A DUALITY BETWEEN CERTAIN SPHERES AND ARCS IN S^3

BY

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1. **Introduction.** Harrold and Moise [18] have shown that if a 2-sphere in the standard 3-sphere S^3 is locally polyhedral except at one point, then one of its closed complementary domains is a closed 3-cell. Cantrell [6] has shown that the other open complementary domain is an open 3-cell and [8], [9] that if an $(n-1)$-sphere Σ in the standard n-sphere S^n is locally flat (see [4] and [5, p. 49] for definitions) except at one point for $n > 3$, then Σ is flat in S^n. Fox and Artin [13] have given the first examples of arcs which are locally flat except at one point.

The main result of this paper is a duality between 2-spheres which are locally flat in S^3 except at one point and arcs which are locally flat in S^3 except at one endpoint. Roughly, if Σ is a 2-sphere which is locally flat in S^3 except possibly at one point p, then we associate with Σ any arc in Σ which has p for an endpoint. Conversely, if α is an arc which is locally flat in S^3 except possibly at one endpoint p, then we "blow up" α into a little 2-sphere which tapers down to p just as α does and we associate this sphere with α. We make this precise in §3.

We extend this result to a duality theorem concerning nearly flat 2-manifolds in a 3-manifold. As an application of this duality theorem, in §4 we prove a uniqueness theorem in a class of decomposition spaces. In §5 we extend a result of Lininger [22] by characterizing a class of crumpled cubes.

In §6 pseudo-half spaces are characterized. An *n-pseudo-half space* M^n is an n-manifold with boundary such that the interior of M^n is homeomorphic to R^n and the boundary of M^n is homeomorphic to R^{n-1}. Cantrell [7] and Doyle [11] have shown that, for $n \neq 3$, every n-pseudo-half space is homeomorphic to the closed half-space R^n_+. Kwun and Raymond [21] give an example of a 3-pseudo-half space which is not homeomorphic to the closed half-space R^3_+. It follows from [1], [14] that uncountably many topologically different 3-pseudo-half spaces exist. In Theorem 7 we prove the following:

M^n is an n-pseudo-half space if and only if M^n is homeomorphic to $B^n - \alpha$ where α is arc in the standard closed n-ball B^n such that α intersects its boundary S^{n-1} at one endpoint and $S^{n-1} \cup \alpha$ is locally flat except possibly at the other endpoint.

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Harrold [17] has given a sufficient condition for an arc in S^3 to be cellular (see [3] for definition) and Doyle [12] has given a sufficient condition for an arc in S^n to be cellular. McMillan [23] has shown that, for $n \neq 4$, a subarc of a cellular arc is cellular. Stewart [26] has given an example of a cellular arc in S^3 which is wild at every point. In §7 we prove the following:

If α is an arc in S^3 such that α contains a subarc β both of whose endpoints are isolated wild points of β, then α is not cellular.

2. **Preliminary results.** If X is locally flat at point x in a triangulated n-manifold N^n, then X is locally tame at x. Thus it follows from Bing's Approximation Theorem [2] that if X, a closed subset of a triangulated 3-manifold N^3, is locally flat except on a set Y, then X is equivalent to a set K which contains Y such that $K - Y$ is locally polyhedral. Hence if a 2-sphere Σ in S^3 is locally flat except at one point, then by [18] one of the closed complementary domains of Σ is a closed 3-cell. Moreover, if X is a 2-sphere or an arc in a 3-manifold N^3, then the following statements are equivalent:

(1) X is locally flat at x,

(2) X is locally tame at x.

Also we will use the facts established in [3], [4] that if Σ is an $(n-1)$-sphere in S^n, then the following statements are equivalent:

(1) Σ is locally flat at every point of Σ,

(2) Σ is flat,

(3) Σ is bi-collared.

The two theorems in this section seem to be folk theorems in this subject. The proof of Theorem 1 is standard but the proof of Theorem 2 is often incomplete so that we will include it here.

THEOREM 1. *Let α and β be arcs in an n-manifold M which are locally flat except at the common endpoint p such that α is a proper subarc of β and let U be a neighborhood of $\beta - p$. Then there is a pseudo-isotopy $\phi_t (t \in I)$ of M onto itself such that:*

(1) $\phi_0 = 1$,

(2) $\phi_t | (M - U) = 1$,

(3) $\phi_1(\alpha) = p$,

(4) $\phi_1(\beta) = \alpha$,

(5) $\phi_1 | (M - \alpha)$ *is a homeomorphism onto* $M - p$.

