THE LATTICE OF TOPOLOGIES: STRUCTURE AND COMPLEMENTATION

BY

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The family of all topologies on a set is a complete, bounded lattice. The purpose of this paper is to study the structure of the lattice of topologies, employing the notion of ultraspace introduced by Fröhlich [7] and to show that this lattice is complemented.

The set of ultraspaces may naturally be divided into two classes, each of which generates a sublattice. One of these sublattices is the lattice of T_1-topologies. The other, which is studied in §2, is the lattice of principal topologies. Principal topologies are defined in terms of ultraspaces and are then characterized by properties of open sets.

Some topological properties of ultraspaces are investigated in §4 and maximal regular (maximal T_1, maximal normal, etc.) topologies are characterized in terms of ultraspaces.

The problem of complementation in the lattice of topologies has been outstanding for some time. However, several partial solutions have been provided. Hartmanis [11] first showed the lattice was complemented if the ground set was finite and asked whether this was true in the infinite case. Gaifman [8] gave a positive answer for denumerable sets and Berri [3], using this fact, was able to provide complements for certain special topologies such as a topological group with a dense, nonopen countable subgroup.

It is shown in §5 that the lattice of principal topologies is complemented. Gaifman [9] established that the complementation problem can be reduced to verifying that each T_1-topology has a complement. In §6, it is proved that if every T_1-topology has a lattice complement which is a principal topology, then every topology does. In §7 it is shown that the lattice of topologies on an arbitrary set is complemented by proving that every topology in the sublattice of T_1-topologies has a lattice complement which lies in the sublattice of principal topologies.

Preliminary definitions and remarks. Throughout this paper, E will denote an arbitrary set, and τ, with or without subscripts, will denote a topology on E,

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whose members are open sets. The symbol $|E|$ will denote the cardinality of the set E.

The definitions of T_0, T_1, T_2, *regular*, *completely regular* and *normal* topologies are those found in Kelley [12]. The definitions of *partially ordered set*, *lattice*, *sublattice*, *infimum* and *supremum* are those found in Szász [16] and Birkhoff [5]. The definitions and properties of filters and ultrafilters are those of Cartan [6], Samuel [14] and Schmidt [15].

The family Σ of all topologies definable on a set E, partially ordered by set inclusion, is a complete lattice (Vaidyanathaswamy [17]). The greatest element of Σ, the discrete topology, will be denoted by 1 and the least element, the trivial topology, by 0.

A topology τ on E is an *ultraspace* if the only topology on E strictly finer than τ is the discrete topology.

For a filter $\mathcal{F}$ on E, Fröhlich [7] defined $\mathfrak{S}(p,\mathcal{F})$ to be the family of sets $\mathfrak{P}(E-\{p\})\cup\mathcal{F}$, where $\mathfrak{P}(E-\{p\})$ is the collection of all subsets of E which do not contain p. Hence, $\mathfrak{S}(p,\mathcal{F})$ is a topology on E such that for every $x\in E$, $x\neq p$, the set $\{x\}$ is open and the open sets containing p are just the sets in $\mathcal{F}$ which contain p.

Fröhlich proved that the ultraspaces on E are exactly the topologies of the form $\mathfrak{S}(x,\mathcal{U})$ where $x\in E$ and $\mathcal{U}$ is an ultrafilter on $E, \mathcal{U}\neq\mathcal{U}(x)$; that the correspondence between ultraspaces and ordered pairs $(x,\mathcal{U})$ is one-to-one; and that every topology τ on E is the infimum of the ultraspaces on E which are finer than τ.

Since an ultrafilter is either principal or nonprincipal (i.e. it either contains a finite set or it does not), the class of ultraspaces on E is composed of the class of *nonprincipal ultraspaces* (those in which the ultrafilter is nonprincipal) and the class of *principal ultraspaces* (those in which the ultrafilter is principal).

A point $x\in E$ is called an *isolated point* of τ if $\{x\}\in\tau$. If x_0 is not an isolated point, then $\tau\leqq\mathfrak{S}(x_0,\mathcal{U})$ for some ultrafilter $\mathcal{U}$.

An *infraspace* is a topology such that the only topology strictly coarser than it is the trivial topology. Every infraspace has the form $\{E,A,\varnothing\}$ where $A\subset E$, $A\neq\varnothing$, $A\neq E$. Clearly every topology τ is the supremum of infraspaces coarser than τ.

1. **T_1-topologies.** A topology on E is a T_1-topology if for every $x\in E$, the set $\{x\}$ is closed. For every $x\in E$ and for every nonprincipal ultrafilter $\mathcal{U}$ on E, the set $E-\{x\}\in\mathcal{U}$. Hence $\{x\}$ is closed in every nonprincipal ultraspace. In a principal ultraspace $\mathfrak{S}(x,\mathcal{U}(y))$, the set $\{y\}$ is not closed. Therefore, an ultraspace is a T_1-topology if and only if it is a nonprincipal ultraspace.

THEOREM 1.1. *A topology τ on E is a T_1-topology if and only if it is the infimum of nonprincipal ultraspaces.*

Proof. Any topology finer than a T_1-topology must also be a T_1-topology; so a T_1-topology can be the infimum of only nonprincipal ultraspaces.

For each $x \in E$, the set $E - \{x\}$ is open in each nonprincipal ultraspace, and thus in the infimum of any family of nonprincipal ultraspaces, so $\{x\}$ is closed in this infimum.

The T_1-topologies form a lattice Λ, which is a complete sublattice of Σ, and which has been studied previously by Bagley [1], and Hartmanis [11]. The finest T_1-topology is the discrete topology and the coarsest is the *cofinite topology* $\mathscr{C}$, in which a nonempty set is open if and only if its complement is a finite set. On a finite set the only T_1-topology is the discrete topology.

2. **Principal topologies.** Every topology τ on E is the infimum of all ultraspaces on E finer than τ. If also $\tau = \inf\{\mathfrak{S}(x,\mathscr{U}(y)) \mid \tau \leqq \mathfrak{S}(x,\mathscr{U}(y))\}$, then τ is said to be a *principal topology*. If $x = y$, then $\mathfrak{S}(x,\mathscr{U}(y))$ is the discrete topology and if $x \neq y$, then $\mathfrak{S}(x,\mathscr{U}(y))$ is a principal ultraspace.

A principal ultraspace $\mathfrak{S}(p, \mathscr{U}(q))$ on E is a principal topology in which every open set containing p must also contain q.

The topology $\tau = \mathfrak{S}(p,\mathscr{U}(q)) \wedge \mathfrak{S}(q,\mathscr{U}(r))$ is a principal topology in which for each $x \in E$, $x \neq p$, $x \neq q$, the set $\{x\} \in \tau$. Every open set p must contain q and every open set containing p must also contain r. Therefore $\tau \leqq \mathfrak{S}(p,\mathscr{U}(r))$.

THEOREM 2.1. *If τ is a principal topology on E, the set*

$$B_x = \{y \in E \mid \mathfrak{S}(x,\mathscr{U}(y)) \geqq \tau\}$$

is open for each $x \in E$.

Proof. If $\{x\} \in \tau$ then $B_x = \{x\}$, since if there is a $y \in E$, $y \neq x$, such that $\tau \leqq \mathfrak{S}(x,\mathscr{U}(y))$, then $\{x\} \notin \tau$. Let $\tau \leqq \mathfrak{S}(p,\mathscr{U}(q))$. If $p \notin B_x$, then B_x is open in $\mathfrak{S}(p,\mathscr{U}(q))$. If $p \in B_x$, then $\tau \leqq \mathfrak{S}(x,\mathscr{U}(p)) \wedge \mathfrak{S}(p,\mathscr{U}(q)) \leqq \mathfrak{S}(x,\mathscr{U}(q))$ (Fröhlich [7]). Hence $q \in B_x$ and B_x is open in $\mathfrak{S}(p,\mathscr{U}(q))$. Since B_x is open in each $\mathfrak{S}(p,\mathscr{U}(q))$ finer than τ, $B_x \in \tau$.

THEOREM 2.2. *If τ is a principal topology on E, then for each $x \in E$, every set in τ containing x must also contain the set*

$$B_x = \{y \in E \mid \mathfrak{S}(x,\mathscr{U}(y)) \geqq \tau\}.$$

Proof. If S is a set containing x and $y \notin S$ for some $y \in B_x$, then S is not in $\mathfrak{S}(x,\mathscr{U}(y))$. Since $\tau \leqq \mathfrak{S}(x,\mathscr{U}(y))$, S is not in τ.

A subcollection $\mathscr{B}$ of τ is a *base of open sets minimal at each point* if for every $x \in E$ there is a $V \in \mathscr{B}$ such that $x \in V$ and every set in τ containing x must contain V.

THEOREM 2.3. *A topology τ is a principal topology if and only if it has a base of open sets minimal at each point.*

Proof. Let τ be a principal topology. For each $x \in E$, the set

$$B_x = \{y \in E \mid \mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau\}$$

is open (Theorem 2.1) and every set in τ containing x must contain B_x (Theorem 2.2).

