LINEAR TRANSFORMATIONS OF GAUSSIAN MEASURES(1)

BY
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Introduction. In spite of increased study in recent years, our knowledge of measures in function spaces remains poor. This is true even in the simplest case of Gaussian measures. In a recent and excellent survey article [14], A. M. Yaglom attributes this state of affairs to a lack of concrete theorems about the integral calculus of function spaces. Of all measures in function spaces, we know most about Wiener measure especially because of a long series of papers by R. H. Cameron and W. T. Martin on the Wiener integral. One result, the linear transformation theorem [3], has recently been generalized but still for Wiener measure by one of Cameron's students, D. A. Woodward [13]. The purpose of our paper is to state and prove an analog of Woodward's theorem for general Gaussian measures. We mention that our results are technically related to the elegant and highly abstract work of I. E. Segal [9] but in our opinion cannot be easily deduced from it.

Some notation and background material are needed to set the stage for our main theorem. By a Gaussian process (sometimes symbolized $\{x(t),\ t\in I\}$), we shall mean a triple $\{X,B,\lambda_{rm}\}$ where $X\equiv X(I)$ is a set of real valued functions defined on an interval $I\equiv[a,b]$, B is the Borel field of subsets of X generated by sets of the form

$$\{x\in X: x(t_0)<c,\ t_0\in I\}$$

and λ_{rm} is a Gaussian probability measure on B determined by a covariance function r and a mean function m [5, pp. 71–74]. In this paper we shall always take the mean function m to be identically zero; hence without confusion we may write λ_r in place of λ_{rm}. We will assume that r is continuous on $I\times I$ and also that it is regular enough so that $X(I)$ may be taken as $C(I)$, the space of continuous functions on I. For a discussion of sufficient conditions on r which make this possible, see [7, pp. 519–522] in the general case and [2] in the stationary case.

As a special but important example, we mention the Wiener process $\{C,B,\lambda_w\}$

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determined by the covariance $w(s,t) \equiv \min(s,t)$. It is for this process that a host of transformation theorems have been obtained, in particular, the afore mentioned theorem of Woodward. We will state this theorem and our generalization presently but first we need a definition of bounded variation for a function of two variables.

We say that $M \in \mathrm{BV}H$ if there exists (t_0,s_0) in $I \times I$ such that $M(t_0,s)$ and $M(t,s_0)$ are of bounded variation (BV) on I and if $\mathrm{var}(M)$ on $I \times I$ is finite where

$$\mathrm{var}(M) = \sup \sum_{i=1}^{m} \sum_{j=1}^{n} \left| M(t_i,s_j) - M(t_i,s_{j-1}) + M(t_{i-1},s_{j-1}) - M(t_{i-1},s_j) \right|.$$

This definition which is given by Woodward is due to Hardy and Krause (see [1], [4] for a discussion of properties of functions in BVH and the double Riemann-Stieltjes integral based on them and [6] for the n-dimensional generalization). We need also a symbol for a certain Fredholm determinant. Let

$$(0.1) \qquad \mathscr{D} \equiv \mathscr{D}_M = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_I \cdots \int_I \begin{vmatrix} M(s_1,s_1) \cdots M(s_1,s_n) \\ \vdots \qquad\qquad \vdots \\ M(s_n,s_1) \cdots M(s_n,s_n) \end{vmatrix} ds_1 \cdots ds_n.$$

THEOREM 1 (WOODWARD [13]). *Let $\{C, B, \lambda_w\}$ be the Wiener process on $I \equiv [0,1]$ and let*

$$(Tx)(t) = x(t) + \int_0^1 \int_0^t M(u,s)\, du\, dx(s)$$

be a transformation defined on C where $M \in \mathrm{BV}H$ and $\mathscr{D}_M \neq 0$. Then T carries C onto C in a one-to-one manner and if F is a measurable function for which either side of the following equation exists, both sides exist and are equal.

$$(0.2) \qquad E\{F(x)\} = |\mathscr{D}|\, E\{F(Tx) \exp[-\Psi(x)/2]\}$$

where

$$\Psi(x) = \int_0^1 \int_0^1 \left[2M(s,t) + \int_0^1 M(u,t)M(u,s)\, du \right] dx(s)\, dx(t).$$

Some differences between this theorem and the one in Woodward's original paper should be noted. First, Woodward allows the kernel M to have a special kind of jump discontinuity on the diagonal $s = t$, the so called Volterra case. Second, since

$$\lambda_w\{x \in C: x(0) = 0\} = 1,$$

he considers the space of continuous functions which vanish at 0 rather than our C. Lastly he uses $w(s,t) = \min(s,t)/2$ rather than $\min(s,t)$ so that there is an extra factor of 2 in the exponential of formula (0.2) in his paper.

