1.5, $r \in \langle J(Y,Z), s_3\rangle \subseteq \langle J(Y,Z), s\rangle$.

*Case* 2. $|Q - Q_s| \neq |Q - Q_r|$, $|P_s| < Z$. This occurs only if $Z \geqq d$. Now $|Q_{s^{-1}}| = |P_s| = |P_r| = |Q_{r^{-1}}| < Z$, so $|Q - Q_{s^{-1}}| = |Q - Q_{r^{-1}}|$. Thus Case 1 may be applied to show $r^{-1} \in \langle J(Y,Z), s^{-1}\rangle = \langle J(Y,Z), s\rangle$, so $r \in \langle J(Y,Z), s\rangle$.

*Case* 3. $|Q - Q_s| \neq |Q - Q_r|, |P_s| = Z$. Note that $Y > Z \geqq d$ and $|Q_s| = Z = |Q_r|$. It may be assumed that $s$ has the properties implied by Lemma 1.3. Denote $Q_s = \{x_\alpha\}_{\alpha \in A}$ and partition $A$ into $A_1, A_2$ with $|A_1| = |A_2| = Z$. Choose $\{y_\alpha\}_{\alpha \in A_2} \subset P_0(s)$ and define $s_1 \in S(P, Y)$ by $s_1 = \prod_{\alpha \in A_2}(s(x_\alpha), y_\alpha)$. Then $s_2 = s^{-1} s_1 s \in \langle J(Y,Z), s\rangle$, $Q_{s_2} = \{x_\alpha\}_{\alpha \in A_2}$, $\{x_\alpha\}_{\alpha \in A_1} \subseteq Q - Q_{s_2}$, and $P_{s_2} = \{y_\alpha\}_{\alpha \in A_2}$. Thus $|Q_{s_2}| = |P_{s_2}| = |Q - Q_{s_2}| = Z$. (Actually $s_2 = \prod_{\alpha \in A_2}(x_\alpha, y_\alpha)$.) If $|Q - Q_r| = Z$, set $r_1 = s_2$. If $|Q - Q_r| = W < Z$, partition $Q - Q_{s_2}$ into $Q_1, Q_2$ such that $|Q_1| = Z$, $|Q_2| = W$. Denote $Q_1 = \{x'_\alpha\}_{\alpha \in A_2}$. Define $s_3 \in S(Q, Y)$ by $s_3 = \prod_{\alpha \in A_2}(x_\alpha, x'_\alpha)$. Choose $\{y'_\alpha\}_{\alpha \in A_2} \subseteq P_0(s) - \{y_\alpha\}_{\alpha \in A_2}$, and define $s_4 = \prod_{\alpha \in A_2}(y_\alpha, y'_\alpha) \in S(P, Y)$. Then $s_5 = s_2 s_4 s_3 s_2 \in \langle J(Y,Z), s_2\rangle \subseteq \langle J(Y,Z), s\rangle$, $Q_{s_5} = Q_{s_2} \cup Q_1$, $Q - Q_{s_1} = Q_2$, $P_{s_5} = \{y_\alpha\}_{\alpha \in A_2} \cup \{y'_\alpha\}_{\alpha \in A_2}$. Hence $|Q_{s_5}| = Z = |P_{s_5}|$ and $|Q - Q_{s_5}| = W$. Put $r_1 = s_5$.

Let $r_1$ be as defined for the pertinent situation, then $|Q_{r_1}| = |Q_r|$, $|P_{r_1}| = |P_r|$, and $|Q - Q_{r_1}| = |Q - Q_r|$. Case 1 may now be applied to yield $r \in \langle J(Y,Z), r_1\rangle \subseteq \langle J(Y,Z), s\rangle$.

1.9. Lemma. *Let $Z < X$ or $Y < X^+$, $s \in S(X, Y) - J(Y,Z)$, with $|Q_s| = W \geqq d$. Let $0 \leqq V \leqq W$. Then there is $r \in \langle J(Y,Z), s\rangle$ such that*:

(1) $|P_0(r)| = X$.

(2) $r(Q_r) \cap P_r = \varnothing$.

(3) $|Q_r| = W$.

(4) $|Q_0(r)| = Z$ *where* $Q_0(r) = fs\ r \cap Q$.

(5) $|P_r| = V$.

(6) $r(P_r) \cap Q_r = \varnothing$.

**Proof.** Suppose $W < Z$. Then $Z > d$. By Lemma 1.6, there is $s_1 \in \langle J(Y,Z), s\rangle$ with $|Q_{s_1}| = W$, $|P_{s_1}| = V$. Furthermore it may be assumed that $s_1$ has the properties implied by Lemma 1.3. Thus $s_1$ satisfies (1), (2), (3), (5). Now $|Q_{s_1} \cup s_1(P_{s_1})| < Z$, so consider the cyclic decomposition of $s_1$. Let $C_x$ denote the cycle in $s_1$ in which $x$ occurs and let $Q_1 = \{x \in Q : C_x \cap (Q_{s_1} \cup s_1(P_{s_1})) = \varnothing\}$. Since there are less than $Z$ cycles which intersect $Q_{s_1} \cup s_1(P_{s_1})$, $|Q_1| = Z$. Define $s_2 \in S(Q,Y)$ by $s_2(x) = x$ if $x \in Q - Q_1$ and $s_2(x) = s_1^{-1}(x)$ if $x \in Q_1$. Then $r_1 = s_2 s_1 \in \langle J(Y,Z), s_1\rangle \subseteq \langle J(Y,Z), s\rangle$, $Q_1 = fs\ r_1 \cap Q$, $Q_{r_1} = Q_{s_1}$, $P_{r_1} = P_{s_1}$, $r_1(Q_{r_1}) = s_1(Q_{s_1})$, $P_0(r_1) = P_0(s_1)$. Thus $r_1$ satisfies (1), (2), (3), (4) and (5).

