ON COMPLEX QUADRATIC FIELDS WITH CLASS NUMBER EQUAL TO ONE(1)

BY
HAROLD STARK

Let $R(\sqrt{-p})$ be a quadratic extension of the rationals, where p is a positive square free integer. For nine values of p, namely 1, 2, 3, 7, 11, 19, 43, 67, 163, the integers of $R(\sqrt{-p})$ form a unique factorization domain. Heilbronn and Linfoot [1] have shown that there is at most one more such value of p, and Lehmer [2] has shown that p must be a prime greater than $5 \cdot 10^9$. In the present paper we verify and extend the lower bound of $5 \cdot 10^9$ for p. The result is

THEOREM 1. *If the ring of integers of $R(\sqrt{-p})$ (p square free) forms a unique factorization domain, and $p > 10^4$, then $p > \exp(2.2 \cdot 10^7)$.*

It will be assumed throughout that p is an integer satisfying the hypothesis of Theorem 1. We start with a formula equivalent to that given by Lemma 2 of [1].

$$\zeta(s)L(s) - \zeta(2s) = 2^{2s-1}p^{(1/2)-s}\zeta(2s-1)\sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} + h(s), \tag{1}$$

valid for $\sigma > \frac{1}{2}$, where $\zeta(s)$ is the Riemann zeta function, $L(s)$ is the Dirichlet L-series formed with the quadratic character (mod p), and

$$h(s) = \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \left(x - [x] - \frac{1}{2}\right) \frac{d}{dx}\left\{\left[\left(x + \frac{j}{2}\right)^2 + \frac{pj^2}{4}\right]^{-s}\right\} dx.$$

Let $x + (j/2) = u(j\sqrt{p})/2$ give a change of variable from x to u and integrate by parts $2m-1$ times; we get, for $m = 1, 2, \cdots$,

$$h(s) = -\sum_{j=1}^{\infty} \left(\frac{2}{j\sqrt{p}}\right)^{2m+2s-1} \int_{-\infty}^{\infty} \frac{B_{2m}(x-[x])}{(2m)!} \frac{d^{(2m)}}{du^{(2m)}} \{(u^2+1)^{-s}\}du, \tag{2}$$

where $B_k(x)$ is the kth Bernoulli polynomial ($B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}, \cdots$) and the series converges for $\operatorname{Re} s > 1 - m$.

It is well known ([3, p. 245], Jordan's $\varphi_n(x) = B_n(x)/n!$) that for $0 \leqq x \leqq 1$, $k > 1$,

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$$\left|\frac{B_k(x)}{(k)!}\right| \leqq \frac{2}{(2\pi)^k}\zeta(k), \tag{3}$$

so that to estimate $h(s)$, we need only estimate $d^{(2m)}/du^{(2m)}\{(u^2+1)^{-s}\}$. Using the fact that

$$\frac{d^2}{du^2}\{(u^2+1)^{-s}\} = 2s(2s+1)(u^2+1)^{-s-1} - 2s(2s+2)(u^2+1)^{-s-2},$$

we see inductively that we can write

$$\frac{d^{(2m)}}{du^{(2m)}}\{(u^2+1)^{-s}\} = \sum_{k=0}^{m} c_{mk}(s)(u^2+1)^{-s-m-k}. \tag{4}$$

Again, by induction we have

$$|c_{mk}(s)| \leqq \binom{m}{k}\prod_{j=0}^{2m-1}(2|s|+2j). \tag{5}$$

Thus

$$\begin{aligned}\left|\frac{d^{(2m)}}{du^{2(m)}}\{(u^2+1)^{-s}\}\right| &\leqq \sum_{k=0}^{m}\binom{m}{k}(u^2+1)^{-\sigma-m-k}\prod_{j=0}^{2m-1}(2|s|+2j)\\ &\leqq 2^m(u^2+1)^{-\sigma-m}\prod_{j=0}^{m-1}(2|s|+2j)(2|s|\\ &\qquad +2(2m-1-j))\\ &\leqq 2^m(u^2+1)^{-\sigma-m}(2|s|+2m-1)^{2m}.\end{aligned} \tag{6}$$