Our extension of this theorem to general Gaussian processes is

THEOREM 2. *Let $\{C, B, \lambda_r\}$ be a Gaussian process on $I \equiv [a,b]$ determined by a covariance function r which is continuous on $I \times I$ (see remarks in second paragraph of this section). Let*

$$(Tx)(t) = x(t) + \int_a^b \int_a^b x(s)r(t,u)dK(u,s)$$

where $K \in \mathrm{BV}H$. Finally let

$$D \equiv D_K = 1 + \sum_{m=1}^{\infty} A_m/m! \tag{0.4}$$

where

$$A_m = \int_a^b \int_a^b \cdots \int_a^b \int_a^b \begin{vmatrix} r(s_1,t_1) \cdots r(s_1,t_m) \\ \vdots \qquad\qquad \vdots \\ r(s_m,t_1) \cdots r(s_m,t_m) \end{vmatrix} dK(t_1,s_1) \cdots dK(t_m,s_m) \tag{0.5}$$

and suppose that $D_K \neq 0$. Then T maps C onto C in a one-to-one manner and if F is a measurable function for which either side of the following equation exists, both sides exist and are equal.

$$E\{F(x)\} = |D| E\{F(Tx)\exp[-\Phi(x)/2]\} \tag{0.6}$$

where

$$\Phi(x) = 2\int_a^b \int_a^b x(s)x(t)dK(s,t) + \int_a^b \int_a^b \int_a^b \int_a^b x(s)x(t)r(u,v)dK(u,s)dK(v,t). \tag{0.7}$$

After proving Theorem 2, we will reinterpret it for processes with triangular covariance functions (Theorem 3) and finally obtain Theorem 1 as a very special case via the simple substitutions $r(s,t) = \min(s,t)$.

1. **Some preliminary lemmas.** In order to demonstrate the one-to-one and onto properties of the transformation of Theorem 2, it is necessary to study the Riemann-Stieltjes integral equation

$$x(t) = y(t) + \lambda \int_a^b \int_a^b x(s)r(t,u)dK(u,s). \tag{1.1}$$

A search of the literature failed to uncover previous study of this equation but it is easily attacked using the classical Fredholm approach. We summarize the results that we need in

LEMMA 1. *If*
(i) *y is continuous on $I \equiv [a,b]$,*
(ii) *r is continuous on $I \times I$,*
(iii) *$K \in \mathrm{BV}H$,*
(iv) *$D(\lambda) \equiv D_K(\lambda) \neq 0$,*

then the integral equation (1.1) *has one and only one solution given by*

$$x(t) = y(t) + \frac{1}{D(\lambda)} \int_a^b \int_a^b y(s)D(t,u;\lambda)\,dK(u,s).$$

Here

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} (-\lambda)^m A_m/m!\ ,$$

$$D(t,u;\lambda) = \lambda r(t,u) + \lambda \sum_{m=1}^{\infty} (-\lambda)^m B_m(t,u)/m!$$

where A_m *is given by* (0.5) *and*

$$B_m(t,u) = \int_a^b \int_a^b \cdots \int_a^b \int_a^b \begin{vmatrix} r(t,u) r(t,t_1) \cdots r(t,t_m) \\ r(s_1,u) r(s_1,t_1) \cdots r(s_1,t_m) \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ r(s_m,u) r(s_m,t_1) \cdots r(s_m,t_m) \end{vmatrix} dK(t_1,s_1) \cdots dK(t_m,s_m).$$

The series for $D(\lambda)$ *and* $D(t,u;\lambda)$ *converge absolutely for all* λ *and the second converges uniformly in* (t,u) *on* $I \times I$.

We omit the proof of this lemma since it is so similar to that for the classical Fredholm equation as outlined for example in [12]. We mention only that the following identities play a role analogous to those in [12, p. 216] and are proved in the same manner.

$$D(t,u;\lambda) = \lambda r(t,u)D(\lambda) + \lambda \int_a^b \int_a^b r(s,u)\,D(t,v;\lambda)dK(v,s)$$

$$= \lambda r(t,u)D(\lambda) + \lambda \int_a^b \int_a^b r(t,v)D(s,u;\lambda)dK(v,s).$$

Applying Lemma 1 with $\lambda = -1$ and $D = D(-1)$, we conclude that the transformation T of Theorem 2 is one-to-one and onto C.