Suppose $W = Z$. It may be assumed that $s$ has the properties implied by Lemma 1.3. Denote $Q_s = \{x_\alpha\}_{\alpha \in A}$ and partition $A$ into $A_1$, $A_2$ such that $|A_1| = \angle = |A_2|$. Choose $\{y_\alpha\}_{\alpha \in A_2} \subset P_0(s)$, and define $s_1 \in S(P,Y)$ by $s_1 = \prod_{\alpha \in A_2}(s(x_\alpha), y_\alpha)$. Then $s_2 = s^{-1}s_1 s = \prod_{\alpha \in A_2}(x_\alpha, y_\alpha) \in \langle J(Y,Z), s_1\rangle \subseteq \langle J(Y,Z), s\rangle$, $Q_{s_2} = \{x_\alpha\}_{\alpha \in A_2}$, $fs$ $s_2 \cap Q = Q - Q_{s_2} \supseteq \{x_\alpha\}_{\alpha \in A_1}$, $P_{s_2} = \{y_\alpha\}_{\alpha \in A_2}$, $fs\ s_2 \cap P = P - P_{s_2}$. Thus $s_2$ satisfies (1), (3), (4). Note that $|P_{s_2}| = Z$. Choose $\{y'_\alpha\}_{\alpha \in A_2} \subset P - \{y_\alpha\}_{\alpha \in A_2}$ and define $s_3 = \prod_{\alpha \in A_2}(y_\alpha, y'_\alpha)$. Then $s_4 = s_3 s_2 \in \langle J(Y,Z), s_2\rangle \subseteq \langle J(Y,Z), s\rangle$, $Q_{s_4} = Q_{s_2}$, $P_{s_4} = P_{s_2}$, $Q_0(s_4) = Q_0(s_2)$. Thus $s_4$ satisfies (1), (2), (3), (4). If $V = Z$, then $s_4$ also satisfies (5), so put $r_1 = s_4$. If $V < Z$, apply to $s_4$ the construction used in the proof of Lemma 1.6 to yield $s_5 \in \langle J(Y,Z), s_4\rangle \subseteq \langle J(Y,Z), s\rangle$, $|P_{s_5}| = V$, $|Q_{s_5}| = Z$, $Q_0(s_4) \subseteq Q_0(s_5)$. Thus $s_5$ satisfies (3), (4), (5). Now apply to $s_5$ the appropriate construction from the proof of Lemma 1.3 to yield $s_6 \in \langle J(Y,Z), s_5\rangle \subseteq \langle J(Y,Z), s\rangle$ satisfying (1), (2), (5). Since this construction composes $s_5$ only with permutations in $S(P,Y)$, properties (3) and (4) are preserved. Put $r_1 = s_6$. Then $r_1$ satisfies (1), (2), (3), (4), and (5).

Let $r_1 \in \langle J(Y,Z), s\rangle$, satisfying (1) through (5), be defined as above for the appropriate situation. Partition $Q_0(r_1)$ into $Q_1, Q_2$ such that $|Q_1| = Z = |Q_2|$. Denote $P_{r_1} = \{y_\beta\}_{\beta \in B}$, choose $\{x_\beta\}_{\beta \in B} \subseteq Q_1$, and define $s_7 = \prod_{\beta \in B}(r_1(y_\beta), x_\beta) \in S(Q,Y)$. Then $r = s_7 r_1 \in \langle J(Y,Z), r_1\rangle \subseteq \langle Y(J,Z), s\rangle$, $Q_r = Q_{r_1}$, $P_r = P_{r_1}$, $r(P_r) \subseteq Q_1$, $Q_2 \subseteq Q_0(r)$, $r(Q_r) = r_1(Q_{r_1})$, $P_0(r) = P_0(r_1)$. Thus $r$ satisfies all conditions.

1.10. Lemma. *Let $Z < X$ or $Y < X^+$, $s \in S(X,Y) - J(Y,Z)$, $|Q_s| = W \geqq d$, $0 \leqq m \leqq n < d$. Then there is $r \in \langle J(Y,Z), s\rangle$ such that $|Q_r| = n$ and $|P_r| = m$.*

**Proof.** If $n = 0$, choose $r \in J(Y,Z)$. If $n > 0$, it may be assumed that $s$ has all properties implied by Lemma 1.9, in particular that $|P_s| = m$. Denote $Q_s = \{x_\alpha\}_{\alpha \in A}$ and $P_s = \{y_i\}_{i=1}^m$. Choose $\{x_i\}_{i=1}^m \subset Q_0(s)$ and define $s_1 = \prod_{i=1}^m(s(y_i), x_i) \in S(Q,Y)$. Partition $A$ into $A_1, A_2$ such that $|A_1| = W$, $|A_2| = n - m$. Partition $P_0(s)$ into $P_1, P_2, P_3$ such that $|P_1| = n - m$, $|P_2| = d$, $|P_3| = X$. Define $s_2 \in S(P,Y)$ such that $s_2(\{s(x_\alpha)\}_{\alpha \in A_1}) = \{s(x_\alpha)\}_{\alpha \in A}$, $s_2(\{s(x_\alpha)\}_{\alpha \in A_2}) = P_1$, $s_2(P_1 \cup P_2) = P_2$, $s_2$ fixes $P_3$. Then $r = s^{-1}s_2 s_1 s \in \langle J(Y,Z), s\rangle$, $Q_r = \{x_i\}_{i=1}^m \cup \{x_\alpha\}_{\alpha \in A_2}$, $P_r = P_s$. Thus $|Q_r| = n$, $|P_r| = m$.

1.11. Lemma. *Let $Z < X$ or $Y < X^+$, $s \in S(X,Y) - J(Y,Z)$, $T(s) = W$, $d \leqq W \leqq Z$. Let $r \in S(X,Y) - J(Y,Z)$, $T(r) \leqq W$. Then $r \in \langle J(Y,Z), s\rangle$.*

**Proof.** It may be assumed that $|Q_s| \geqq |P_s|$ and $|Q_r| \geqq |P_r|$, since otherwise one may consider inverses of $s$ or $r$.

*Case* 1. $|Q_r| = W$, $|P_r| = V \leqq W$. By Lemma 1.6, there is $s_1 \in \langle J(Y,Z), s\rangle$ such that $|Q_{s_1}| = W$, $|P_{s_1}| = V$. Then by Lemma 1.8, $r \in \langle J(Y,Z), s_1\rangle \subseteq \langle J(Y,Z), s\rangle$.