Suppose τ is not a principal topology. Then there is a nonprincipal ultraspace $\mathfrak{S}(x, \mathscr{U})$ such that $\tau \leqq \mathfrak{S}(x, \mathscr{U})$. If there is a minimal open set V containing x, then $V \in \mathscr{U}$ and every open set containing x must contain V. Hence $\tau \leqq \mathfrak{S}(x, \mathscr{U}(y))$ for all $y \in V$ and $\tau \leqq \bigwedge\{\mathfrak{S}(x, \mathscr{U}(y)) \mid y \in V\} \leqq \mathfrak{S}(x, \mathscr{U})$. Thus, if τ has a base of open sets minimal at each point, then for each nonprincipal ultraspace $\mathfrak{S}(x, \mathscr{U})$ finer than τ there is a principal topology finer than τ and coarser than $\mathfrak{S}(x, \mathscr{U})$ which implies τ is a principal topology. Therefore, a nonprincipal topology can not have a base of open sets minimal at each point.

THEOREM 2.4. *A topology τ is a principal topology if and only if arbitrary intersections of open sets are open.*

Proof. If arbitrary intersections of open sets are open, for each $x \in E$ the intersection of all open sets containing x is a minimal open set at x and the family $\mathscr{B}$ of open sets minimal at x for all $x \in E$, is a base for τ. Hence τ is a principal topology.

If τ is a principal topology and $\mathscr{N}$ is a subcollection of τ, then $\bigcap \mathscr{N}$ is open since for each $x \in \bigcap \mathscr{N}$, the minimal open set containing x is also in $\bigcap \mathscr{N}$.

COROLLARY. *An infraspace is a principal topology.*

THEOREM 2.5. *The principal topologies form a sublattice of the lattice Σ.*

Proof. If τ_1 and τ_2 are principal topologies, then certainly $\tau_1 \wedge \tau_2$ is a principal topology. For each x, the set $U_x = V_x^1 \cap V_x^2$, where V_x^i is a minimal open set in τ_i containing x for $i = 1, 2$, is a minimal open set in $\tau_1 \vee \tau_2$ containing x and the family $\mathscr{B} = \{U_x \mid U_x \in \tau_1 \vee \tau_2$ is minimal at x, for all $x \in E\}$ is a base for $\tau_1 \vee \tau_2$, minimal at each point. Hence $\tau_1 \vee \tau_2$ is a principal topology.

The lattice of principal topologies on E will be denoted by Π.

The lattice Π is a complete lattice since Π is meet complete and is bounded above by 1. For any subset $\mathscr{H} \subset \Pi$, $\inf_\Sigma \mathscr{H} = \inf_\Pi \mathscr{H}$. However, Π is not a complete sublattice of Σ. Let τ be a T_1-ultraspace, $\tau \neq 1$. Then $\tau = \sup\{\gamma_i \mid i \in \Gamma\}$, where each γ_i, $i \in \Gamma$, is an infraspace coarser than τ. Each γ_i is a principal topology and $\tau = \sup_\Sigma\{\gamma_i \mid i \in \Gamma\} \leqq \sup_\Pi\{\gamma_i \mid i \in \Gamma\}$. But $\sup_\Pi\{\gamma_i \mid i \in \Gamma\}$, being finer than a T_1-topology, must also be a T_1-topology, so $\sup_\Pi\{\gamma_i \mid i \in \Gamma\} = 1 \neq \tau$.

A subset G of $E \times E$ is a *preorder relation* if and only if

(1) $(x, x) \in G$ for all $x \in E$; and

(2) $(x, y) \in G$ and $(y, z) \in G$ imply $(x, z) \in G$ for all $x, y, z \in E$.

Subsequently, the notation xGy will be used to denote that $(x, y) \in G$.

There is a *chain of length n from x to y* in G if and only if there are elements $x = x_0, x_1, \cdots, x_{n-1}, x_n = y$, $n \geqq 0$, such that $x_0Gx_1, x_1Gx_2, \cdots, x_{n-1}Gx_n$.

Operations $\wedge$ and $\vee$ are defined on preorder relations by

(1) $G_1 \wedge G_2 = G_1 \cap G_2$,

(2) $G_1 \vee G_2 = (G_1 \cup G_2)$, where $\hat{G} = \{(x,y) \mid$ there is a chain of length n from x to y in G, $n \geqq 0\}$.

Under these operations, the family $\mathscr{G}$ of preorder relations, partially ordered by set inclusion, is a lattice.

Each preorder relation G defines a topology τ_G: a set $S \subseteq E$ is open if and only if for each $x \in S$, if xGy then $y \in S$. Moreover, the topology τ_G determined by the preorder relation G is a principal topology and $\tau_G = \inf\{\mathfrak{S}(x, \mathscr{U}(y)) \mid xGy\}$. Each principal topology τ determines a preorder relation G_τ by $G_\tau = \{(x,y) \mid \tau \leqq \mathfrak{S}(x, \mathscr{U}(y))\}$.

THEOREM 2.6. *The lattice* Π *of principal topologies is anti-isomorphic to the lattice* $\mathscr{G}$ *of preorder relations.*

Proof. There is a one-to-one correspondence between principal topologies in Π and preorder relations in $\mathscr{G}$ since the mappings $\eta : \Pi \to \mathscr{G}$ and $\phi : \mathscr{G} \to \Pi$ defined by

$$\eta(\tau) = G_\tau = \{(x,y) \mid \mathfrak{S}(x, \mathscr{U}(y)) \geqq \tau\}$$

and

$$\phi(G) = \tau_G = \inf\{\mathfrak{S}(x, \mathscr{U}(y)) \mid xGy\}$$

are inverses. That is, $\eta(\phi(G)) = G$ and $\phi(\eta(\tau)) = \tau$.

Since $xG_\tau y$ holds if and only if $\tau \leqq \mathfrak{S}(x, \mathscr{U}(y))$, if $\tau_1 \leqq \tau_2$ then $G_{\tau_1} \geqq G_{\tau_2}$. And since $\tau_G = \bigwedge \{\mathfrak{S}(x, \mathscr{U}(y)) \mid xGy\}$, then $G_1 \leqq G_2$ implies $\tau_{G_1} \geqq \tau_{G_2}$. Hence η and $\eta^{-1} = \phi$ are antitone.

Thus, the lattice of preorder relations is a complete lattice (since Π is) with a least element, namely the relation $\Delta = \{(x,x) \mid x \in E\}$ corresponding to the discrete topology and a greatest element $E \times E$ corresponding to the trivial topology.

Let G be a relation on E. There is a *path from* x_0 *to* x_n in G if there is a finite collection of elements $x_1, x_2, \cdots, x_{n-1}$ of E such that x_iGx_{i-1} or $x_{i-1}Gx_i$, $i = 1, 2, \cdots, n$. If there is a path from x_0 to x_n, then there is a path from x_n to x_0.

A subset $F \subseteq E$ is a *component* of E relative to a relation G if for every pair $x, y \in F$ there is a path from x to y in G and for $w \notin F$ there is no path from x to w for any $x \in F$.

Two distinct components of E relative to G are disjoint and every point of E is in some component.

A *relation* G on E is *connected* if E is a component relative to G.

A *topology* τ on E is *connected* if E is not the union of two nonempty, disjoint open sets.

THEOREM 2.7. *A principal topology τ_G is connected if and only if G is a connected relation.*

Proof. Suppose τ_G is not connected. Then $E = U \cup V$ where U, $V \in \tau_G$, $U \neq \varnothing$, $V \neq \varnothing$ and $U \cap V = \varnothing$. Suppose G is a connected relation. Let $x \in U$ and $y \in V$. Then there is a finite subset $\{x = x_1, x_2, \cdots, x_n = y\} \subset E$ such that $x_i G x_{i+1}$ or $x_{i+1} G x_i$ for $i = 1, 2, \cdots, n-1$. Denote by x_k the first x_i such that $x_i \in V$. Since $x_1 \in U$, $k > 1$. Thus $x_{k-1} \in U$ and $x_k \in V$. If $x_{k-1} G x_k$, then U is not open in τ_G and if $x_k G x_{k-1}$, then V is not open in τ_G. This is a contradiction so G is not a connected relation.

Suppose G is not a connected relation. For some $x_0 \in E$ let $U = \{y \in E \mid \text{there is a path between } x_0 \text{ and } y\}$. The set U is open. For each $x \in U$, the set $\{y \in E \mid xGy\} \subset U$ because if there is a path between x_0 and y and yGz holds, then there is a path between x_0 and z. Since G is not connected there is some $y_0 \in E - U$ and $E - U$ is also open. Otherwise for some $y \in E - U$, yGz and $z \in U$, but since there is a path between x_0 and z and yGz, then there is a path between x_0 and y. Thus $E - U$ must be open, and τ is not connected.