Before stating our next two lemmas, we will need to introduce some further notation. Roughly speaking, our plan is to approximate the transformation

$$y(t) = (Tx)(t) = x(t) + \int_a^b \int_a^b x(s)r(t,u)dK(u,s)$$

by

$$y(t_i) = x(t_i) + \sum_{j=1}^{n} \sum_{k=1}^{n} x(t_k)r(t_i,t_j)$$

$$\cdot\ [K(t_j,t_k) - K(t_j,t_{k-1}) - K(t_{j-1},t_k) + K(t_{j-1},t_{k-1})]$$

where $t_i = a + i(b-a)/n$, $i = 0,1,2,\cdots,n$.

More briefly we write

$$(1.2)\qquad y_i = x_i + \sum_{k=1}^{n} P_{ik}x_k, \qquad i = 1,2,\cdots n,$$

where

$$P_{ik} = \sum_{j=1}^{n} r_{ij}\Delta_{jk},$$

$$r_{ij} = r(t_i,t_j),$$

$$\Delta_{jk} = K(t_j,t_k) - K(t_j,t_{k-1}) - K(t_{j-1},t_k) + K(t_{j-1},t_{k-1})$$

and x_i and y_i have their obvious meanings.

Now (1.2) may be thought of as a linear transformation in Euclidean n-space with determinant D_n given by (see [8, pp. 23–24])

$$(1.3)\qquad D_n = \begin{vmatrix} 1+P_{11} & P_{12}\cdots P_{1n} \\ P_{21} & 1+P_{22}\cdots P_{2n} \\ \vdots & \vdots \quad \vdots \\ P_{n1} & P_{n2}\cdots 1+P_{nn} \end{vmatrix} = 1 + \sum_{m=1}^{n} A_{mn}/m!$$

where

$$(1.4)\qquad \begin{aligned} A_{mn} &= \sum_{j_1,\ldots,j_m=1}^{n} \begin{vmatrix} P_{j_1j_1}\cdots P_{j_1j_m} \\ \vdots \\ P_{j_mj_1}\cdots P_{j_mj_m} \end{vmatrix} \\ &= \sum_{i_1,j_1=1}^{n} \cdots \sum_{i_m,j_m=1}^{n} \begin{vmatrix} r_{j_1i_1}\cdots r_{j_1i_m} \\ \vdots \\ r_{j_mi_1}\cdots r_{j_mi_m} \end{vmatrix} \Delta_{i_1j_1}\cdots\Delta_{i_mj_m}. \end{aligned}$$

This brings us to our next lemma.

LEMMA 2. *If r is continuous and $K \in \mathrm{BV}H$ on $I \times I$, then* $\lim_{n\to\infty} D_n = D$ (*see* (1.3) *and* (0.4) *for the definitions of D_n and D*).

Proof. Let J be a bound for $|r|$ and var (K) on $I \times I$. Then by Hadamard's theorem, A_m and A_{mn} (see (0.5) and (1.4)) may be bounded as follows:

$$|A_m| \leqq m^{m/2}J^{2m},$$

$$|A_{mn}| \leqq m^{m/2}J^{2m}, \qquad n = 1,2,\cdots.$$

Hence if $\varepsilon > 0$ is given, we may choose N so large that

$$\sum_{m=N+1}^{\infty} |A_m|/m! < \varepsilon/3$$

and

$$\sum_{m=N+1}^{n} |A_{mn}|/m! < \varepsilon/3, \quad n = N+1, N+2, \cdots.$$

But $\lim_{n\to\infty} A_{mn} = A_m$. Thus we may choose $M \geqq N$ such that if $n > M$

$$\sum_{m=1}^{N} |A_{mn} - A_m|/m! \ < \varepsilon/3.$$

Hence if $n > M$

$$\begin{aligned} |D_n - D| &\leqq \left| \sum_{m=N+1}^{n} A_{mn}/m! \right| + \left| \sum_{m=1}^{N} (A_{mn} - A_m)/m! \right| + \left| \sum_{m=N+1}^{\infty} A_m/m! \right| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