*Case* 2. $T(r) = V$, $d \leqq V < W$. By Lemma 1.7, there is $s_1 \in \langle J(Y,Z), s\rangle$ such that $T(s_1) = V$. Then by case 1, $r \in \langle J(Y,Z), s_1\rangle \subseteq \langle J(Y,Z), s\rangle$.

*Case* 3. $T(r) = n < d$. In this case $|Q_r| = n$, $|P_r| = m$, $n \geqq m$. By Lemma 1.10, there is $s_1 \in \langle J(Y,Z), s\rangle$ such that $|Q_{s_1}| = n$, $|P_{s_1}| = m$. By Lemma 1.8, $r \in \langle J(Y,Z), s_1\rangle \subseteq \langle J(Y,Z), s\rangle$.

1.12. Theorem. *Let $Z < d$. Then for each $Y$, $d \leqq Y \leqq X^+$, $J(Y,Z)$ is a maximal subgroup of $S(X,Y)$.*

**Proof.** Let $s \in S(X,Y) - J(Y,Z)$, $r \in S(X,Y)$. Then $T(r) = m$, $0 \leqq m \leqq Z$. If $m = 0$, then $r \in J(Y,Z) \subseteq \langle J(Y,Z), s\rangle$. If $m \neq 0$, denote $Q_r = \{x_i\}_{i=1}^m$ and $P_r = \{y_i\}_{i=1}^m$. By Lemma 1.4, $s_1 = \prod_{i=1}^m (x_i, y_i) \in \langle J(Y,Z), s\rangle$. By Lemma 1.5, $r \in \langle J(Y,Z), s_1\rangle \subseteq \langle J(Y,Z), s\rangle$. Thus $S(X,Y) \subseteq \langle J(Y,Z), s\rangle$. The reverse inclusion is obvious.

1.13. Theorem. *If $J(d,Z)$ is a maximal subgroup of $S(X,d)$, then $J(A,Z)$ is a maximal subgroup of $A(X)$.*

**Proof.** Let $s, r \in A(X) - J(A,Z)$. Now $r \in \langle J(d,Z), s\rangle$, so $r = \prod_{i=1}^n s_i$, $n < d$, where $s_i \in J(d,Z)$ or $s_i = s^m$ for some integer $m$, $i = 1, \cdots, n$. Let $P_0 = P - (\bigcup_{i=1}^n \operatorname{spt} s_i)$ $|P_0| = X$. Choose a transposition $t \in S(P,d) - A(X)$ such that $\operatorname{spt} t \subset P_0$. Then $r = (\prod_{i=1}^n s_i')t^k$ where $s_i' = s_i$ if $s_i \in A(X)$ and $s_i' = s_i t$ if $s_i \notin A(X)$. Now $\prod_{i=1}^n s_i' \in \langle J(A,Z), s\rangle$, so $\prod_{i=1}^n s_i' \in A(X)$ and $t^k = (\prod_{i=1}^n s')^{-1} r \in A(X) \cap S(P, d) \subseteq J(A,Z)$. Hence $r \in \langle J(A,Z), s\rangle$.

1.14. Theorem. *Let $d \leqq Z \leqq X$. Then $J(d,Z)$ is maximal in $S(X,d)$ and $J(A,Z)$ is maximal in $A(X)$.*

**Proof.** Let $s, r \in S(X,d) - J(d,Z)$. By Theorem 1.1, $|Q_r| = |P_r| = n < d$. Denote $Q_r = \{x_i\}_{i=1}^n$ and $P_r = \{y_i\}_{i=1}^n$. By Lemma 1.4, $s_1 = \prod_{i=1}^n (x_i, y_i) \in \langle J(d,Z), s\rangle$. By Lemma 1.5, $r \in \langle J(d,Z), s_1\rangle \subseteq \langle J(d,Z), s\rangle$. Thus $J(d,Z)$ is maximal in $S(X,d)$, and by Theorem 1.13, $J(A,Z)$ is maximal in $A(X)$.

1.15. Lemma. *For each $s, r \in S(X, X^+)$, $T(rs) \leqq T(s) + T(r)$.*

**Proof.** $Q_{rs} = [Q_s \cap s^{-1}(P_r')] \cup [Q_s' \cap s^{-1}(Q_r)]$. Hence $|Q_{rs}| \leqq |Q_s| + |Q_r| \leqq T(s) + T(r)$. Similarly $|P_{rs}| \leqq T(s) + T(r)$. Thus $T(rs) \leqq T(s) + T(r)$.

1.16. Theorem. *If $Z \geqq d$, $Y > d$, then $J(Y,Z)$ is not maximal in $S(X,Y)$.*

**Proof.** Let $s \in S(X,Y) - J(Y,Z)$ with $T(s) < d$. By Lemma 1.15, $r \in \langle J(Y, Z), s\rangle$ implies that $T(r) < d$. Thus $\langle J(Y,Z), s\rangle$ contains no permutations with infinite transfer index. Since $S(X,Y)$ contains permutations with infinite transfer index, $\langle J(Y,Z), s\rangle \neq S(X,Y)$.

For each $Y$, $d < Y \leqq X^+$, let $C_1(Y)$ be the class of all $J(Y,Z)$ subgroups with $1 \leqq Z < d$. Let $C_1(d)$ and $C_1(A)$ be the class of $J(d,Z)$ subgroups, respectively $J(A,Z)$ subgroups, with $1 \leqq Z \leqq X$. By Theorems 1.12 and 1.14, if $H \in C_1(Y)$

or $C_1(A)$, then $H$ is maximal in $S(X, Y)$, respectively $A(X)$. Now $H \in C_1(Y)$ implies that $H$ is intransitive and contains no subgroup normal in $S(X, Y)$; $H \in C_1(A)$ implies $H$ is intransitive.