A topology on E which is neither a T_1-topology nor a principal topology is a *mixed topology*. A mixed topology can be represented as the infimum of a T_1-topology and a principal topology, but this representation need not be unique. The supremum of two mixed topologies can be a T_1-topology or a principal topology. For example, if $\tau = \mathfrak{S}(x, \mathscr{U}) \wedge \mathfrak{S}(p, \mathscr{U}(q))$, $\tau' = \mathfrak{S}(x, \mathscr{U}) \wedge \mathfrak{S}(q, \mathscr{U}(p))$, and $\tau'' = \mathfrak{S}(y, \mathscr{V}) \wedge \mathfrak{S}(p, \mathscr{U}(q))$ where $x \neq y$ and $\mathscr{U} \neq \mathscr{V}$, then $\tau \vee \tau' = \mathfrak{S}(x, \mathscr{U})$ and $\tau \vee \tau'' = \mathfrak{S}(p, \mathscr{U}(q))$. The infimum of two mixed topologies cannot be a T_1-topology, but it can be a principal topology.

Let $\mathscr{U}$ and $\mathscr{V}$ be distinct nonprincipal ultrafilters. Then there is a set A such that $A \in \mathscr{U}$ and $E - A \in \mathscr{V}$. Let $\tau_1 = \bigwedge \{\mathfrak{S}(x, \mathscr{U}(q)) \mid q \in A - \{x\}\}$ and $\tau_2 = \bigwedge \{\mathfrak{S}(y, \mathscr{U}(q)) \mid q \in E - A - \{x\}\}$. Then $\tau_1 \leqq \mathfrak{S}(x, \mathscr{U})$ and $\tau_2 \leqq \mathfrak{S}(y, \mathscr{V})$, and $\mathfrak{S}(x, \mathscr{U}) \wedge \tau_2$ and $\mathfrak{S}(y, \mathscr{V}) \wedge \tau_1$ are mixed topologies. But

$$[\mathfrak{S}(x, \mathscr{U}) \wedge \tau_2] \wedge [\mathfrak{S}(y, \mathscr{V}) \wedge \tau_1] = [\mathfrak{S}(x, \mathscr{U}) \wedge \tau_1] \wedge [\mathfrak{S}(y, \mathscr{V}) \wedge \tau_2] = \tau_1 \wedge \tau_2,$$

which is a principal topology.

The reader who is interested only in the complementation of Σ, may skip §§3, 4 and 5.

3. **Lattice properties of Σ.** Vaidyanathaswamy [17] gave an example to show that the lattice of topologies on a set of cardinality c is not distributive. The following proof uses ultraspaces and is very straightforward.

THEOREM 3.1. *The lattice of topologies Σ on a set E is distributive if E has fewer than three elements. If E has three or more elements, Σ is not even modular.*

Proof. If E has one element or two elements, Σ is a distributive lattice. Let $E = \{p, q, r\}$ and let $\mathscr{K} = \mathfrak{S}(p, \mathscr{U}(q)) \wedge \mathfrak{S}(p, \mathscr{U}(r))$, $\mathscr{L} = \mathfrak{S}(p, \mathscr{U}(r))$, $\mathscr{N} = \mathfrak{S}(r, \mathscr{U}(q))$ be topologies on E. Thus $\mathscr{K} \leqq \mathscr{L}$. But $(\mathscr{K} \vee \mathscr{N}) \wedge \mathscr{L} = 1 \wedge \mathscr{L} = \mathscr{L}$ and $\mathscr{K} \vee (\mathscr{N} \wedge \mathscr{L}) \leqq \mathfrak{S}(p, \mathscr{U}(q))$ since $\mathscr{K} \leqq \mathfrak{S}(p, \mathscr{U}(q))$ and $\mathscr{N} \wedge \mathscr{L} \leqq \mathfrak{S}(p, \mathscr{U}(q))$. Thus $(\mathscr{K} \vee \mathscr{N}) \wedge \mathscr{L} \nleqq \mathscr{K} \vee (\mathscr{N} \wedge \mathscr{L})$ since $\mathscr{L} \nleqq \mathfrak{S}(p, \mathscr{U}(q))$ and Σ is not modular.

A lattice L is *self-dual* if and only if there is a one-to-one mapping ϕ of L onto itself such that $\phi(a \wedge b) = \phi(a) \vee \phi(b)$ and $\phi(a \vee b) = \phi(a) \wedge \phi(b)$.

If Σ is self-dual, there is a one-to-one map ϕ of Σ onto itself. If $a \leqq b$, then $\phi(a) = \phi(a \wedge b) = \phi(a) \vee \phi(b)$ which implies $\phi(b) \leqq \phi(a)$. Thus $\phi(0) = 1$ and $\phi(1) = 0$ and infraspaces map onto ultraspaces and conversely, so the number of infraspaces and ultraspaces must be equal.

In the lattice of topologies on a set E, if $|E| = n < \infty$, there are $n(n-1)$ ultraspaces (all principal) and $2^n - 2$ infraspaces. If $|E| \geqq \aleph_0$, there are $2^{|E|}$ infraspaces and $2^{2^{|E|}}$ ultraspaces on E (there are $2^{2^{|E|}}$ ultrafiters on E [2], [10]). Thus the number of ultraspaces equals the number of infraspaces only when $|E| \leqq 3$.

THEOREM 3.2. *The lattice of topologies on E is self-dual if and only if* $|E| \leqq 3$.

Proof. If $|E| > 3$, then Σ cannot be self-dual by the preceding argument. If $|E| = 1$ or $|E| = 2$, Σ is obviously self-dual. If $|E| = 3$, there are twenty-nine topologies on E, but it can be seen by rotating the diagram on page 386 by 180° that this lattice is also self-dual.

4. **Topological properties.** Ultraspaces may be studied easily because of their point-ultrafilter representation. In this section some topological properties of the ultraspaces are investigated and the maximal T_0, T_1, T_2, regular and normal topologies are characterized. Other topological properties of T_1-ultraspaces are considered in Gillman and Jerrison [10].

THEOREM 4.1. *Every ultraspace is* T_0, *normal and extremally disconnected.*

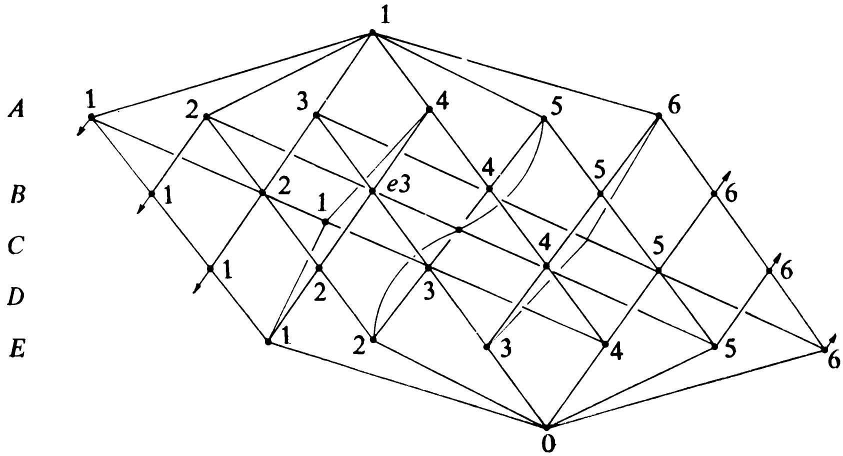
Proof. Let $\mathfrak{S}(x, \mathscr{U})$ be an ultraspace on E. Then $\mathfrak{S}(x, \mathscr{U})$ is a T_0-topology since for any two points of E at least one as a set is open.

If A and B are disjoint closed sets, then $x \notin A$ or $x \notin B$. If $x \notin A$ then A and $E - A$ are open and $\mathfrak{S}(x, \mathscr{U})$ is normal.

If V is an open set and $x \in V$ then $V = \bar{V}$. If $x \notin V$ then either $E - V \in \mathscr{U}$ or $V \in \mathscr{U}$. If $E - V \in \mathscr{U}$ then $V = \bar{V}$. If $V \in \mathscr{U}$ then $\bar{V} = V \cup \{x\}$, but $V \cup \{x\} \in \mathscr{U}$ and hence is open. Thus $\mathfrak{S}(x, \mathscr{U})$ is extremally disconnected.

THEOREM 4.2. *No ultraspace on a set E is connected if* $|E| \geqq 3$.

Lattice of topologies on a three point set $E = \{a, b, c\}$



0 Trivial Topology

1 Discrete Topology

A*

1	$\{b\}$	$\{c\}$	$\{b,c\}$	$\{c,a\}$
2	$\{c\}$	$\{b\}$	$\{a,b\}$	$\{b,c\}$
3	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b,c\}$
4	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{a,c\}$
5	$\{a\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$
6	$\{a\}$	$\{c\}$	$\{c,a\}$	$\{c,b\}$

B

1	$\{b\}$	$\{c\}$	$\{b,c\}$
2	$\{b\}$	$\{b,a\}$	$\{b,c\}$
3	$\{a\}$	$\{b\}$	$\{a,b\}$
4	$\{a\}$	$\{a,b\}$	$\{a,c\}$
5	$\{a\}$	$\{c\}$	$\{a,c\}$
6	$\{c\}$	$\{c,a\}$	$\{c,b\}$

C

1	$\{a,c\}$	$\{b\}$
2	$\{a,b\}$	$\{c\}$
3	$\{a\}$	$\{b,c\}$

D

1	$\{b\}$	$\{b,c\}$
2	$\{b\}$	$\{b,a\}$
3	$\{a\}$	$\{a,b\}$
4	$\{a\}$	$\{a,c\}$
5	$\{c\}$	$\{c,a\}$
6	$\{c\}$	$\{c,b\}$

E

1	$\{b\}$
2	$\{a,b\}$
3	$\{a\}$
4	$\{a,c\}$
5	$\{c\}$
6	$\{b,c\}$

* ϕ, E omitted.