LEMMA 3. *Let r be continuous and $K \in \mathrm{BVH}$ on $I \times I$. Suppose that $D \neq 0$ and let N be so large that for $n \geqq N$, $|D_n| > |D|/2$ (see Lemma 2). Then if*

$$\sup_{1 \leqq i \leqq n} \left| x_i + \sum_{k=1}^{n} P_{ik} x_k \right| \leqq M,$$

there exists a constant $B(M)$ independent of n ($n \geqq N$) such that

$$\sup_{1 \leqq i \leqq n} |x_i| \leqq B(M).$$

Proof. Let $y_i = x_i + \sum_{k=1}^{n} P_{ik} x_k$. Then since $D_n \neq 0$ ($n \geqq N$), we may solve for x_i by Cramer's rule, the result being

$$x_i = y_i \frac{D_n(i,i)}{D_n} + \sum_{j=1\ (j\neq i)}^{n} y_j \frac{D_n(j,i)}{D_n}$$

where $D_n(j,i)$ is the cofactor of the jith element of D_n. Now $D_n(i,i)$ may be expanded in a form very similar to that of D_n (see (1.3) and (1.4)) and is easily shown to be bounded independently of n.

For $i \neq j$, $D_n(j,i) = P_{ij} + \sum_{m=1}^{n-2} C_{mn}/m!$ where

$$C_{mn} = \sum_{j_1,\dots,j_m=1}^{n} \begin{vmatrix} P_{ij} & P_{ij_1} \cdots P_{ij} \\ P_{j_1 j} & P_{j_1 j_1} \cdots P_{j_1 j} \\ \vdots & \vdots \\ P_{j_m j} & P_{j_m j_1} \cdots P_{j_m j_m} \end{vmatrix}.$$

Hence

$$\left|\sum_{j=1(j\neq i)}^{n} y_j D_n(j,i)\right| \leqq \sum_{j=1}^{n}\left|y_j\right|\left|P_{ij}\right| + \sum_{m=1}^{n-2}\sum_{j=1}^{n}\left|y_j\right|\left|C_{mn}\right|/m!$$

$$\leqq \sum_{j=1}^{n}\sum_{k=1}^{n}\left|y_j r_{ik}\Delta_{kj}\right|$$

$$+\sum_{m=1}^{n-2}\frac{1}{m!}\sum_{j,k=1}^{n}\cdots\sum_{j_m,k_m=1}^{n}\left\|\begin{matrix} r_{ik}y_j & r_{ik_1}\cdots r_{ik_m} \\ r_{j_1k}y_j & r_{j_1k_1}\cdots r_{j_1k_m} \\ \vdots & \vdots \qquad \vdots \\ r_{j_mk}y_j & r_{j_mk_1}\cdots r_{j_mk_m}\end{matrix}\right\|\left|\Delta_{kj}\cdots\Delta_{k_mj_m}\right|.$$

Using the fact that $|y_j| \leqq M$ and that $|r|$ is bounded and $K \in \mathrm{BV}\,H$ together with Hadamard's theorem, it follows that the above expression is bounded independently of n. The conclusion of the lemma is now a trivial consequence.

2. **Proof of Theorem 2.** In the interest of greater compactness of notation, we introduce the matrix form of (1.2). Hence let $\boldsymbol{x}$ and $\boldsymbol{y}$ be the $n \times 1$ matrices (i.e. vectors) with components x_i and y_i and $\boldsymbol{R}$ and $\boldsymbol{\Delta}$ be the $n \times n$ matrices with elements r_{ij} and Δ_{jk} $(i,j,k = 1,2,\cdots n)$. Then the transformation (1.2) may be written as

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{R}\boldsymbol{\Delta}\boldsymbol{x}\,. \tag{2.1}$$

Also we need functions H_M and H_{Mn} with domains C and Euclidean n-space E^n respectively. Letting $|||\cdots|||$ denote the sup norm, we define

$$H_M(x) = \begin{cases} 1 & \text{if } |||x||| < M-1, \\ 1-a & \text{if } |||x||| = M-1+a, \qquad 0 \leqq a \leqq 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_{Mn}(\boldsymbol{u}) = \begin{cases} 1 & \text{if } |||\boldsymbol{u}||| < M-1, \\ 1-a & \text{if } |||\boldsymbol{u}||| = M-1+a, \qquad 0 \leqq a \leqq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We note that for all $x \in C$,

$$\lim_{n\to\infty} H_{Mn}(\boldsymbol{x}) \equiv \lim_{n\to\infty} H_{Mn}[x(t_1),\cdots,x(t_n)] = H_M(x)$$

and

$$\lim_{M\to\infty} H_M(x) = 1.$$

In establishing a relation like (0.6), it is usually convenient to begin with some restricted class of functions F and then extend to the class of all measurable functions. The idea of our proof is first to establish (0.6) for the function

$$F(x) = \exp\int_a^b x(t)\,dp(t)$$

which we shall call the moment generating functional. Once this is done, it is almost trivial to extend to the class of measurable functions. We break the proof into two cases.