1.17. THEOREM. *Let $G$ be $S(X, Y)$ or $A(X)$ and let $H$ be an intransitive maximal subgroup of $G$. Then $H \in C_1(Y)$ or $C_1(A)$, respectively.*

**Proof.** Since $H$ is intransitive, the sets of transitivity of $H$ form a nontrivial partition of $M$. Choose $Q$ and $P, |Q| \leqq |P|$, partitioning $M$ such that every set of transitivity of $H$ is a subset of $Q$ or a subset of $P$. Denote $|Q| = Z$, and let $J(Z) = S(Q)\cdot S(P)$. If $G = S(X, Y)$, $Y > d$, then $H \leqq J(Y,Z) < S(X, Y)$, so $H = J(Y,Z)$. By Theorem 1.16 and Theorem 1.12, $1 \leqq Z < d$ and $H \in C_1(Y)$. If $G = S(X,d)$, then $H \leqq J(d,Z) < S(X,d)$, so $H = J(d,Z)$. By Theorem 1.12 and Theorem 1.14, $H \in C_1(d)$. If $G = A(X)$, then $H \leqq J(A,Z) < A(X)$, so $H = J(A,Z)$. By Theorem 1.13 and Theorem 1.14, $H \in C_1(A)$.

Theorem 1.17 shows that the preceding constructions yield all possible intransitive maximal subgroups of the infinite symmetric groups. Theorem 3.12 will show that $C_1(Y)$ is the class of all maximal subgroups of $S(X, Y)$ containing no subgroups normal in $S(X, Y)$ if $Y > d$.

For the finite symmetric groups, let $|M| = n < d$. Partition $M$ into sets $P$ and $Q$ such that $|P| \geqq |Q|$. Let $p = |P|$, $q = |Q|$, $J(q) = S(Q)\cdot S(P)$.

1.18. THEOREM. *$H$ is an intransitive maximal subgroup of $S_n$ if and only if $H = J(q)$, $q < p$.*

**Proof.** Suppose $H = J(q)$, $q < p$. Let $s \in S_n - J(q)$ and let $(x, y)$ be any transposition not in $J(q)$. It may be assumed that $x \in Q$ and $y \in P$. Let $x_1 \in Q_s$, $y_1 \in P - P_s$. Then $(x_1, y_1) = s^{-1}(s(x_1), s(y_1))s \in \langle J(q), s\rangle$. Thus $(x, y) = (y, y_1)(x, x_1)(x_1, y_1)(x, x_1)(y, y_1) \in \langle J(q), s\rangle$. Since an arbitrary transposition is in $\langle J(q), s\rangle$, $\langle J(q), s\rangle = S_n$. Thus $H$ is maximal and intransitive.

Suppose $H$ is an intransitive maximal subgroup of $S_n$. Then $H \leqq J(q)$ for some $q$, so $H = J(q)$ by maximality. If $q = p$, let $s \in S_n - H$ be chosen such that $s(Q) = P$. Then $\langle H, s\rangle \neq S_n$. Hence $q < p$.

Thus all possible intransitive subgroups of $S_n$ are also determined.

## II. TRANSITIVE MAXIMAL SUBGROUPS

2.1. DEFINITION. If $d \leqq Z \leqq Y \leqq X^+$, $Z < X$, define $L(Y,Z) = \{s \in S(X, Y): T(s) < Z\}$. If $d < Y \leqq Z \leqq X$ and $Y$ has a predecessor $Y^-$, define

$$L(Y,Z) = \{s \in S(X, Y): T(s) < Y^-\}.$$

It will be noted that no attempt is made to define $L(Y,Z)$ in the case where $d < Y \leqq Z \leqq X$ where $Y$ has no direct predecessor. This will be clarified later by Theorem 3.17.

The following theorem is one of the principal results of this paper; however the proofs are straightforward applications of Lemmas 1.11 and 1.15 and will be omitted.

2.2. THEOREM. *The following statements are true*:

(1) $L(Y,Z) < S(X,Y)$.

(2) *If* $d \leqq W \leqq Z, W < Y$, *then* $S(X,W) \subseteq L(Y,Z)$.

*In particular* $S(X,d) \subseteq L(Y,Z)$.

(3) *If* $W > Z$ *or* $W \geqq Y$, *then* $S(X,W) \nsubseteq L(Y,Z)$.

(4) *If* $d < Z < Y$, *then* $L(Y,Z) = J(Y,Z) \cdot S(X,Z)$.

(5) *If* $d^+ < Y \leqq Z \leqq X$, *then* $L(Y,Z) = J(Y,Z) \cdot S(X,Y^-)$.

(6) *If* $d = Z < Y$, *then* $J(Y,Z) \cdot S(X,d) < L(Y,Z)$.

(7) *If* $d^+ = Y \leqq Z \leqq X$, *then* $J(Y,Z) \cdot S(X,d) < L(Y,Z)$.

(8) *If* $Z = W^+ < Y$, *then* $L(Y,Z) = \langle J(Y,Z), s\rangle$ *for any* $s \in L(Y,Z)$ *such that* $T(s) = W$. *If* $Y^- = W^+ < Z \leqq X$, *then* $L(Y,Z) = \langle J(Y,Z), s\rangle$ *for any* $s \in L(Y,Z)$ *such that* $T(s) = W$.

(9) $L(Y,Z)$ *is maximal in* $S(X,Y)$.

It is also possible to generate maximal subgroups over partitions of $M$ in the case where $Z = X$, $Y = X^+$.

2.3. DEFINITION. For each $s \in S(X,X^+)$, define the *remainder index of* $s = R(s) = \max\{|Q'_s|, |P'_s|\}$.

The next lemma is clear, so the proof is omitted.

2.4. LEMMA. *The following equations are true for every* $s, r \in S(X,X^+)$:

(1) $Q_{rs} = [Q_s \cap s^{-1}(P'_r)] \cup [Q'_s \cap s^{-1}(Q_r)]$.

(2) $P_{rs} = [P_s \cap s^{-1}(Q'_r)] \cup [P'_s \cap s^{-1}(P_r)]$.

(3) $Q'_{rs} = [Q_s \cap s^{-1}(P_r)] \cup [Q'_s \cap s^{-1}(Q'_r)]$.

(4) $P'_{rs} = [P_s \cap s^{-1}(Q_r)] \cup [P'_s \cap s^{-1}(P'_r)]$.

In the following lemma statement (1) is a direct restatement of Lemma 1.15. The proofs of statements (2) through (8) follow from Lemma 2.4 in the manner of the proof for Lemma 1.15.

2.5. LEMMA. *The following inequalities are true for every* $s, r \in S(X,X^+)$:

(1) $T(rs) \leqq T(s) + T(r)$.