Proof. Let $\mathfrak{S}(p,\mathscr{V})$ be an ultraspace on E. If $|E| \geqq 3$, there is an $x \in E$ such that $x \neq p$ and $\mathscr{V} \neq \mathscr{U}(x)$. Thus $\{x\}$ and $E-\{x\}$ are both open.

An ultraspace is a T_1-topology if and only if it is a nonprincipal ultraspace (cf. Theorem 1.1).

THEOREM 4.3. *An ultraspace is a T_2-topology if and only if is it a nonprincipal ultraspace.*

Proof. For any two points, at least one as a set is open and its complement is also open and contains the other point.

A nonprincipal ultraspace is normal and T_1, hence is regular and completely regular. A principal ultraspace $\mathfrak{S}(x,\mathscr{U}(y))$ is not regular (since $\{x\}$ is a closed set, $y \notin \{x\}$, but every open set containing x contains y), and therefore not completely regular. An ultraspace is totally disconnected (i.e. no connected subset contains more than one point) if and only if it is a nonprincipal ultraspace. Similarly, being nonprincipal is a necessary and sufficient condition for an ultraspace to be zero-dimensional (i.e. there is a base for the topology such that each set in the base is both open and closed).

A topological space is a door space if every subset is either open or closed. It is easy to see that every ultraspace is a door space. Actually it is not difficult to characterize door spaces.

THEOREM 4.4. *A space is a door space if and only if the ultraspaces in its representation have either a common point or a common ultrafilter.*

Proof. Let $\tau_1 = \bigwedge\{\mathfrak{S}(x,\mathscr{U}_i) \mid i \in I_1\}$ and $\tau_2 = \bigwedge\{\mathfrak{S}(y_i,\mathscr{V}) \mid i \in I_2\}$. A set not containing x is in τ_1 and if a set contains x, its complement is in τ_1. Thus τ_1 is a door space. Every set in $\mathscr{V}$ is in τ_2 and if a set is not in $\mathscr{V}$, its complement is. Hence τ_2 is a door space.

If $\tau \leqq \mathfrak{S}(x,\mathscr{U}) \wedge \mathfrak{S}(y,\mathscr{V})$ where $x \neq y$ and $\mathscr{U} \neq \mathscr{V}$, then there is a set A such that $A \in \mathscr{U}$ and $E - A \in \mathscr{V}$. The set $(A \cup \{y\}) \cap (E-\{x\})$ is not in $\mathfrak{S}(y,\mathscr{V})$ and its complement is not in $\mathfrak{S}(x,\mathscr{U})$. Thus $(A \cup \{y\}) \cap (E-\{x\})$ is neither open nor closed in τ.

Vaidyanathaswamy [17] erroneously stated that there are no maximal T_1-topologies. Liu [13] gave a necessary and sufficient condition for a topology to be a maximal T_1-topology and then proved that maximal T_1-topologies do exist by constructing an ultraspace with the aid of Zorn's lemma.

An immediate consequence of Theorem 1.1 is that every maximal T_1-topology is of the form $\mathfrak{S}(p,\mathscr{U})$ where $p \in E$, $\mathscr{U}$ is a nonprincipal ultrafilter on E, $\mathscr{U} \neq \mathscr{U}(p)$. Then every subset of $E-\{p\}$ and every set in $\mathscr{U}$ is open.

The maximal T_1-topologies are precisely the maximal T_2-topologies. Since every nondiscrete topology is coarser than some ultraspace and every ultraspace is a T_0-topology, the ultraspaces are the maximal T_0-topologies. Similarly, a normal topology is a maximal normal topology if and only if it is an ultraspace.

All nonprincipal ultraspaces are maximal regular spaces, but there are also principal topologies which are regular.

A collection $\mathscr{D}$ of principal ultraspaces

$$\{\mathfrak{S}(x_1,\mathscr{U}(x_2)),\ \mathfrak{S}(x_2,\mathscr{U}(x_3)),\cdots,\mathfrak{S}(x_n,\mathscr{U}(x_1))\}$$

is called a *cycle of order n*.

LEMMA 4.5. *A topology which is the intersection over a cycle of order 2 is a maximal regular topology.*

Proof. Let $\tau=\mathfrak{S}(p,\mathscr{U}(q))\wedge\mathfrak{S}(q,\mathscr{U}(p))$. Then τ is regular since every closed set is also open. The only nondiscrete topologies finer than τ are $\mathfrak{S}(p,\mathscr{U}(q))$ and $\mathfrak{S}(q,\mathscr{U}(p))$, neither of which is regular.

LEMMA 4.6. *A topology whose representation contains a cycle of order n is coarser than a maximal regular topology of the form* $\mathfrak{S}(x,\mathscr{U}(y))\wedge\mathfrak{S}(y,\mathscr{U}(x))$.

Proof. Let $\mathscr{D}$ be a cycle of order n in the representation of τ. Then $\tau\leqq\wedge\mathscr{D}$ and since $\mathfrak{S}(x_1,\mathscr{U}(x_2))\wedge\mathfrak{S}(x_2,\mathscr{U}(x_3))\leqq\mathfrak{S}(x_1,\mathscr{U}(x_3))$, $\wedge\mathscr{D}\leqq\mathfrak{S}(x_1,\mathscr{U}(x_n))\wedge\mathfrak{S}(x_n,\mathscr{U}(x_1))$.

LEMMA 4.7. *A principal topology with no cycle in its representation is not regular.*

Proof. By deleting (x,x) from a relation G corresponding to τ, the topology τ_G remains the same. So assume $(x,x)\notin G$ for all $x\in E$.

Let $\tau=\wedge_{i\in I}\{\mathfrak{S}(x_i,\mathscr{U}_i)\}$ where $\{\mathfrak{S}(x_i,\mathscr{U}_i)\mid i\in I\}$ contains no cycle. If $B_{x_0}=\{y\in E\mid \text{there is a chain from } x_0 \text{ to } y \text{ in } G\}$ for some point x_0, then $x_0\notin B_{x_0}$ since for $x_0\in B_{x_0}$ and $(x_0,x_0)\notin G$, there is a cycle of order $n>1$ in $\{\mathfrak{S}(x_i,\mathscr{U}_i)\mid i\in I\}$ which is a contradiction. The set B_{x_0} is open, so $E-B_{x_0}$ is closed. For $y\in B_{x_0}$ there is no open set containing $E-B_{x_0}$ which does not contain y since $x_0\in E-B_{x_0}$. Thus τ is not regular.

THEOREM 4.8. *A regular topology is a maximal regular topology if and only if it is a nonprincipal ultraspace or it is of the form* $\mathfrak{S}(x,\mathscr{U}(y))\wedge\mathfrak{S}(y,\mathscr{U}(x))$ *for some* $x,y\in E$.

Proof. If τ is a nonprincipal ultraspace or $t\tau=\mathfrak{S}(x,\mathscr{U}(y))\wedge\mathfrak{S}(y,\mathscr{U}(x))$, then τ is a maximal regular topology. Every T_1-topology and every mixed topology is coarser than some nonprincipal ultraspace and every regular principal topology is coarser than a maximal regular topology of the form

$$\mathfrak{S}(x,\mathscr{U}(y)\wedge\mathfrak{S}(y,\mathscr{U}(x)).$$

5. **Principal complements for principal topologies.** In 1958, Hartmanis [11] proved that the lattice of topologies on a finite set is complemented. Since every topology on a finite set is a principal topology, this implies that the lattice Π of principal topologies on a finite set is complemented.

In this section it will be shown that the lattice $\mathcal{G}$ of preorder relations is a complemented lattice and hence, that the lattice Π of principal topologies on an arbitrary set is complemented. Actually, this result is just a special case of the more general theorem concerning the complementation of Σ and follows from Theorem 7.8.

A *lattice complement of a preorder relation* G is a preorder relation G' such that $G \vee G' = E \times E$ and $G \wedge G' = \Delta = \{(x,x) \mid x \in E\}$.

The set complement $G^* = (E \times E) - G$ is not necessarily transitive and in general $G^* \wedge G \neq \Delta$.

THEOREM 5.1. *The lattice $\mathcal{G}$ of preorder relations on a set E is a complemented lattice.*

Proof. Let G be a preorder relation on a set E and let $E = \bigcup_{\alpha \in \theta} E_\alpha$ where each E_α is a component of E relative to G. Then for $x, y \in E, x$ and y are in the same component if and only if there is a path from x to y in G, i.e., if xGy or yGx.