Case 1. $\boldsymbol{R}$ is nonsingular (at least for sufficiently large n). Making use of the notation introduced above and letting $\boldsymbol{p}$ denote the vector with components $p(t_i) - p(t_{i-1})$, $i = 1, 2, \cdots, n$, we have for any function p of BV,

$$\begin{aligned} &E\left\{\exp \int_a^b x(t)\,dp(t)\right\} \\ &= \lim_{M\to\infty} E\left\{H_M(x)\exp\int_a^b x(t)\,dp(t)\right\} \qquad \text{(monotone convergence)} \\ &= \lim_{M\to\infty}\lim_{n\to\infty} E\{H_{Mn}(\boldsymbol{x})\exp(\boldsymbol{x}'\boldsymbol{p})\} \qquad \text{(bounded convergence)} \\ &= \lim_{M\to\infty}\lim_{n\to\infty}[(2\pi)^n \det \boldsymbol{R}]^{-1/2}\int_{E^n} H_{Mn}(\boldsymbol{v})\exp(\boldsymbol{v}'\boldsymbol{p} - \tfrac{1}{2}\boldsymbol{v}'\boldsymbol{R}^{-1}\boldsymbol{v})\,d\boldsymbol{v}. \end{aligned}$$

The integral above is just an ordinary n-fold Lebesgue integral. In it we make the transformation $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{R}\Delta\boldsymbol{u}$ for which D_n (see (1.3)) is the Jacobian. We get

$$\lim_{M\to\infty}\lim_{n\to\infty}[(2\pi)^n \det \boldsymbol{R}]^{-1/2}|D_n| \cdot \int_{E^n} H_{Mn}(\boldsymbol{u} + \boldsymbol{R}\Delta\boldsymbol{u})\exp[(\boldsymbol{u} + \boldsymbol{R}\Delta\boldsymbol{u})'\boldsymbol{p} - \tfrac{1}{2}(\boldsymbol{u} + \boldsymbol{R}\Delta\boldsymbol{u})'\boldsymbol{R}^{-1}(\boldsymbol{u} + \boldsymbol{R}\Delta\boldsymbol{u})]\,d\boldsymbol{u}$$

which in turn is equal to

$$|D|\lim_{M\to\infty}\lim_{n\to\infty} E\{H_{Mn}(\boldsymbol{x} + \boldsymbol{R}\Delta\boldsymbol{x})\exp[(\boldsymbol{x} + \boldsymbol{R}\Delta\boldsymbol{x})'\boldsymbol{p} - \boldsymbol{x}'\Delta\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}'\Delta'\boldsymbol{R}\Delta\boldsymbol{x}]\}. \tag{2.2}$$

We would like to pass the limit on n inside the expected value. Now the expectand is easily shown to be bounded independently of n by Lemma 3 so things look promising. However, we do not know if $\lim_{n\to\infty} H_{Mn}(\boldsymbol{x} + \boldsymbol{R}\Delta\boldsymbol{x})$ exists or not. To get around this difficulty, we apply Fatou's Lemma. Denoting the expectand by $F_{Mn}(\boldsymbol{x})$, we have

$$E\left\{\liminf_{n\to\infty} F_{Mn}(\boldsymbol{x})\right\} \leqq \lim_{n\to\infty} E\{F_{Mn}(\boldsymbol{x})\} \leqq E\left\{\limsup_{n\to\infty} F_{Mn}(\boldsymbol{x})\right\}.$$