(2) $T(rs) \leqq R(s) + R(r)$.

(3) $T(rs) \leqq T(s) + R(s)$.

(4) $T(rs) \leqq T(r) + R(r)$.

(5) $R(rs) \leqq T(s) + R(r)$.

(6) $R(rs) \leqq T(s) + R(s)$.

(7) $R(rs) \leqq T(r) + R(s)$.

(8) $R(rs) \leqq T(r) + R(r)$.

The following lemma is an immediate consequence of Lemma 2.5.

2.6. LEMMA. *Let* $s,r \in S(X,X^+)$. *If* $\min\{T(s),R(s)\} < W$ *and* $\min\{T(r),R(r)\} < W$, *then* $T(rs) < 2W$ *or* $R(rs) < 2W$.

2.7. DEFINITION. If $Z = X$, define $L(X^+,X) = \{s \in S(X,X^+): T(s) < X$ or $R(s) < X\}$.

2.8. LEMMA. *Let* $Z = X, s, r \in S(X,X^+) - L(X^+,X)$ *such that* $|Q_r| = |P_r| = |Q'_r| = |P'_r| = X$. *Then* $r \in \langle L(X^+,X), s\rangle$.

**Proof.** Since $s \in S(X,X^+) - L(X^+,X)$, $T(s) = X = R(s)$, so $s$ must satisfy one of the cases listed in the following table:

| Case | $\lvert Q_s\rvert$ | $\lvert Q'_s\rvert$ | $\lvert P_s\rvert$ | $\lvert P'_s\rvert$ |
|---|---|---|---|---|
| 1 | $X$ | $X$ | $X$ | $<X$ |
| 1′ | $X$ | $<X$ | $X$ | $X$ |
| 2 | $X$ | $X$ | $<X$ | $X$ |
| 2′ | $<X$ | $X$ | $X$ | $X$ |
| 3 | $X$ | $X$ | $X$ | $X$ |

By the obvious symmetry induced by $Z = X$, case 1′ is essentially the same as case 1, and case 2′ is essentially the same as case 2. Therefore it suffices to consider cases 1, 2, and 3.

If case 1 holds, partition $Q_s$ into $Q_1$, $Q_2$ and $Q'_s$ into $Q_3, Q_4$, all of cardinality $X$. Choose $s_1 \in S(Q)$ such that $s_1(s(Q'_s)) = Q_1 \cup Q_3$ and $s_1(s(P_s)) = Q_2 \cup Q_4$. Put $s_2 = ss_1s$. If case 2 holds, partition $s(P'_s)$ into $P_1$, $P_2$ and $s(Q_s)$ into $P_3, P_4$, all of cardinality $X$. Choose $s_1 \in S(P)$ such that $s_1(s(Q_s)) = P_1 \cup P_3$ and $s_1(s(P'_s)) = P_2 \cup P_4$. Put $s_2 = s^{-1}s_1s$. If case 3 holds, put $s_2 = s$.

In whichever case applies, $s_2 \in \langle L(X^+,X), s\rangle$ and $|Q_{s_2}| = |P_{s_2}| = |Q'_{s_2}| = |P'_{s_2}| = X$. Choose $s_3 \in S(Q)$ and $s_4 \in S(P)$ such that $s_3(Q_r) = Q_{s_2}, s_3(Q'_r) = Q'_{s_2}$, $s_4(P_r) = P_{s_2}, s_4(P'_r) = P'_{s_2}$. Then $s_5 = s_2s_4s_3 \in \langle L(X^+,X), s_2\rangle \subseteq \langle L(X^+,X), s\rangle$, $Q_{s_5} = Q_r$, $P_{s_5} = P_r$. By Lemma 1.5, $r \in \langle J(X^+,X), s_5\rangle \subseteq \langle L(X^+,X), s\rangle$.

2.9. LEMMA. *Let* $Z = X, s, r \in S(X,X^+) - L(X^+,X)$, *then* $r \in \langle L(X^+,X), s\rangle$.

**Proof.** By Lemma 2.8, choose $s_1 \in \langle L(X^+,X), s\rangle$ such that $|Q_{s_1}| = |P_{s_1}| = |Q'_{s_1}| = |P'_{s_1}| = X$. Now $r$ must fit one of the cases displayed in the table listed in the proof of Lemma 2.8. Again by the obvious symmetry, it suffices to consider cases 1, 2, and 3.

If case 1 holds, denote $|P_r'| = Y < X$. Partition $Q_{s_1}'$ into $Q_1, Q_2$ with $|Q_1| = X = |Q_2|$. Choose $s_2 \in S(Q)$ such that $s_2(s_1(Q_{s_1}')) = Q_1 \cup Q_{s_1}$ and $s_2(s_1(P_{s_1})) = Q_2$. Partition $s_1(P_{s_1}')$ into $P_1, P_2$ and $s_1(Q_{s_1})$ into $P_3$, $P_4$ such that $|P_1| = Y$ and $|P_2| = |P_3| = |P_4| = X$. Partition $P_{s_1}'$ into $P_5$, $P_6$ and $P_{s_1}$ into $P_7$, $P_8$ such that $|P_5| = Y$ and $|P_6| = |P_7| = |P_8| = X$. Choose $s_3 \in S(P)$ such that $s_3(P_1) = P_5$, $s_3(P_2) = P_7$, $s_3(P_3) = P_6$, $s_3(P_4) = P_8$. Put $s_4 = s_1 s_3 s_2 s_1 \in \langle L(X^+, X), s_1 \rangle \subseteq \langle L(X^+, X), s \rangle$.

If case 2 holds, denote $|P_r| = Y < X$. Partition $s_1(P_{s_1})$ into $Q_1, Q_2$ and $s_1(Q_{s_1}')$ into $Q_3, Q_4$ such that $|Q_1| = Y$ and $|Q_2| = |Q_3| = |Q_4| = X$. Partition $Q_{s_1}'$ into $Q_5, Q_6$ and $Q_{s_1}$ into $Q_7, Q_8$ such that $|Q_5| = Y$ and $|Q_6| = |Q_7| = |Q_8| = X$. Choose $s_2 \in S(Q)$ such that $s_2(Q_1) = Q_5$, $s_2(Q_2) = Q_7$, $s_2(Q_3) = Q_6$, $s_2(Q_4) = Q_8$. Partition $P_{s_1}'$ into $P_1, P_2$ such that $|P_1| = X = |P_2|$. Choose $s_3 \in S(P)$ such that $s_3(s_1(Q_{s_1})) = P_{s_1} \cup P_1$ and $s_3(s_1(P_{s_1}')) = P_2$. Put $s_4 = s_1 s_3 s_2 s_1 \in \langle L(X^+, X), s_1 \rangle \subseteq \langle L(X^+, X), s \rangle$.