Let $G_1 = \{(y,x) \mid xGy \text{ and not } yGx\} \cup \Delta$. Select one point from each E_α and denote it by x_α (the x_α's, $\alpha \in \theta$, will now remain fixed). Let $G_2 = \{(x_\alpha, x_\beta) \mid \alpha, \beta \in \theta\} \cup \Delta$. The relations G_1 and G_2 are both preorder relations so $G' = G_1 \vee G_2$ is also a preorder relation.

If xGy, then x and y are in the same component E_α of E and by definition, $(x,y) \notin G_1$, hence $(x,y) \notin G'$. Therefore $G \wedge G' = \Delta$.

For $(x,y) \in E \times E$, x and y are in components E_α and E_β, respectively, of E. If $(x, x_\alpha) \notin G$ then $(x, x_\alpha) \in G_1$ and if $(x_\beta, y) \notin G$ then $(x_\beta, y) \in G_1$. Since $(x_\alpha, x_\beta) \in G_2$, $\{(x, x_\alpha), (x_\alpha, x_\beta), (x_\beta, y)\} \subset G \cup G'$ so $(x,y) \in G \vee G'$. Thus $G \vee G' = E \times E$.

THEOREM 5.2. *The lattice Π of principal topologies is a complemented lattice.*

This follows directly from Theorem 2.6, since Π and $\mathcal{G}$ are anti-isomorphic.

6. **Reduction to T_1-topologies.** Hartmanis [11], after concluding that the lattice of topologies on a finite set is complemented, posed the question: "Is the lattice of topologies on an infinite set complemented?" Gaifman [8] showed that this lattice is complemented if the set E is countable. To obtain this result, he proved that if every T_1-topology on a set has a complement, then every topology on that set has one.

In this section it will be shown that if every T_1-topology on a set has a complement which is a principal topology, then every topology on that set has a principal complement. It will also be established that every T_1-topology on a countable set has a principal complement.

A topology τ is said to have a *principal complement* if there is a principal topology τ such that $\tau \vee \tau' = 1$ and $\tau \wedge \tau' = 0$.

If τ is a topology on a set E and $F \subset E$, then $\tau | F$ denotes the restriction of τ to F (i.e., the relative topology on F induced by τ).

THEOREM 6.1. *If every T_0-topology (T_1-topology) τ on E has a (principal) complement and $F \subset E$, then every T_0-topology (T_1-topology) τ_0 on F has a (principal) complement.*

Proof. Let τ_0 be a T_0-topology on F, and define a topology τ on E by $\tau = \{(A \subset E) | A \cap F \in \tau_0\}$. Then τ is a T_0-topology and hence has a complement τ'. For $x \in F$ there is a $U \in \tau$, $V \in \tau'$ such that $\{x\} = U \cap V$. But $U \cap F \in \tau_0$ and $V \cap F \in \tau' | F$ so $\{x\} = (U \cap F) \cap (V \cap F) \in \tau_0 \vee \tau' | F$. Thus $\tau_0 \vee \tau' | F = 1$. For $U \in \tau_0 \wedge \tau' | F$, $U = V \cap F$ for some $V \in \tau'$, hence $V \in \tau$ (by definition of τ). Since $\tau \wedge \tau' = 0$, it follows that $V = \varnothing$ or $V = E$ and consequently that $U = \varnothing$ or $U = F$. Thus $\tau_0 \wedge \tau' | F = 0$, and $\tau' | F$ is a complement of τ_0. If τ' is principal, then for each $x \in F$ there is a minimal open set $B_x \in \tau'$ containing x. Thus $B_x \cap F$ is a minimal open set in $\tau' | F$ containing x, so $\tau' | F$ is a principal topology.

THEOREM 6.2. *If every T_0-topology on E has a principal complement, then every topology on E has a principal complement.*

Proof. Gaifman [9] showed that if every T_0-topology on a set has a complement then every topology does. He did this in the following way:

Let τ be a topology on E. Define $x \approx y$ by: for every $V \in \tau$, $x \in V$ if and only if $y \in V$. Obviously $\approx$ is an equivalence relation. There exists a subset E_1 of E whose intersection with every equivalence class consists of exactly one point. Now, $\tau_1 = \tau | E_1$ is a T_0-topology and if every T_0-topology on E has a complement then every T_0-topology on E_1 has a complement since $|E_1| \leqq |E|$. Denote this complement by τ_1'. Then $\tau' = \{A \subset E | A \cap E_1 \in \tau_1'\}$ is a complement for τ.

In Gaifman's notation, if every T_0-topology on E has a principal complement then every T_0-topology on E_1 has a principal complement (Theorem 6.1). Thus, let τ_1' be a principal complement of τ_1. For each $x \in E - E_1$, $\{x\} \in \tau'$. For each $x \in E_1$, there is a minimal set in τ_1' containing x and this set is thus a minimal set in τ' containing x. Therefore, τ' is a principal topology (Theorem 2.3).

THEOREM 6.3. *If every T_1-topology on a set E has a principal complement, then every topology on E has a principal complement.*

Proof. By Theorem 6.2 it suffices to show that if every T_1-topology on E has a principal complement, then every T_0-topology on E has a principal complement. So let τ be a T_0-topology on E.

Under the assumption that every T_1-topology on E has a complement, Gaifman [9] deduced that every T_0-topology on E has a complement in the following way:

He defined an ordered family of disjoint sets E_α, such that $\tau_\alpha = \tau \mid E_\alpha$ is a T_1-topology (the other properties of the E_α are not essential to this proof), and constructed the topology τ' in the following way:

Since each τ_α is a T_1-topology on E_α, by Theorem 6.1 τ_α has a complement τ'_α.

The topology τ' is generated by sets of the form:

(i) $\{x\}$, where $x \notin \bigcup_\alpha E_\alpha$.

(ii) V, $V \in \tau'_\alpha$ where α is such that for every $\beta > \alpha$ and for every $W \in \tau$ such that $E_\alpha \subseteq W$, $W \cap [\bigcup_{\beta \leqq \nu} E_\nu] \neq \varnothing$.

(iii) $V \cup \bigcup_{\beta \leqq \nu} E_\nu$ where $V \in \tau'_\alpha$ and α, β are such that $\beta > \alpha$ and for some $W \in \tau$, $E_\alpha \subseteq W$, and $W \cap [\bigcup_{\beta \leqq \nu} E_\nu] = \varnothing$.

Gaifman proved $\tau \vee \tau' = 1$ and $\tau \wedge \tau' = 0$.

Now, if τ^* is defined in a modified form as the topology generated by the following sets:

(i) and (ii) as above,

(iii) $V \cup \bigcup_{\beta_\alpha \leqq \nu} E_\nu$ where $V \in \tau'_\alpha$ and β_α, α are such that $\beta_\alpha > \alpha$ (β_α fixed for each α) and for some $W \in \tau$, $E_\alpha \subseteq W$, and $W \cap [\bigcup_{\beta_\alpha \leqq \nu} E_\nu] = \varnothing$, then it can be shown that τ^* is a principal topology if each τ'_α is. First, τ^* is a union of sets of the form (i), (ii) and (iii). The intersection of sets of the form (i) with different sets of the form (i), (ii) or (iii) is empty. The intersection of sets of the form (ii) with sets of the form (ii) or (iii) is of the form (ii) and the intersection of two sets of the form (iii) is again of form (iii). To elaborate this last point let $U_i = V_i \cup \bigcup_{\beta_{\alpha_i} \leqq \nu} E_\nu$, $i = 1, 2$, be two sets of form (iii) where $V_i \in \tau_{\alpha_i}$, $i = 1, 2$. If $\alpha_1 = \alpha_2$ then $\beta_{\alpha_1} = \beta_{\alpha_2}$ and $U_1 \cap U_2 = (V_1 \cap V_2) \cup \bigcup_{\beta_{\alpha_1} \leqq \nu} E_\nu$ and this is of form (iii). If $\alpha_1 < \alpha_2$ then $U_1 \cap U_2 = U_2$ if $\beta_{\alpha_1} \leqq \alpha_2$, and $U_1 \cap U_2 = \bigcup_{\beta_3 \leqq \nu} E_\nu$ if $\alpha_2 < \beta_{\alpha_1}$, where $\beta_3 = \max(\beta_{\alpha_1}, \beta_{\alpha_2})$. This last set is of the form (iii) also, where $V = \varnothing$.