In this inequality, let $M \to \infty$. Using monotone convergence, we may pass the limit on M inside the expected values in the two extreme members. But for M large enough, $H_{Mn}(\boldsymbol{x} + \boldsymbol{R}\Delta\boldsymbol{x}) = 1$ and hence

$$\lim_{M\to\infty}\liminf_{n\to\infty} F_{Mn}(\mathbf{x}) = \lim_{M\to\infty}\limsup_{n\to\infty} F_{Mn}(\mathbf{x})$$
$$= \lim_{n\to\infty}\exp[(\mathbf{x}+\mathbf{R}\Delta\mathbf{x})'\mathbf{p}-\mathbf{x}'\Delta\mathbf{x}-\tfrac{1}{2}\mathbf{x}'\Delta'\mathbf{R}\Delta\mathbf{x}]$$
$$= \exp\left[\int_a^b (Tx)(t)\,dp(t)-\Phi(x)/2\right].$$

Thus continuing from (2.2), we obtain

$$E\left\{\exp\int_a^b x(t)\,dp(t)\right\} = |D|\,E\left\{\exp\left[\int_a^b (Tx)(t)\,dp(t)-\Phi(x)/2\right]\right\}. \tag{2.3}$$

To extend to the class of all B measurable functions F, we proceed as follows. Let $T^{-1}B$ denote the class of all subsets of C which have the form $T^{-1}M$, for some $M\in B$. Now $T^{-1}B$ is a Borel field and $T^{-1}B\subset B$. To see the later, note that $T^{-1}\{x\in C: x(t_0)<c\}\in B$ and use the fact B is generated by sets of the form $\{x\in C: x(t_0)<c\}$.

Now let $M\in B$ and define a set function λ^* on B by

$$\lambda^*(M) = E\{\chi_{T^{-1}M}(x)\,|D|\exp[-\Phi(x)/2]\}$$
$$= E\{\chi_M(Tx)\,|D|\exp[-\Phi(x)/2]\},$$

χ_A denoting the set characteristic function of A. λ^* is a probability measure on B. Consider the probability space $\{C,B,\lambda^*\}$. Denoting expected values on it by E^*, we see that for any $M\in B$

$$E^*\{\chi_M(x)\} = E\{\chi_M(Tx)\,|D|\exp[-\Phi(x)/2]\}.$$

This result readily extends to any B measurable function F, i.e.,

$$E^*\{F(x)\} = E\{F(Tx)\,|D|\exp[-\Phi(x)/2]\}. \tag{2.4}$$

But $\int_a^b x(t)\,dp(t)$ is such a function and hence comparing (2.3) and (2.4), we see that $\{C,B,\lambda\}$ and $\{C,B,\lambda^*\}$ have the same moment generating functional. However, this functional uniquely determines the measures of all sets in B and thus the expected values of all B measurable functions so that $E\{F(x)\}=E^*\{F(x)\}$. Applying (2.4), we have the desired result.

Case 2. $\mathbf{R}$ is singular (for some values of n arbitrarily large). Let $\{C,B,\lambda_w\}$ be the Wiener process with covariance function $w(s,t)\equiv\min(s,t)$. On $C\times C$, let λ denote the product measure $\lambda_r\times\lambda_w$ and let $x_m(t)=x(t)+z(t)/m$ where $x\in\{C,B,\lambda_r\}$ and $z\in\{C,B,\lambda_w\}$. Then $\{x_m(t),\ a\leqq t\leqq b\}$ is a Gaussian process with covariance function

$$r_m(s,t)=r(s,t)+w(s,t)/m^2.$$

Moreover $\mathbf{R}_m$, the $n\times n$ matrix with elements $r_m(t_i,t_j)$, is nonsingular. Thus adding a subscript m where appropriate, we may proceed much as in Case 1. We obtain

$$E\left\{\exp\int_a^b x(t)\,dp(t)\right\}$$
$$= \lim_{M\to\infty}\lim_{n\to\infty}\lim_{m\to\infty} E\{H_{Mn}(x_m)\exp(x'_m p)\}$$
$$= \lim_{M\to\infty}\lim_{n\to\infty}\lim_{m\to\infty} |D_{nm}|\,E\{H_{Mn}(x_m + R_m\Delta x_m)$$
$$\cdot \exp[(x_m + R_m\Delta x_m)'p - x'_m\Delta x_m - \tfrac{1}{2}x'_m\Delta' R_m\Delta x_m]\}.$$

It is easy to show that $\lim_{m\to\infty} D_{nm} = D_n$. Also one may prove an analog of Lemma 3 which allows us to pass the limit on m inside the expected value. From here on the proof proceeds as in Case 1.