If case 3 holds, put $s_4 = s_1$. Then in any case, $|Q_{s_4}| = |Q_r|$, $|Q_{s_4}'| = |Q_r'|$, $|P_{s_4}| = |P_r|$, $|P_{s_4}'| = |P_r'|$. Choose $s_5 \in S(Q)$ and $s_6 \in S(P)$ such that $s_5(Q_r) = Q_{s_4}$, $s_5(Q_r') = Q_{s_4}'$, $s_6(P_r) = P_{s_4}$, $s_6(P_r) = P_{s_4}$. Then $s_7 = s_4 s_6 s_5 \in \langle L(X^+, X), s_4 \rangle \subseteq \langle L(X^+, X), s \rangle$, $Q_{s_7} = Q_r$, $P_{s_7} = P_r$. By Lemma 1.5, $r \in \langle J(X^+, X), s_7 \rangle \subseteq \langle L(X^+, X), s \rangle$.

2.10. Theorem. *Let $Z = X$. Then the following statements are true*:

(1) $S(X, X) < L(X^+, X)$.

(2) $L(X^+, X) < S(X, X^+)$.

(3) *If* $s \in S(X, X^+) - L(X^+, X)$, *then* $\langle L(X^+, X), s \rangle = S(X, X^+)$.

(4) $L$ *is a maximal subgroup of* $S(X, X^+)$.

**Proof.** If $s \in S(X, X)$, then $T(s) < X$, so $s \in L(X^+, X)$. There is $r \in L(X^+, X)$ such that $T(r) = X$, so $r \notin S(X, X)$. Thus (1) holds.

By Lemma 2.6, if $s, r \in L(X^+, X)$, then $rs \in L(X^+, X)$. For every $s \in S(X, X^+)$, $T(s) = T(s^{-1})$ and $R(s) = R(s^{-1})$. Thus $L(X^+, X)$ is a subgroup of $S(X, X^+)$. Furthermore there is $s \in S(X, X^+)$ such that $T(s) = X = R(s)$. Hence (2) holds. Statement (3) holds by Lemma 2.9, and statement (4) is a consequence of (2) and (3).

In summary, when $Z = X$, defining $L(X^+, X) = \{s \in S(X, X^+) : T(s) < X$ or $R(s) < X\}$ yields a transitive maximal subgroup of $S(X, X^+)$. The usual definition of $\{s \in S(X, X^+) : T(s) < X\}$ does not yield a maximal subgroup since it is a proper subgroup of $L(X^+, X)$. For each $Y$, $d < Y \leqq X^+$, let $C_2(Y)$ denote the class of all possible $L(Y, Z)$ subgroups. If $H \in C_2(Y)$ then $H$ is a transitive maximal subgroup of $S(X, Y)$ and $S(X, d) \subset H$.

## III. Properties of Transitive Maximal Subgroups

An obvious conjecture to follow the construction of transitive maximal subgroups of $S(X, Y)$, $d < Y \leqq X^+$, is whether or not $C_2(Y)$ is the class of all such subgroups of $S(X, Y)$. This is still an open question. Failing to answer this question, a logical conjecture is whether or not all transitive maximal subgroups of $S(X, Y)$ contain $S(X, d)$. Other conjectures are whether $A(X)$ contains any transitive maximal subgroups and whether $S(X, d)$ contains any transitive maximal subgroups other than $A(X)$. These questions will be answered in this chapter.

3.1. DEFINITION. Let $M$ be a set, $|M|$ arbitrary, $H \leqq S(M)$. A relation $F$ in $M \times M$ is an *H-relation* if whenever $(x, y) \in F$ and $h \in H$, then $(h(x), h(y)) \in F$. If an $H$-relation is an equivalence relation, it is called an *H-equivalence.*

It is immediate that an $H$-equivalence induces a partitioning of $M$ into equivalence classes called *sets of primitivity of* $H$.

The following material through Theorem 3.6 is taken from lecture notes by Wielandt [2], so proofs are omitted.

3.2. THEOREM. *The equivalence classes of an H-equivalence have equal cardinality.*

3.3. DEFINITION. A transitive permutation group $H$ on a set $M$ is called *primitive* if each $H$-equivalence on $M$ is trivial, that is if each $H$-equivalence is either $M \times M$ or $\Delta$, where $(x, y) \in \Delta$ if and only if $x = y$. Otherwise $H$ is called *imprimitive.*

3.4. THEOREM. *If H is n-ply transitive for some finite* $n \geqq 2$, *then H is primitive.*

3.5. THEOREM. *If H is n-set transitive for some finite* $n \geqq 2$ *and* $|M| > n + 1$, *then H is primitive.*

3.6. THEOREM. *If* $|M| \geqq d$, *H primitive on M, and* $H \cap S(X, d) \neq \{e\}$, *then* $A(X) \subseteq H$.

3.7. LEMMA. *Let* $d < Y \leqq X$, *H a maximal subgroup of* $S(X, Y)$. *Then* $H \cap S(X, d) \neq \{e\}$.

**Proof.** Suppose $H \cap S(X, d) = \{e\}$. Then $H \cdot A(X) = S(X, Y)$, so if $s \in S(X, d)$, then $s = hr$ with $h \in H$ and $r \in A(X)$. Hence $h = sr^{-1} \in S(X, d)$, so $h = e$ and $r = s$. Thus $s \in A(X)$ and $S(X, d) \subseteq A(X)$, which is not true.

3.8. THEOREM. *Let* $d \leqq Y \leqq X^+$, *H a transitive maximal subgroup of* $G = S(X, Y)$ or $A(X)$. *Then the following statements are equivalent:*

(1) *H is n-ply transitive for some* $n \geqq 2$.