Second, $\tau \vee \tau^* = 1$. If $x \notin \bigcup_\alpha E_\alpha$, then $\{x\} \in \tau^*$. If $x \in E_\alpha$, then $\{x\} = V \cap V'$ where $V \in \tau'_\alpha$, $V' \in \tau'_\alpha$ and $V = U \cap E_\alpha$ for some $U \in \tau$. If, for every $W \in \tau$ and every $\beta > \alpha$, $E_\alpha \subseteq W$ implies $W \cap [\bigcup_{\beta \leqq \nu} E_\nu] \neq \varnothing$ then $V' \in \tau^*$ so $\{x\} = (U \cap E_\alpha) \cap V' = U \cap V' \in \tau \vee \tau^*$. Otherwise there is a $\beta_\alpha > \alpha$ and some $W \in \tau$ such that $E_\alpha \subseteq W$ and $W \cap [\bigcup_{\beta_\alpha \leqq \nu} E_\nu] = \varnothing$. Then $V'' = V' \cup \bigcup_{\beta_\alpha \leqq \nu} E_\nu \in \tau^*$ and $(U \cap W) \cap V'' = \{x\}$. Hence $\tau \vee \tau^* = 1$. Since $\tau^* \subseteq \tau'$, $\tau \wedge \tau' = 0$ implies $\tau \wedge \tau^* = 0$. Thus τ^* is a complement of τ.

Since τ_α is a T_1-topology, by Theorem 6.1, τ'_α is a principal topology for each α.

For $x \notin \bigcup_\alpha E_\alpha$, there is a minimal open set containing x in τ^*, namely $\{x\}$.

For $x \in E_\alpha$, there is a minimal open set $B_x \in \tau'_\alpha$ containing x. In case (ii) $B_x \in \tau^*$. Any set of form (ii) in τ^* containing x must be in τ'_α and hence contains B_x. Any set of form (iii) in τ^* containing x, contains E_α, hence B_x. So B_x is a minimal open set in τ^* containing x. In case (iii), $V_x = B_x \cup \bigcup_{\beta_\alpha \leqq \nu} E_\nu \in \tau^*$. Let $W \in \tau^*$ such that $x \in W$. W is the union of sets of the form (i), (ii) and (iii) but only the part of form (iii) can contain x. So we assume $W = U \cup \bigcup_{\beta_{\alpha_1} \leqq \gamma} E_\nu$ where $U \in \tau'_{\alpha_1}$.

If $\alpha_1 = \alpha$, then $B_x \subset U$ so $V_x \subset W$. If $\alpha_1 < \alpha$ and $\beta_{\alpha_1} \leqq \alpha$ then $W \cap V_x = V_x$ so $V_x \subset W$. If $\alpha_1 < \alpha < \beta_{\alpha_1}$ then $x \notin W$ and if $\alpha < \alpha_1$ then $x \notin W$. So V_x is a minimal open set in τ^* containing x. Thus τ^* has a base of open sets, minimal at each point and τ^* is a principal topology (Theorem 2.3).

THEOREM 6.4. *A T_1-topology on a countable set has a principal complement.*

Proof. Let τ be a T_1-topology on a countable set E. Let E be ordered as $\{x_i \mid i = 1,2,\cdots\}$ where $\{x_1\} \notin \tau$. If $\{x_i\} \in \tau$, $i = 1,2,\cdots, \tau = 1$ and has a principal complement, 0. Let G be a relation on E defined as $G = \{(x_i, x_j) \in E \times E \mid j < i\}$. Then the topology τ' defined by G is a principal topology and is a complement of τ.

For each $x_i \in E$, $i \neq 1$, the set $B_{x_i} = \{x_j \in E \mid j \leqq i\}$ is open in τ'. Since B_{x_i} is finite, $U = E - \{B_{x_i} - \{x_i\}\}$ is an open set in τ containing x_i. Thus $\{x_i\} = B_{x_i} \cap U \in \tau' \vee \tau$. And since $\{x_1\} \in \tau'$, $\tau \vee \tau' = 1$.

Let $U \in \tau \wedge \tau'$, $U \neq \varnothing$. If $x_i \in U$, then $B_{x_i} \subset U$ since $U \in \tau'$. Hence $x_1 \in U$ and since τ is a T_1-topology, U must be infinite. For every $x_k \in E$ there is a $j \geqq k$ such that $x_j \in U$. But $B_{x_j} \subset U$ implies $x_k \in U$. Thus $U = E$ and $\tau \wedge \tau' = 0$.

THEOREM 6.5. *Every topology on a countable set has a principal complement.*

7. **The general theorem.** In this section it will be shown that every topology has a principal lattice complement. Several preliminary theorems are proved first.

THEOREM 7.1. *Let τ be a topology on a set E, where $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \varnothing$ such that $\tau_1 = \tau \mid E_1$ and $\tau_2 = \tau \mid E_2$ have (principal) lattice complements τ_1 and τ_2' respectively. Then τ has a (principal) complement.*

Proof. If either E_1 or E_2 is empty, the theorem is trivial. So suppose both are nonempty.

Case (a). Suppose E_1 and E_2 are open sets in τ. Let $x_1 \in E_1$ and $x_2 \in E_2$. Let τ' be the topology generated by sets of the form:

(i) U, where $U \in \tau_1'$ and $x_1 \notin U$,

(ii) V, where $V \in \tau_2'$ and $x_2 \notin V$,

(iii) $U \cup V$, where $x_1 \in U \in \tau_1'$ and $x_2 \in V \in \tau_2'$.

(If τ_1' and τ_2' are principal topologies, then τ' is the principal topology defined by the relation $G = G_1 \cup G_2 \cup \{(x_1, x_2), (x_2, x_1)\}$ where G_1 defines τ_1' and G_2 defines τ_2'.)

It is not difficult to see that the restriction of an open set in τ' to E_1 (or E_2) is open in τ_1' (or τ_2').

For $x \in E$, suppose $x \in E_1$. Then there are open sets $U \in \tau_1$, and $U' \in \tau_1'$ such that $\{x\} = U \cap U'$. Since E_1 is open, $U \in \tau$, and either $U' \in \tau'$, if $x_1 \notin U'$, or $U' \cup V' \in \tau'$ if $x_1 \in U'$ where $x_2 \in V' \in \tau_2'$. In either case, $\{x\} \in \tau \vee \tau'$ for each $x \in E_1$. Similarly $\{x\} \in \tau \vee \tau'$ for each $x \in E_2$, so $\tau \vee \tau' = 1$.

Let $A \in \tau \wedge \tau'$, $A \neq \varnothing$. Thus $A \cap E_1 \in \tau_1 \wedge \tau_1'$ and $A \cap E_2 \in \tau_2 \wedge \tau_2'$. Some x is in A, and if $x \in E_1$, then $A \cap E_1 = E_1$. But since $x_1 \in A$ and $A \in \tau'$, $x_2 \in A$. Thus $A \cap E_2 = E_2$ and $A = E$. Hence $\tau \wedge \tau' = 0$.

Case (b). Suppose neither E_1 nor E_2 is open. Let τ' be the topology generated by sets of the form:

(i) U, where $U \in \tau_1'$,

(ii) V, where $V \in \tau_2'$.

(If τ_1' and τ_2' are principal topologies, then τ' is the principal topology defined by the relation $G = G_1 \cup G_2$ where G_1 defines τ_1' and G_2 defines τ_2'.)

For $x \in E$, if $x \in E_1$ then there is a $U \in \tau_1$ and $U' \in \tau_1'$ such that $\{x\} = U \cap U'$. But $U = V \cap E_1$ where $V \in \tau$. Hence $\{x\} = V \cap E_1 \cap U' = V \cap U' \in \tau \vee \tau'$ since $U' \in \tau'$. Similarly, if $x \in E_2$, $\{x\} \in \tau \vee \tau'$ so $\tau \vee \tau' = 1$.

Let $A \in \tau \wedge \tau'$, $A \neq \varnothing$. If $x \in E_1 \cap A$, $A \cap E_1 \in \tau_1 \wedge \tau_1'$ implies $A \cap E_1 = E_1$. But $E_1 \notin \tau$ so some $x \in E_2$ is in A. Thus $A \cap E_2 = E_2$ and $A = E$. Similarly if $A \cap E_2 \neq \varnothing$, then $A = E$. Therefore $\tau \wedge \tau' = 0$.

Case (c). Suppose E_1 is open but E_2 is not open. Let $x_1 \in E_1$ and $x_2 \in E_2$ and let τ' be the topology generated by sets of the form:

(i) U, where $U \in \tau_1'$ and $x_1 \notin U$,

(ii) V, where $V \in \tau_2'$,

(iii) $U \cup V$, where $x_1 \in U \in \tau_1'$ and $x_2 \in V \in \tau_2'$. (If τ_1' and τ_2' are principal topologies, then τ' is the principal topology defined by the relation $G = G_1 \cup G_2 \cup \{(x_1, x_2)\}$, where G_1 defines τ_1' and G_2 defines τ_2'.)

The restriction of an open set in τ' to E_1 (or E_2) is open in τ_1' (or τ_2').

For $x \in E_1$, $\{x\} = U \cap U'$ where $U \in \tau_1$ and $U' \in \tau_1'$. But $U \in \tau$ and if $x_1 \notin U'$ then $U' \in \tau'$ and if $x_1 \in U'$ then $U' \cup V' \in \tau'$ where $x_2 \in V' \in \tau_2'$. In either case, $\{x\} \in \tau \vee \tau'$. For $x \in E_2$, $\{x\} = U \cap U'$ where $U \in \tau_2$ and $U' \in \tau_2$. But $U' \in \tau'$ and $U = V \cap E_2$ where $V \in \tau$. Thus $\{x\} = V \cap U'$, hence $\{x\} \in \tau \vee \tau'$. Therefore $\tau \vee \tau' = 1$.