3. **Applications to triangular covariance functions.** We consider the class of covariance functions of the form

$$r(s,t) = \begin{cases} \theta(s)\phi(t) & s \leqq t, \\ \theta(t)\phi(s) & s \geqq t, \end{cases}$$

where

(3.1) $\theta(a) \geqq 0$ and $\phi(t) > 0$ on $I \equiv [a,b]$,

(3.2) θ'' and ϕ'' exist and are continuous on I,

(3.3) $\phi(t)\theta'(t) - \theta(t)\phi'(t) > 0$ on I.

This class of covariance functions and the corresponding class of Gaussian processes have already been the objects of considerable study in two papers of the author [10], [11]. It is known, for example, that such processes have representations with continuous sample functions. For this class of processes we may reformulate Theorem 2 in a form just like Theorem 1.

THEOREM 3. *Let $\{C, B, \lambda_r\}$ be a Gaussian process determined by a triangular covariance function*

$$r(s,t) = \begin{cases} \theta(s)\phi(t) & s \leqq t, \\ \theta(t)\phi(s) & s \geqq t, \end{cases}$$

where conditions (3.1), (3.2) *and* (3.3) *hold. Let*

$$(Tx)(t) = x(t) + \int_a^b \int_a^t M(u,s)\,du\,dx(s)$$

be a transformation defined on C with $M \in \mathrm{BV}H$ and $\mathscr{D} \neq 0$ (see 0.1). Then T carries C onto C in a one-to-one manner and if F is a measurable function for which either side of the following equation exists, both sides exist and are equal.

$$E\{F(x)\} = |\mathscr{D}|E\{F(Tx)\exp[-\Psi(x)/2]\} \tag{3.4}$$

where

$$\begin{aligned}\Psi(x) = 2 &\int_a^b \int_a^b N(t,s)\, d[x(t)/\phi(t)]\, dx(s) \\ &+ \int_a^b \int_a^b \int_a^b N(u,s)N(u,t)\, d[\theta(u)/\phi(u)]\, dx(s)\, dx(t)\end{aligned} \tag{3.5}$$

and

$$N(t,s) = \left[\phi(t)M(t,s) - \phi'(t)\int_a^t M(u,s)du\right]/[\phi(t)\theta'(t) - \phi'(t)\theta(t)]. \tag{3.6}$$

REMARK. Theorem 1 is an immediate consequence of this theorem. One need only note that for $r(s,t) = \min(s,t)$, $\theta(s) = s$, $\phi(s) = 1$ and $N(t,s) = M(t,s)$.

Before proving Theorem 3, we state a technical lemma which will allow us to manipulate various Stieltjes integrals.

LEMMA 4. *If x is continuous on I, r is continuous on $I \times I$ and $K \in \mathrm{BV}H$, then*

$$\int_a^b \int_a^b x(s)r(t,u)dK(u,s) = \int_a^b x(s)d_s\left[\int_a^b r(t,u)d_uK(u,s)\right].$$

Proof. Let $\varepsilon > 0$ be given and let $S = [a = s_0 < s_1 < \cdots < s_n = b]$ and $U = [a = u_0 < u_1 < \cdots < u_m = b]$ be partitions of $[a,b]$. Choose $\delta > 0$ so small that $|S| < \delta$ and $|U| < \delta$ imply that

$$A(S) = \left|\sum_{i=1}^n x(s_i)\int_a^b r(t,u)d_u[K(u,s_i) - K(u,s_{i-1})] - \int_a^b x(s)d_s\left[\int_a^b r(t,u)d_uK(u,s)\right]\right| < \varepsilon/3$$

and

$$B(S,U) = \left|\int_a^b \int_a^b x(s)r(t,u)dK(u,s) - \sum_{i=1}^n \sum_{j=1}^m x(s_i)r(t,u_j)[K(u_j,s_i) - K(u_j,s_{i-1}) - K(u_{j-1},s_i) + K(u_{j-1},s_{i-1})]\right| < \varepsilon/3.$$

Next let S_0 be a fixed partition with $|S_0| < \delta$ and choose a partition U_0 with $|U_0| < \delta$ and such that

$$C(S_0,U_0) = \left|\sum_{i=1}^n x(s_i)\int_a^b r(t,u)d_u[K(u,s_i) - K(u,s_{i-1})] - \sum_{i=1}^n x(s_i)\sum_{j=1}^m r(t,u_j)[K(u_j,s_i) - K(u_j,s_{i-1}) - K(u_{j-1},s_i) + K(u_{j-1},s_{i-1})]\right| < \varepsilon/3.$$

Then

$$\left| \int_a^b \int_a^b x(s)r(t,u)dK(u,s) - \int_a^b x(s)d_s \left[\int_a^b r(t,u)d_uK(u,s) \right] \right|$$
$$\leqq B(S_0, U_0) + C(S_0, U_0) + A(S_0) < \varepsilon$$

from which the result follows.