(2) *H is n-set transitive for some* $n \geqq 2$.

(3) *H is primitive.*

(4) $A(X) \subseteq H$.

(5) *H is n-ply transitive for all n,* $0 < n < d$.

**Proof.** Clearly (1) implies (2), (4) implies (5), and (5) implies (1). By Theorem

3.5, (2) implies (3). If $G = S(X,d)$ or $A(X)$, then $H \cap S(X,d) \neq \{e\}$; and if $G = S(X,Y)$, $Y > d$, then by Lemma 3.7, $H \cap S(X,d) \neq \{e\}$. Then by Theorem 3.6, (3) implies (4).

As a consequence of Theorem 3.8, it is immediately clear that $A(X)$ has no maximal subgroups which are primitive, and that the only primitive maximal subgroup of $S(X,d)$ is $A(X)$.

3.9. Definition. Let $M$ be partitioned into $\{M_\alpha\}_{\alpha \in A}$. For each $s \in S(X, X^+)$ let $A_s = \{\alpha \in A : s(M_\alpha) \neq M_\beta$ for all $\beta \in A\}$, $A_s' = A - A_s$. Define $I(s) = |A_s|$ = *mixing index of* $s$.

3.10. Lemma. *For every* $s, r \in S(X, X^+)$, $I(rs) \leqq I(s) + I(r)$.

**Proof.** Now $s$ induces a one-to-one function $\bar{s}$ from $A_s'$ into $A$, defined by $\bar{s}(\alpha) = \beta$ if $s(M_\alpha) = M_\beta$. Then $A_{rs} \subseteq A_s \cup [A_s' \cap \{\alpha \in A:\ r(s(M_\alpha)) \neq M_\beta$ for all $\beta \in A\}] = A_s \cup [A_s' \cap \{\alpha \in A : r(M_{\bar{s}(\alpha)}) \neq M_\beta$ for all $\beta \in A\}] = A_s \cup [A_s' \cap \bar{s}^{-1}(A_r \cap \bar{s}(A_s'))]$. Thus $|A_{rs}| \leqq |A_s| + |A_r|$.

3.11. Lemma. *Let* $K = \prod_{\alpha \in A} S(M_\alpha)$ *and* $N(K)$ = *normalizer of* $K$ *in* $S(X, X^+)$. *Define* $M_{\alpha\beta}(s) = \{m \in M_\alpha:\ s(m) \in M_\beta\}$. *Then* $s \in N(K)$ *if and only if both of the following conditions hold for each* $\alpha, \beta \in A$.

(1) $M_{\alpha\beta}(s) = \varnothing$ *or* $M_{\alpha\beta}(s) = M_\alpha$.

(2) $s(M_{\alpha\beta}(s)) = \varnothing$ or $s(M_{\alpha\beta}(s)) = M_\beta$.

In view of Lemma 3.11, $s \in N(K)$ if and only if $I(s) = 0$.

3.12. Theorem. *Let* $d < Y \leqq X^+$, $H$ *a transitive maximal subgroup of* $S(X,Y)$. *Then* $A(X) \subset H$.

**Proof.** Suppose not. Then by Theorem 3.8, $H$ is imprimitive, so there is a nontrivial $H$-equivalence $F$ in $M \times M$ inducing a partition $\{M_\alpha\}_{\alpha \in A}$ on $M$ consisting of the equivalence classes of $F$. By Theorem 3.2, $|M_\alpha| = |M_\beta|$ for every $\alpha, \beta \in A$, and by the nontriviality of $F$, $|A| > 1$. and $|M_\alpha| > 1$. Let $K(Y) = [\prod_{\alpha \in A} S(M_\alpha)] \cap S(X,Y)$ and $N(K,Y) = N(K) \cap S(X,Y) = N(K(Y))$ in $S(X,Y)$. By Lemma 3.11, $H \leqq N(K,Y) < S(X,Y)$. Thus by the maximality of $H$, $H = N(K,Y)$.

Suppose $|A| \geqq d$. Select a countable subfamily $\{M_i\}_{i=1}^\infty$ of $\{M_\alpha\}_{\alpha \in A}$. Let $x_i \in M_i$, $s = (x_1, x_2)$, $r = \prod_{i=1}^\infty (x_{2n-1} x_{2n})$. Since $|M_i| \geqq 2$, $s, r \in S(X,Y) - N(K,Y)$. Now $I(s) = 2$, $I(r) = d$. Thus by Lemma 3.10, $r \notin \langle N(K,Y), s\rangle$.

Suppose $A = \{1, 2, \cdots, n\}$, $n < d$, then $|M_i| = X \geqq d$, $1 \leqq i \leqq n$. Let $s = (x,y)$ where $x \in M_1$, and $y \in M_2$. Now $s \in S(X,Y) - N(K,Y)$ and has the property that for each $i \in A$, $s(M_i) \cap M_j$ is infinite for exactly one $j \in A$. Observe that every permutation in $N(K,Y)$ also has this property. Suppose $s_1, s_2 \in S(X,Y)$ satisfy this property. Let $i \in A$, then there is unique $j \in A$ such that $|s_1(A_i) \cap A_j| \geqq d$, and there is unique $k \in A$ such that $|s_2(A_j) \cap A_k| \geqq d$. Then $k$ is the unique element of $A$ such that $|s_2 s_1(A_i) \cap A_k| \geqq d$. Thus $s_2 s_1$ satisfies the same property, and in particular every permutation in $\langle N(K, Y), s\rangle$ must satisfy this property. Now let $M_1$ be partitioned into $Q_1$ and $Q_2$, $M_2$ be partitioned into $P_1$ and $P_2$ such that $|P_1| = d = |Q_1|$ and $|P_2| = X = |Q_2|$. Choose $r \in S(X, Y)$ such that $r(Q_1) = P_1$, $r(P_1) = Q_1$, $r$ leaves $P_2$ and $Q_2$ fixed. Then $r$ fails to have the above property, so $r \notin \langle N(K, Y), s\rangle$.

In either case, $H = N(K, Y)$ is not maximal in $S(X, Y)$, which is contrary to the hypothesis. Hence $A(X) \subset H$.