In view of (2), (3) and (6), we have for $\sigma \geqq \frac{1}{2}$ and $m \geqq 1$:

$$\begin{aligned}|h(s)| &\leqq \sum_{y=1}^{\infty}\left(\frac{2}{y\sqrt{p}}\right)^{2m+2\sigma-1}\cdot\frac{2}{(2\pi)^{2m}}\zeta(2m)\cdot 2^m(2|s|+2m-1)^{2m}\int_{-\infty}^{\infty}(u^2+1)^{-1}du\\ &\leqq 2\pi\zeta^2(2m)\left(\frac{4|s|+4m-2}{\pi\sqrt{(2p)}}\right)^{2m}\\ &< 2\pi\left(\frac{2m}{2m-1}\right)^2\left(\frac{4|s|+4m-2}{\pi\sqrt{(2p)}}\right)^{2m}.\end{aligned} \tag{7}$$

Letting $m = 30$, we see that if $|s| \leqq 22$ and $\sigma \geqq \frac{1}{2}$ (and of course $p > 10{,}000$), then

$$|h(s)| < 10^{-19}.$$

Let θ denote a number, complex or real, not necessarily the same each time it occurs, which satisfies $|\theta| \leqq 1$. We find that for $|s| < 22$ and $\sigma \geqq \frac{1}{2}$, (1) becomes

$$\zeta(s)L(s) - \zeta(2s) = \zeta(2-2s)\frac{\Gamma(1-s)}{\Gamma(s)}\left(\frac{\sqrt{p}}{2\pi}\right)^{1-2s} + 10^{-19}\theta, \tag{8}$$

where the functional equation for $\zeta(s)$ was used to obtain the first term on the right.

Let

$$s_n = \frac{1}{2} + i\gamma_n \tag{9}$$

denote the nth zero of $\zeta(s)$ above the real axis. It is known that $\gamma_1 \approx 14$ and $\gamma_2 \approx 21$ (see Appendix); in particular, $|s_1| < |s_2| < 22$ and thus

$$\zeta(2s_n) = -\zeta(2-2s_n)\frac{\Gamma(1-s_n)}{\Gamma(s_n)}\left(\frac{p}{4\pi^2}\right)^{-i\gamma_n} + 10^{-19}\theta, \qquad (n = 1,2). \tag{10}$$

Multiplying both sides by $(p/4\pi^2)^{i\gamma_n}(1/\zeta(2s_n))$ and using the fact that $|\zeta(2s_n)| > \frac{1}{2}$ for $n = 1,2$ (see Appendix), we get

$$\begin{aligned}\left(\frac{p}{4\pi^2}\right)^{i\gamma_n} &= -\frac{\zeta(2-2s_n)}{\zeta(2s_n)}\frac{\Gamma(1-s_n)}{\Gamma(s_n)} + 2\cdot 10^{-19}\theta \\ &= -\frac{\zeta(1-2i\gamma_n)\Gamma\left(\frac{1}{2}-i\gamma_n\right)}{\zeta(1+2i\gamma_n)\Gamma\left(\frac{1}{2}+i\gamma_n\right)}(1+2\cdot 10^{-19}\theta), \qquad (n=1,2).\end{aligned} \tag{11}$$

Taking arguments of both sides of (11) gives

$$\gamma_n \log\left(\frac{p}{4\pi^2}\right) = a_n + 2\pi x_n + 3\cdot 10^{-19}\theta, \qquad (n = 1,2) \tag{12}$$

where x_n is an integer and

$$\begin{aligned} a_n &\equiv \pi - 2\arg\zeta(2s_n) - 2\arg\Gamma(s_n) \pmod{2\pi}, \\ 0 &\leqq a_n < 2\pi. \end{aligned} \tag{13}$$