Proof of Theorem 3. Let $n(t,s) = N(t,s)$ on the interior of the rectangle $I \times I$ and let $n(t,s) = 0$ on the boundary so that $n(a,s) = n(s,a) = n(b,s) = n(s,b) = 0$ for $s \in I$. We observe that $n \in \mathrm{BV}\,H$ since N has this property as is easily checked from (3.6). Define K by

$$K(u,s) = \int_a^u [1/\phi(v)]d_v n(v,s)$$

and note that $K \in \mathrm{BV}\,H$ and $K(u,a) = K(u,b) = 0$. Thus

$$\begin{aligned} &\int_a^b \int_a^b x(s)r(t,u)dK(u,s) \\ &\quad = \int_a^b x(s)d_s \left[\int_a^b r(t,u)d_uK(u,s) \right], \qquad \text{(Lemma 4)} \\ &\quad = - \int_a^b \int_a^b r(t,u)d_uK(u,s)dx(s). \end{aligned}$$

But

$$\begin{aligned} &- \int_a^b r(t,u)d_uK(u,s) \\ &= - \int_a^b [r(t,u)/\phi(u)]d_un(u,s) \\ &= - \phi(t) \int_a^t [\theta(u)/\phi(u)]d_un(u,s) - \theta(t) \int_t^b d_un(u,s) \\ &= \phi(t) \int_a^t n(u,s)d[\theta(u)/\phi(u)] \\ &= \phi(t) \int_a^t \frac{d}{du} \left[\int_a^u M(v,s)dv/\phi(u) \right] du \\ &= \int_a^t M(v,s)\, dv \end{aligned} \tag{3.7}$$

and so the transformation of Theorem 2 reduces to that of Theorem 3.

Moreover

$$\int_a^b\int_a^b x(s)x(t)dK(t,s) = \int_a^b x(s)d_s\left[\int_a^b x(t)d_tK(t,s)\right], \quad \text{(see Lemma 4)}$$
$$= -\int_a^b\int_a^b x(t)d_tK(t,s)dx(s)$$
$$= -\int_a^b\int_a^b [x(t)/\phi(t)]d_tn(t,s)dx(s)$$
$$= \int_a^b\int_a^b n(t,s)d[x(t)/\phi(t)]dx(s)$$
$$= \int_a^b\int_a^b N(t,s)d[x(t)/\phi(t)]dx(s).$$

Also

$$\int_a^b\int_a^b\int_a^b\int_a^b x(s)x(t)r(u,v)dK(u,s)dK(v,t)$$
$$= \int_a^b\int_a^b x(s)x(t)d\left[\int_a^b\int_a^b r(u,v)d_uK(u,s)d_vK(v,t)\right],$$

(by an argument similar to that in Lemma 4)

$$= \int_a^b\int_a^b\left[\int_a^b\int_a^b r(u,v)d_uK(u,s)d_vK(v,t)\right]dx(s)dx(t).$$

But

$$\int_a^b\int_a^b r(u,v)d_vK(v,t)d_uK(u,s)$$
$$= -\int_a^b\int_a^u M(v,t)dvd_uK(u,s), \quad \text{(see (3.7))}$$
$$= -\int_a^b\left[\int_a^u M(v,t)dv/\phi(u)\right]d_un(u,s)$$
$$= \int_a^b n(u,s)d_u\left[\int_a^u M(v,t)dv/\phi(u)\right]$$
$$= \int_a^b n(u,s)n(u,t)d[\theta(u)/\phi(u)]$$
$$= \int_a^b N(u,s)N(u,t)d[\theta(u)/\phi(u)].$$

The above results imply that (0.7) is the same as (3.5).

It remains to be shown that $D = \mathscr{D}$. It seems intuitively clear that this must be true since this is correct in the Wiener case. Moreover, it can be shown rigorously by manipulating D_n in an appropriate way, taking the limit as $n \to \infty$ and discovering that it is $\mathscr{D}$. The calculations are so notationally complicated that we have decided not to include them.

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