Let $A \in \tau \wedge \tau'$, $A \neq \varnothing$. If $A \cap E_1 \neq \varnothing$, then $A \cap E_1 = E_1$ since $A \cap E_1 \in \tau_1 \wedge \tau_1'$. But $x_1 \in E_1$ implies $x_2 \in A$ since $A \in \tau'$ and thus $A \cap E_2 = E_2$. If $A \cap E_2 \neq \varnothing$, then $A \cap E_2 = E_2$ since $A \cap E_2 \in \tau_2 \wedge \tau_2'$. But $E_2 \notin \tau$ so some $x \in E_1$ must be in A. Hence $A \cap E_1 = E_1$ and $A = E$. Therefore $\tau \wedge \tau' = 0$.

THEOREM 7.2. *If τ is a topology on E and $E = I \cup (E - I)$, where I is the set of isolated points of τ, and if $\tau \mid (E - I)$ has a (principal) lattice complement, then τ has a (principal) lattice complement.*

This is an immediate consequence of Theorem 7.1, since $\tau \mid I$ is the discrete topology and as such has a principal complement.

THEOREM 7.3. *If every topology (T_1-topology) with no isolated points has a principal complement, then every topology (T_1-topology) has a principal complement.*

Proof. Let τ be a topology (T_1-topology) on a set E and let $I = I_1$ be the set of isolated points of τ. It can be assumed that $I_1 \neq \emptyset$. Let I_2 be the set of isolated points of $\tau | (E - I_1)$.

Let λ be an ordinal number and suppose I_λ has been defined for all $\lambda < \nu$. Define I_ν as the set of isolated points of $\tau | (E - \bigcup_{\lambda<\nu} I_\lambda)$.

Thus a family of disjoint sets $\{I_\lambda | \lambda < \mu\}$ is inductively defined where $|E| < |\mu|$ so that, for each $\nu < \mu$, I_ν is the set of isolated points of $\tau | (E - \bigcup_{\lambda<\nu} I_\lambda)$.

There is some $\gamma < \mu$ such that $I_\gamma = \emptyset$. Let γ be the first ordinal such that $I_\gamma = \emptyset$ and let $E_1 = \bigcup_{\lambda<\gamma} I_\lambda$ and $E_2 = E - E_1$. Since $I_\gamma = \emptyset$, the topology $\tau | E_2$ has no isolated points. Let $\tau_1 = \tau | E_1$.

Select $x_\lambda \in I_\lambda$ for each $\lambda < \gamma$. Let

$$G = \bigcup_{\lambda<\gamma} (I_\lambda \times I_\lambda) \cup \{(x_\lambda, x_\nu) | \lambda < \nu < \gamma\}.$$

Then the principal topology τ' on E_1 defined by G is a complement of τ_1.

Let $V \in \tau_1 \wedge \tau'$, $V \neq \emptyset$. If $V \cap I_\lambda \neq \emptyset$, $\lambda < \nu$, then $I_\lambda \subset V$ since $V \in \tau'$. It follows that $x_\lambda \in V$, hence $x_\nu \in V$ and $I_\nu \subset V$, for $\lambda < \nu < \gamma$. Thus $V = \bigcup_{\delta \leq \lambda < \gamma} I_\lambda$ for the least ordinal δ such that $V \cap I_\delta \neq \emptyset$.

Since $V \in \tau_1$, $V = E_1 \cap V'$ for some $V' \in \tau$. As $x_\delta \in V$ is isolated in $\tau | (E - \bigcup_{\lambda<\delta} I_\lambda)$, $\{x_\delta\} = W \cap (E - \bigcup_{\lambda<\delta} I_\lambda)$ for some $W \in \tau$. But $V \subset V' \subset E - \bigcup_{\lambda<\delta} I_\lambda$, and hence $W \cap V' = \{x_\delta\}$ is isolated in τ. Thus $\delta = 1$ and $V = E_1$. Therefore $\tau_1 \vee \tau' = 0$.

For each $x \in E_1$, x is in some I_ν. Thus x is an isolated point in $\tau | (E - \bigcup_{\lambda<\nu} I_\lambda)$. There is then a $W \in \tau$ such that $\{x\} = W \cap (E - \bigcup_{\lambda<\nu} I_\lambda)$, but $\{E - \bigcup_{\lambda<\nu} I_\lambda\} \in \tau'$ and $W \cap E_1 \in \tau_1$. Therefore $\tau_1 \vee \tau' = 1$.

It has been shown that $\tau | E_1$ has a principal complement τ'. If $E_2 = \emptyset$ then τ' is a principal complement of τ. If $E_2 \neq \emptyset$, then $\tau | E_2$ has no isolated points, and hence by assumption a principal complement. (If τ is a T_1-topology, so is $\tau | E_2$.) But then by Theorem 7.1, τ has a principal complement.

The next theorem in this section is an extension of the following result of Berri [3], [4]: A topology on a set E has a complement if there is a decomposition of E into countable sets such that no union of any proper subcollection is open.

THEOREM 7.4. *Let τ be a topology on a set E such that*

(i) $E = \bigcup_{\alpha \in \theta} E_\alpha$, *where the E_α's are pairwise disjoint,*

(ii) $\tau_\alpha = \tau | E_\alpha$ *has a (principal) lattice complement τ'_α, and*

(iii) *for all $V \in \tau$, if $V \neq E, \emptyset$, then V is not the union of E_α's.*

Then τ has a (principal) complement τ'. If some τ'_α has an isolated point, so does τ'.

Proof. Let τ' consist of all sets of the form $\bigcup_{\alpha \in \theta} V_\alpha$ where $V_\alpha \in \tau'_\alpha$ for all $\alpha \in \theta$. (If each τ'_α is princinal, τ' is the principal topology defined by the union of the relations representing the τ'_α.)

Let $x \in E$. Then $x \in E_\alpha$ for some $\alpha \in \theta$. There is a $V \in \tau_\alpha$ and a $V' \in \tau'_\alpha$ such that $\{x\} = V \cap V'$. Since $V' \in \tau'$ and $V = U \cap E_\alpha$ for some $U \in \tau$, $\{x\} = (U \cap E_\alpha) \cap V' = U \cap V' \in \tau \vee \tau'$. Thus $\tau \vee \tau' = 1$.

If $V \in \tau \wedge \tau'$ then $V \cap E_\alpha \in \tau_\alpha \wedge \tau'_\alpha$, hence for all $\alpha \in \theta$, either $V \cap E_\alpha = E_\alpha$ or $V \cap E_\alpha = \varnothing$. If $V \cap E_\alpha = \varnothing$ for all $\alpha \in \theta$, then $V = \varnothing$. If $V \cap E_\alpha = E_\alpha$ for some α, then $V \cap E_\alpha = E_\alpha$ for all $\alpha \in \theta$ by hypothesis, and $V = E$. Therefore $\tau \wedge \tau' = 0$. If $\{x\} \in \tau'_\alpha$ for some α, then $\{x\} \in \tau'$.

Theorem 7.5. *Let τ be a T_1-topology on a set E containing a proper open set S with at least two points, such that $\tau_S = \tau | S$ has a (principal) complement τ'_S with an isolated point. Then τ has a (principal) complement with an isolated point.*

Proof. Let $x_0 \in S$ be an isolated point, i.e., $\{x_0\} \in \tau'_S$, and let $y_0 \in S$, $y_0 \neq x_0$. Define τ' as unions of sets of the form:

(i) V, where $V \in \tau'_S$ and $y_0 \notin V$,

(ii) $V \cup (E - S) \cup \{x_0\}$, where $y_0 \in V \in \tau'_S$,

(iii) $\{z\} \cup \{x_0\}$, for each $z \in E - S$.

The intersection of sets of form (i) with sets of the form (i), (ii) or (iii) is again a set of form (i). A set of form (ii) intersected with a set of form (ii) is a set of form (ii) and intersected with a set of form (iii) is of form (iii). The intersection of two distinct sets of form (iii) is a set of form (i). Thus these sets form a base for a topology.