THEOREM 2. *Let M be a manifold with boundary F. Add a closed collar $F \times [-1,0]$ to M by identifying $(x,0)$ with x for $x \in F$. Then $M \cup (F \times [-1,0]) \approx M$ and $M \cup (F \times (-1,0]) \approx \operatorname{Int} M$.*

Proof. By Theorem 2 of [4] F is collared in M, so that there is a homeomorphism $H: F \times [-1,1] \to M \cup (F \times [-1,0])$ such that:

$$H(x,t) = (x,t), \quad x \in F, t \in [-1,0],$$
$$H(x,t) \in M, \quad x \in F, t \in [0,1].$$

Let $Y = H(F \times 1)$. If F is compact, then Y is closed in M and

$$H(F \times [-1,1]) \cap (M - H(F \times I')) = Y,$$

where $I' = [0,1)$. Thus there is a canonical map which pushes F out to $F \times \{-1\}$ and which is the identity on $M - H(F \times I')$. However, if F is not compact, Y may not be closed in M nor separate M.

For each $x \in F$, let $\delta_x = \frac{1}{2}\rho(x, M - H(F \times I'))$, where ρ is a metric in M. Then $U = \bigcup_{x \in F} V_{\delta_x}(x)$ is a neighborhood of F in M and the triangle inequality insures that $\mathrm{Cl}\, U \subset H(F \times I')$.

Given a map $\lambda: F \to (0,1]$, we define the spindle neighborhood $S(F,\lambda)$ by:

$$S(F,\lambda) = \{(x,t) \in F \times I' \mid x \in F \text{ and } t < \lambda(x)\}.$$

By [4] the spindle neighborhoods form a neighborhood basis for $F \times 0$ in $F \times I'$. So let $S(F,\lambda)$ be a spindle neighborhood of $F \times 0$ such that $S(F,\lambda) \subset H^{-1}(U)$. Define $G: F \times [-1,1] \to M \cup (F \times [-1,0])$ by $G(x,t) = H(x, t\lambda(x))$ and let $X = G(F \times 1)$. Then X is closed in M and

$$G(F \times [-1,1]) \cap (M - G(F \times I')) = X.$$

Thus there is a canonical map which pushes F out to $F \times \{-1\}$ and which is the identity on $M - G(F \times I')$. Moreover, $\mathrm{Int}\, M$ is mapped onto $M \cup (F \times (-1,0])$.

We conclude this section with several lemmas.

LEMMA 1. *Let K be a disk in R^3 that is locally polyhedral except at an interior point p. Then there is a polyhedral disk D with boundary F such that $D \cap K = F$ and F separates p from* Bd *K in K.*

Proof. This is a generalization of Lemma 1 of [18] and the proof is essentially the same.

LEMMA 2. *Let K be a disk in R^3 that is locally flat except at an interior point p. Then there is a homeomorphism $g: B^3 \to R^3$ such that:*

(1) $g(S^2_+) \subset K$,

(2) $g(0,0,1) = p$,

(3) $g(B^3 - S^2_+) \subset R^3 - K$,

(4) $g(S^2)$ *is locally flat except at* p,

(5) $(K - g(S^2_+)) \cup g(S^2_-)$ *is locally flat.*

Proof. It follows from a remark above that we can assume with no loss of generality that K is locally polyhedral except at p. By Lemma 1 there is a polyhedral disk D with boundary F such that $D \cap K = F$ and F separates p from BdK in K. Let P be the closed complementary domain of F in K such that P contains p

Since the 2-sphere $\Sigma = P \cup D$ is locally polyhedral except at p, it follows from Theorem 1 of [18] that Σ is collared in one of the closed complementary domains of Σ in R^3. This establishes the existence of g with the required properties.

Let B^n be the closed unit n-ball centered at the origin in R^n and let $B_r(x) = \mathrm{Cl}\, V_r(x)$ be the closed n-ball of radius r centered at x. For the rest of this section and in Theorem 4, we will use the following definitions:

$$\begin{aligned}
a &= (0,0,1),\\
b &= (0,0,-1),\\
J &= [(0,0,1/2),(0,0,1)],\\
D_0 &= \{(x,y,z) \in B^3 \mid z = 1/2\},\\
G_0 &= \{(x,y,z) \in B^3 \mid z \geqq 1/2\},\\
P_0 &= \{(x,y,z) \in S^2 \mid z \geqq 1/2\}.
\end{aligned}$$

LEMMA 3. *Let* $\varepsilon > 0$, *let*

$$A = (\mathrm{Bd}\, B_2(b)) \cup \{(x,y,z) \in B_2(b) \mid z < -\varepsilon\},$$

and let $d \in \mathrm{Int}\, D_0$. *Then there is a homeomorphism h of $B_2(b)$ onto itself such that*:

(1) $h \mid A = 1$,

(2) $h(S^2) = S^2$,

(3) $h(D_0) = B^2$,

(4) $h(d) = 0$.