Eliminating $\log(p/4\pi^2)$ from the equations (12), and solving for x_2, we obtain

$$x_2 = \frac{\gamma_2}{\gamma_1}x_1 + a_0 + 10^{-18}\theta, \tag{14}$$

where

$$a_0 = \frac{1}{2\pi}\left(\frac{\gamma_2}{\gamma_1}a_1 - a_2\right). \tag{15}$$

From the Appendix,

$$\begin{aligned} \frac{\gamma_2}{\gamma_1} &= 1.487\ 262\ 003\ 892\ 890\ 048 + 10^{-18}\theta, \\ a_0 &= a + .4\cdot 10^{-9}\theta \quad \text{where } a = -.461\ 786\ 352. \end{aligned} \tag{16}$$

We can rewrite (14) as

$$x_2 = \frac{\gamma_2}{\gamma_1} x_1 + a + \frac{1}{2} \cdot 10^{-9}\theta. \tag{17}$$

Note that

$$3.999\ 999\ 660 = \frac{\gamma_2}{\gamma_1} \cdot 3 + a + \frac{1}{2} \cdot 10^{-9}\theta. \tag{18}$$

It is not accidental that $3(\gamma_2/\gamma_1) + a$ should be close to an integer; $x_1 = 3$ corresponds to $p = 163$ (see introduction). In fact

$$\begin{aligned} \gamma_1 \log\left(\frac{163}{4\pi^2}\right) &= 20.042\ 984\ 673\ 072\cdots, \\ a_1 + 2\pi \cdot 3 &= 20.042\ 984\ 673\ 470\cdots. \end{aligned}$$

(Compare this with (12), where these numbers would agree to at least 19 decimal places if $p > 10{,}000$.) From (12), we now see that $p > 10^4$ implies that $x_1 > 3$.

Subtracting (18) from (17) gives:

$$x_2 - 4 = \frac{\gamma_2}{\gamma_1}(x_1 - 3) - b + 10^{-9}\theta, \qquad \text{where } b = .000\ 000\ 340. \tag{19}$$

Now let

$$\begin{aligned} p_1 &= 83{,}532{,}765, & p_2 &= 12{,}832{,}922, \\ q_1 &= 56{,}165{,}467, & q_2 &= 8{,}628{,}555. \end{aligned} \tag{20}$$

Then $p_1 q_2 - q_1 p_2 = 1$, so that p_1 and q_1 are relatively prime.
Also,

$$\left|\left(\frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1}\right)\right| < 2.3 \cdot 10^{-16}. \tag{21}$$

Let

$$Q + \frac{R}{q_1} = \frac{p_1}{q_1}(x_1 - 3), \tag{22}$$

where $0 \leqq R < q_1$ and Q and R are integers. Subtracting (22) from (19) gives

$$x_2 - Q - 4 = \left(\frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1}\right)(x_1 - 3) + \left(\frac{R}{q_1} - b\right) + 10^{-9}\theta. \tag{23}$$

If $x_1 \leqq 5.1 \cdot 10^7$, then

$$\left|\left(\frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1}\right)(x_1 - 3)\right| < 12 \cdot 10^{-9}, \tag{24}$$

and thus $x_1 \leqq 5.1 \cdot 10^7$ implies

$$|x_2 - Q - 4| < 12 \cdot 10^{-9} + (1 - 340 \cdot 10^{-9}) + 10^{-9} < 1. \tag{25}$$

On the other hand, since

$$\frac{18}{q_1} < 321 \cdot 10^{-9} < b = 340 \cdot 10^{-9} < 356 \cdot 10^{-9} < \frac{20}{q_1},$$

we find that $x_1 \leqq 5.1 \cdot 10^7$ and $R \neq 19$ implies

$$\begin{aligned} |x_2 - Q - 4| &\geqq \left|\frac{R}{q_1} - b\right| - \left|\left(\frac{\gamma_2}{\gamma_1} - \frac{p_1}{q_1}\right)(x_1 - 3)\right| - 10^{-9} \\ &> 16 \cdot 10^{-9} - 12 \cdot 10^{-9} - 10^{-9} > 0. \end{aligned} \tag{26}$$

Inequalities (25) and (26) are contradictory, and therefore $x_1 \leqq 5.1 \cdot 10^7$ implies $R = 19$.