3.13. Theorem. *$H$ is a transitive maximal subgroup of $S(X,d)$ not containing $A(X)$, or of $A(X)$ if and only if there is a partition $\{M_\alpha\}_{\alpha\in A}$ of $M$ such that $1 < |M_\alpha| = |M_\beta| < d$ for each $\alpha,\beta \in A$, and $H = N(K,d) = N(K) \cap S(X, d)$ or $H = N(K,A) = N(K) \cap A(X)$, respectively.*

**Proof.** Let $H$ be a transitive maximal subgroup of $S(X,d)$, $A(X) \nsubseteq H$. By Theorem 3.8, $H$ is imprimitive; and by Theorem 3.2 and the definition of imprimitivity, $1 < |M_\alpha| = |M_\beta|$ for each $\alpha,\beta\in A$ where $\{M_\alpha\}_{\alpha\in A}$ are the sets of primitivity of $H$. By Lemma 3.11 and the maximality of $H$, $H = N(K,d)$. If $|M_\alpha| \geqq d$, then $N(K,d) = K$ which is intransitive, contrary to the transitivity of $H$. Hence $|M_\alpha| < d$ and $|A| = X$. It now suffices to show that $N(K,d)$ is maximal. Let $s \in S(X,d) - N(K,d)$, $(x,y) \in S(X,d)$. There are $\alpha$, $\beta$, $\gamma \in A$, $\alpha \neq \beta$, $x_1 \in M_\alpha$, $y_1 \in M_\beta$ such that $s(x_1)$, $s(y_1) \in M_\gamma$. Let $s_1 = (s(x_1), s(y_1)) \in N(K,d)$. Then $(x_1, y_1) = s^{-1}s_1 s \in \langle N(K,d), s\rangle$. Now $x \in M_\delta$, $y \in M_\lambda$, for some $\delta,\lambda \in A$. If $\delta = \lambda$, then $(x,y) \in K \subset N(K,d) \subset \langle N(K,d), s\rangle$. If $\delta \neq \lambda$, there is $s_2 \in N(K,d)$ such that $s_2(x) = x_1$ and $s_2(y) = y_1$. Then $(x,y) = s_2^{-1}(x_1,y_1)s_2 \in \langle N(K,d), s\rangle$. Thus every transposition is in $\langle N(K,D), s\rangle$ so $S(X,d) \subseteq \langle N(K,d), s\rangle$. Hence $H = N(K,d)$ is maximal in $S(X,d)$.

The proof for $H$ a transitive maximal subgroup of $A(X)$ is the same as the proof of Theorem 1.13.

3.14. Theorem. *Let $d < Y \leqq X^+$, $H$ a maximal subgroup of $S(X,Y)$. If $A(X) \subset H$, then $S(X,d) \subset H$.*

**Proof.** Suppose not. Then $S(X,Y) = H \cdot S(X,d)$. Thus

$$\frac{S(X,Y)}{A(X)} = \frac{H \cdot S(X,d)}{A(X)} \cong \frac{H}{A(X)} \cdot \frac{S(X,d)}{A(X)},$$

so $[S(X,Y)/A(X) : H/A(X)] = 2$. Hence $H/A(X)$ is normal in $S(X,Y)/A(X)$. Denote the natural homomorphism of $S(X,Y)$ onto $S(X,Y)/A(X)$ by $f$. Then $H = f^{-1}(H/A(X))$ is normal and maximal, which is a contradiction.

3.15. Corollary. *Let $d < Y \leqq X^+$, $H$ a transitive maximal subgroup of $S(X,Y)$. Then $S(X,d) \subset H$.*

3.16. Lemma. *If $G$ is a subgroup of $S(X,X^+)$, then $G$ contains a largest subgroup normal in $S(X,X^+)$.*

No attempt was made in Definition 2.1 to treat the case where $d < Y \leqq Z \leqq X$ for $Y$ a cardinal number with no direct predecessor. The following theorem shows that in this case, a maximal subgroup cannot be constructed over the appropriate two-celled partition.

3.17. THEOREM. *If $d < Y \leqq Z \leqq X$, and $Y$ has no predecessor, then there is no transitive maximal subgroup of $S(X, Y)$ containing $J(Y,Z)$.*

**Proof.** Suppose $H$ is a transitive maximal subgroup of $S(X, Y)$ containing $J(Y,Z)$. By Corollary 3.15 and Lemma 3.16, there is $d \leqq W < Y$ such that $S(X, W)$ is the largest normal subgroup of $S(X,X^+)$ contained in $H$. By Lemma 1.11, $H = \{s \in S(X, Y): T(s) < W\}$. Now $W^+ < Y$, so $H < \{s \in S(X,Y): T(s) < W^+\} < S(X,Y)$, contrary to the maximality of $H$. Thus no such $H$ exists.

It should be noted that Theorem 3.13 defines the structure of imprimitive maximal subgroups of $S(X,d)$ and $A(X)$. That the imprimitive maximal subgroups of $S_n$ are similarly structured is shown by the following theorem.

3.18. THEOREM. *$H$ is an imprimitive maximal subgroup of $S_n$ if and only if $H = N(K)$ where $M = \bigcup_{i=1}^{m} M_i$ disjointly, $|M_i| = q$, $1 \leqq i \leqq m$, $1 \neq q \neq n$, $n = qm$, $K = \prod_{i=1}^{m} S(M_i)$.*

**Proof.** If $H$ is an imprimitive maximal subgroup of $S_n$, then there is a nontrivial partition $\{M_i\}_{i=1}^{m}$ of $M$ consisting of the sets of primitivity of $H$. By 3.2, $q = |M_i| = |M_j|$ for every $i$ and $j$, so $n = qm$ and $1 \neq q \neq n$. By maximality $H \supseteq N(K)$. It now suffices to show $N(K)$ is maximal. This follows as in the proof of Theorem 3.13.

3.19. COROLLARY. *The imprimitive maximal subgroups of $S_n$ are completely determined (up to isomorphism) by the proper divisors of $n$. If $n$ is prime, $S_n$ has no imprimitive maximal subgroups.*

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APPALACHIAN STATE TEACHERS COLLEGE
BOONE, NORTH CAROLINA