(If τ'_S is a principal topology with defining relation G_S, then τ' is the principal topology defined by

$$G = G_S \cup \{y_0\} \times (E - S) \cup (E - S) \times \{x_0\}.$$

It must be shown that $\tau \vee \tau' = 1$ and $\tau \wedge \tau' = 0$. Let $x \in E$. If $x \in E - S$, then $\{x\} = \{x, x_0\} \cap (E - \{x_0\}) \in \tau' \vee \tau$. If $x \in S$ and $x \neq y_0$, then $\{x\} = U \cap V$ where $U \in \tau_S$ and $V \in \tau'_S$. If $y_0 \notin V$ then $V \in \tau'$ and since $U \in \tau$, $\{x\} \in \tau \vee \tau'$. If $y_0 \in V$ then $V^* = V \cup (E - S) \cup \{x_0\} \in \tau'$ and $U^* = U \cap (E - \{x_0\}) \in \tau$. Thus $\{x\} = U^* \cap V^* \in \tau \vee \tau'$. If $x = y_0$, then $\{y_0\} = V \cap U$ where $V \in \tau'_S$ and $U \in \tau_S$. Thus $\{y_0\} = (V \cup (E - S) \cup \{x_0\}) \cap (U \cap (E - \{x_0\})) \in \tau' \vee \tau$. Therefore $\tau \vee \tau' = 1$. Let $A \in \tau \wedge \tau'$. Since $A \cap S \in \tau_S \cap \tau'_S$, $A \cap S = S$ or $A \cap S = \varnothing$. If $A \cap S = S$, then since $y_0 \in S \in \tau'_S$, A must contain $E - S$ and hence $A = E$. If $A \cap S = \varnothing$, then $A = \varnothing$, since $z \in A \cap (E - S)$ would imply $x_0 \in A \cap S$. Hence $\tau \wedge \tau' = 0$.

Since $\{x_0\} \in \tau'_S$, $\{x_0\} \in \tau'$. Thus τ' has an isolated point.

Theorem 7.6. *If τ is a T_1-topology on a set E such that*

(i) *E is ordered as $\{x_\alpha\}_{\alpha < \mu}$ where μ is the smallest ordinal of cardinality $|E|$,*

(ii) *each nonempty open set in τ has the same cardinality as E,*

(iii) *for each $x \in E$ there is an open set containing x which does not contain any predecessors of x, then τ has a principal complement with a insolated point.*

Proof. Let τ' be the principal topology defined by the relation $G = \{(x,y) \mid y < x\}$. Let $U \in \tau \wedge \tau'$, $U \neq \varnothing$. For each $x \in E$ there is a $z \in E$ $z > x$, such that $z \in U$. Otherwise it follows from the well-order on E that $|U| < |E|$. Thus $x \in U$ since $U \in \tau'$ and $(z,x) \in G$, and $U = E$. Hence $\tau \wedge \tau' = 0$.

For each $x \in E$, the set $V = \{y \mid y \leqq x\} \in \tau'$ and there is a set $U \in \tau$ which contains x but contains no predecessors of x. Thus $\{x\} = U \cap V \in \tau \vee \tau'$ and $\tau \vee \tau' = 1$.

For the first element x_1 in the well-order, $\{x_1\} \in \tau'$.

THEOREM 7.7. *A T_1-topology with no isolated points has a principal complement with an isolated point.*

Proof. If there is some T_1-topology with no isolated points which does not have a principal complement with an isolated point, then there is a first cardinal $\aleph$ such that some T_1-topology τ on E, where τ has no isolated points; and $|E| = \aleph$, fails to have a principal complement with an isolated point.

No set in τ has isolated points in its relative topology, hence by Theorem 7.5 every nonempty set in τ must have cardinality $\aleph$.

Let E be well ordered as $\{x_\alpha\}_{\alpha < \mu}$ where μ is the smallest ordinal of cardinality $\aleph$.

(a) If for each $x \in E$ there is an open set containing x which contains no predecessors of x, then τ has a principal complement with an isolated point by Theorem 7.6.

Hence, the set $S = \{x \in E \mid$ every open set containing x contains predecessors of $x\}$ is not empty. No nonempty subset of S can be in τ since it would be an open set not containing the predecessors of its first element. Since $S \notin \tau$, $E - S \neq \varnothing$.

(b) Suppose there is a nonempty set $U \in \tau$ such that $U \cap S = \varnothing$. Then $\tau_U = \tau \mid U$ satisfies the conditions of Theorem 7.6, and by Theorem 7.6 and Theorem 7.5, τ has a principal complement with an isolated point.

(c) Thus S must be dense in τ. Also, since no nonempty subset of S is open, $E - S$ is also dense in τ.

Let us say that a subset $M \subset E$ with the induced well-order has *property P* if for every $x \in M$ there is an open set containing x which does not contain any predecessors of x in M.

Consider sets of disjoint, dense subsets of E having property P and let Γ be the collection of all such sets. Since $E - S$ is dense and has property P, Γ is not empty. The union of a chain of sets of disjoint dense subsets of E having property P is again such a set. Thus, by Zorn's Lemma, Γ has a maximal element $\mathfrak{M} = \{W_i \mid i \in \theta\}$.

Denote by F the set $E - \bigcup_{i \in \theta} W_i$, $W_i \in \mathfrak{M}$, and let $\tau_F = \tau | F$. The well-order on E induces a well-order on F. Let $T = \{x \in F | \text{every set in } \tau_F \text{ containing } x$ contains predecessors of x in $F\}$. Then $F = T \cup (F - T)$ where $F - T$ has property P with respect to τ_F. No subset of T is in τ_F so $F - T$ is dense in τ_F. If F is dense in τ, since $F - T$ is dense in τ_F, $F - T$ is dense in τ, which contradicts the maximality of $\mathfrak{M}$.

(d) Thus F is not dense in τ and there is a set $U \in \tau$ such that $U \cap F = \emptyset$. But now $U = \bigcup_{i \in \theta} \{U \cap W_i\}$ where $U \cap W_i \neq \emptyset$ for each $i \in \theta$ and the sets $U \cap W_i$ are disjoint. Let $\tau_U = \tau | U$ and let $\tau_i = \tau_U | (U \cap W_i) = \tau | (U \cap W_i)$.

If it can be shown that each τ_i has a principal complement with an isolated point, then by Theorem 7.4, τ_U has a principal complement with an isolated point.

For $i \in \theta$, τ_i has no isolated points, since if $\{x\} \in \tau_i$, then $\{x\} = (U \cap W_i) \cap V$ where $V \in \tau$. But $U \cap V \cap (E - \{x\}) \in \tau$ and $(U \cap V \cap (E - \{x\})) \cap W_i = \emptyset$ which contradicts the fact that W_i is dense in τ.

If for some $V_i \in \tau_i$, $0 < |V_i| < \aleph$, since $\tau_i | V_i$ has no isolated points ($V_i \in \tau_i$ and τ_i has no isolated points), $\tau_i | V_i$ has a principal complement with an isolated point and hence by Theorem 7.5, so does τ_i.

If for all $V_i \in \tau_i$, $V_i \neq \emptyset$ implies $|V_i| = \aleph$, then $|U \cap W_i| = \aleph$. But $U \cap W_i$ is well-ordered (order induced by E) as $\{x_\beta\}_{\beta < \mu}$ where μ is the smallest ordinal of cardinality $\aleph$, and since W_i has property P with respect to τ, $U \cap W_i$ has property P with respect to τ and hence with respect to τ_i. Therefore by Theorem 7.6, τ_i has a principal complement with an isolated point.

Thus τ_i has a principal complement with an isolated point for each $i \in \theta$, and by Theorem 7.4, τ_U has a principal complement with an isolated point. Now by Theorem 7.5, τ has a principal complement with an isolated point.

THEOREM 7.8. *The lattice of topologies on any set is complemented. Moreover, each topology has a principal complement.*

Proof. Every T_1-topology with no isolated points has a principal complement by Theorem 7.7, hence by Theorem 7.3, every T_1-topology has principal complement. Thus every topology has a principal complement by Theorem 6.3.

REFERENCES

1. R. W. Bagley, *On the characterization of the lattice of topologies*, J. London Math. Soc. **30** (1955), 247–249.

2. B. Banaschewski, *Über den Ultrafilterraum*, Math. Nachr. **13** (1955), 273–281.

3. M. P. Berri, *Lattice complements of certain topologies*, Abstract 65T-94 Notices Amer. Math. Soc. **12** (1965), 227.

4. ———, *The complement of a topology for some topological groups*, Fund. Math. (to appear).

5. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ. Vol. 25, Amer. Math. Soc., Providence, R. I., 1948.

6. H. Cartan, *Théorie des filtres*, C. R. Acad. Sci. Paris **205** (1937), 595–598.

7. O. Fröhlich, *Das Halbordnungssystem der topologischen Räume auf einer Menge*, Math. Ann. **156** (1964), 79–95.

8. H. Gaifman, *The lattice of all topologies on a denumerable set*, Abstract 61T-161, Notices Amer. Math. Soc. **8** (1961), 356.

9. ———, *Remarks on complementation in the lattice of all topologies*, (unpublished paper).

10. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, New York, 1960.

11. J. Hartmanis, *On the lattice of topologies*, Canad. J. Math. **10** (1958), 547–553.

12. J. L. Kelley, *General topology*, Van Nostrand, New York, 1955.

13. Teng-Sun Liu, *A note on maximal T_1-topologies*, Portugal. Math. **18** (1959), 235–236.

14. P. Samuel, *Ultrafilters and compactifications of uniform spaces*, Trans. Amer. Math. Soc. **64** (1948), 100–132.

15. J. Schmidt, *Beiträge zür Filtertheorie*. I, Math. Nachr. **7** (1952), 359–378.

16. G. Szász, *Introduction to lattice theory*, Academic Press, New York, 1963.

17. R. Vaidyanathaswamy, *Set topology*, Chelsea, New York, 1960.

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