But if $R = 19$, then we see from (22) that

$$19 \equiv p_1(x_1 - 3) \qquad (\text{mod } q_1),$$

and this implies

$$x_1 - 3 \equiv 51{,}611{,}611 \qquad (\text{mod } q_1).$$

Thus under all circumstances, $x_1 > 5.1 \cdot 10^7$. Hence by (12),

$$\begin{aligned} \log\left(\frac{p}{4\pi^2}\right) &= \frac{a_1 + 2\pi x_1 + 3 \cdot 10^{-19}\theta}{\gamma_1} \\ &> \frac{2\pi(5.1 \cdot 10^7) - 3 \cdot 10^{-19}}{14.2} \\ &> 2.2 \cdot 10^7, \end{aligned}$$

and Theorem 1 follows.

Appendix

I wish to express my thanks to M.D. Bigg who furnished values of γ_1 and γ_2 to fifty decimal places and to R.S. Lehman who furnished values of $\arg \zeta(2s_n)$ and $|\zeta(2s_n)|$ for $n = 1$ and 2, with a proved accuracy of $\pm 10^{-10}$. The values of γ_1 and γ_2 were confirmed independently by Robert Spira to fifteen decimal places. Their values are:

$$\begin{aligned} \gamma_1 &= 14.134\ 725\ 141\ 734\ 693\ 790\ 457 + 10^{-21}\theta, \\ \gamma_2 &= 21.022\ 039\ 638\ 771\ 554\ 992\ 628 + 10^{-21}\theta, \\ \frac{1}{\pi}\arg\zeta(2s_1) &= -\,.108\ 452\ 737\ 083\ 095 + 10^{-10}\theta \qquad (\text{mod } 2), \\ \frac{1}{\pi}\arg\zeta(2s_2) &= \ .067\ 103\ 865\ 503\ 910 + 10^{-10}\theta \qquad (\text{mod } 2), \\ |\zeta(2s_1)| &= 1.948\ 757\ 313\ 817\ 40 + 10^{-10}\theta, \\ |\zeta(2s_2)| &= .830\ 962\ 021\ 546\ 955 + 10^{-10}\theta. \end{aligned} \tag{A1}$$

All other numbers were calculated on a Monroe desk calculator to fifteen places.

The Euler-Maclaurin sum formula for $\log\Gamma(\frac{1}{2}+i\gamma)$ can be expressed ([4, p. 132]. It should be noted that the exponent $(m+1)$ in Nörlund's remainder term should be replaced by m. See also [5, p. 131]):

$$\log\Gamma\left(\frac{1}{2}+i\gamma\right) = i\gamma\log(i\gamma) - i\gamma + \frac{1}{2}\log(2\pi)$$
$$+ \sum_{j=2}^{11}(-1)^j \frac{B_j\left(\frac{1}{2}\right)}{j(j-1)}\cdot\frac{1}{(i\gamma)^{j-1}}$$
$$- \int_0^\infty \frac{B_{11}\left(x-\frac{1}{2}-\left[x-\frac{1}{2}\right]\right)}{11(x+i\gamma)^{11}}\,dx.$$

From this we see that

$$\text{(A2)}\quad \arg\Gamma\left(\frac{1}{2}+i\gamma\right) = \gamma(\log\gamma-1) + \sum_{k=1}^{5}\frac{(-1)^k B_{2k}\left(\frac{1}{2}\right)}{2k(2k-1)\gamma^{2k-1}}$$
$$- \operatorname{Im}\int_0^\infty \frac{B_{11}\left(x-\frac{1}{2}-\left[x-\frac{1}{2}\right]\right)}{11(x+i\gamma)^{11}}\,dx.$$

From the formula [5, p. 123],

$$B_k\left(\frac{1}{2}\right) = -(1-2^{1-k})B_k,$$

we get the following:

$$\frac{B_2\left(\frac{1}{2}\right)}{2\cdot 1} = -.041\ 666\ 666\ 666\ 667 + 10^{-15}\theta,$$

$$\frac{B_4\left(\frac{1}{2}\right)}{4\cdot 3} = .002\ 430\ 555\ 555\ 556 + 10^{-15}\theta,$$

$$\frac{B_6\left(\frac{1}{2}\right)}{6\cdot 5} = -.000\ 768\ 849\ 206\ 349 + 10^{-15}\theta,$$

$$\frac{B_8\left(\frac{1}{2}\right)}{8\cdot 7} = .000\ 590\ 587\ 797\ 619 + 10^{-15}\theta,$$

$$\frac{B_{10}\left(\frac{1}{2}\right)}{10\cdot 9} = -.000\ 840\ 106\ 797\ 138 + 10^{-15}\theta.$$

From [6], we get

$$\log \gamma_1 = 2.648\ 634\ 545\ 730\ 790 + 10^{-15}\theta,$$

$$\log \gamma_2 = 3.045\ 571\ 393\ 984\ 561 + 10^{-15}\theta.$$

Finally, for $t > 14$, we get from (3),

$$\left| \int_0^\infty \frac{B_{11}\left(x - \frac{1}{2} - \left[x - \frac{1}{2}\right]\right)}{11(x+it)^{11}}\,dx \right| \leqq \frac{(2\pi)\frac{2.11!}{(2\pi)^{11}}\cdot\frac{11}{10}}{11t^9}\int_0^\infty \frac{dx}{x^2+t^2}$$

$$= \frac{11!\pi}{10(2\pi)^{11}t^{10}}$$

$$< 10^{-13}.$$

Putting these numbers into (A2), we obtain

$$\text{(A3)}\qquad \begin{aligned} \frac{1}{\pi}\arg\Gamma(s_1) &= 7.418\ 512\ 651\ 985\ 173 + 2\cdot 10^{-13}\theta, \\ \frac{1}{\pi}\arg\Gamma(s_2) &= 13.688\ 619\ 111\ 000\ 235 + 2\cdot 10^{-13}\theta. \end{aligned}$$

We may rewrite (12) thusly:

$$\frac{a_n}{2\pi} \equiv \frac{1}{2} - \frac{1}{\pi}\arg\zeta\,(2s_n) - \frac{1}{\pi}\arg\Gamma(s_n) \qquad (\text{mod } 1),$$

$$0 \leqq \frac{a_n}{2\pi} < 1.$$

Using (A1) and (A3), we find that

$$\frac{a_1}{2\pi} \equiv .189\ 940\ 085\ 097\ 922 + 1.002\cdot 10^{-10}\theta \qquad (\text{mod } 1),$$

$$\frac{a_2}{2\pi} \equiv .744\ 277\ 023\ 495\ 855 + 1.002\cdot 10^{-10}\theta \qquad (\text{mod } 1).$$

Since these numbers are between 0 and 1, the above is actually an equality. Finally,

$$\text{(A4)}\qquad \frac{\gamma_2}{\gamma_1} = 1.487\ 262\ 003\ 892\ 890\ 048 + 10^{-18}\theta$$

and

$$\text{(A5)}\qquad \begin{aligned} a_0 &= \frac{\gamma_2}{\gamma_1}\left(\frac{a_1}{2\pi}\right) - \frac{a_2}{2\pi} \\ &= -.461\ 786\ 351\ 913\ 533 + 3\cdot 10^{-10}\theta. \end{aligned}$$

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University of Michigan
Ann Arbor, Michigan