LEMMA 4. *Let*

$$A = (\mathrm{Bd}\, B_2(b)) \cup (R^3_- \cap (B_2(b) - V_1(0))).$$

Then there is a map ϕ of $B_2(b)$ onto itself such that:

(1) $\phi \mid A = 1$,

(2) $\phi(0 \times B^1_+) = a$,

(3) $\phi(B^2) = S^2_+$,

(4) $\phi \mid (B_2(b) - (0 \times B^1_+))$ *is a homeomorphism onto* $B_2(b) - a$.

LEMMA 5. *Let G be a closed 3-cell and let $f_1, f_2 \colon B \twoheadrightarrow G$ ("$\twoheadrightarrow$" means "onto") be homeomorphisms such that $f_1(a) = f_2(a)$ and $f_1(b) = f_2(b)$. Then there is a homeomorphism $h \colon G \twoheadrightarrow G$ such that*:

(1) $h \mid \mathrm{Bd}\, G = 1$,

(2) $hf_1 \mid (0 \times B^1) = f_2 \mid (0 \times B^1)$.

Proof. Define $g \colon S^2 \twoheadrightarrow S^2$ by $g = f_2^{-1} f_1 \mid S^2$. Extend it to a homeomorphism $g \colon B^3 \twoheadrightarrow B^3$ by radial extension. Then $g \mid (0 \times B^1) = 1$. Define $h \colon G \twoheadrightarrow G$ by $h = f_2 g f_1^{-1}$. Then

$$\begin{aligned}
h \mid \mathrm{Bd}\, G = f_2 g f_1^{-1} \mid \mathrm{Bd}\, G &= f_2(f_2^{-1} f_1) f_1^{-1} \mid \mathrm{Bd}\, G = 1,\\
hf_1 \mid (0 \times B^1) &= (f_2 g f_1^{-1}) f_1 \mid (0 \times B^1) = f_2 g \mid (0 \times B^1)\\
&= f_2 \mid (0 \times B^1).
\end{aligned}$$

Lemma 6. *If Σ_1 and Σ_2 are disjoint 2-spheres in S^3, $A = [\Sigma_1, \Sigma_2]$ and α is an arc in S^3 such that $\Sigma_i \cap \alpha = p_i$, a point, and $\Sigma_i \cup \alpha$ is locally flat, $i = 1,2$, then $(A, A \cap \alpha) \approx (S^2 \times I, 1 \times I)$.*

Proof. We identify S^3 with the one-point compactification of R^3. Let

$$C = B^2 \times [0,1] \cup (0 \times [1,2]) \cup B^2 \times [2,3].$$

Let G_i be the closed complementary domain of Σ_i which does not contain $A, i = 1,2$. Let $f: C \twoheadrightarrow G_1 \cup \alpha \cup G_2$ be a homeomorphism such that $f(B^2 \times [0,1]) = G_1$, $f(0 \times [1,2]) = \alpha \cap A, f(B^2 \times [2,3]) = G_2$ and so that $f|(B^2 \times [0,1])$ and $f|(B^2 \times [2,3])$ induce the same orientation on S^3. Evidently f is a locally flat embedding of C into S^3.

It follows from the Annulus Theorem in S^3 (see for example [24], [15]) that there is a stable homeomorphism $h: S^3 \twoheadrightarrow S^3$ such that $hf(x,t) = f(x, t+2)$ for all $x \in B^2$, $t \in I$. It follows from Lemma 7.1 of [5] that there is a homeomorphism $g: S^3 \twoheadrightarrow S^3$ such that gf is the inclusion $C \subset R^3 \subset S^3$. Let g_1 be a homeomorphism of $\mathrm{Cl}(S^3 - C)$ onto $S^2 \times I$ such that $g_1(0 \times [1,2]) = 1 \times I$. Then $g_1(g|A)$ is a homeomorphism of $(A, A \cap \alpha)$ onto $(S^2 \times I, 1 \times I)$.

Lemma 7. *If Σ_1 and Σ_2 are flat 2-spheres in S^3 and α is an arc in S^3 such that $\Sigma_i \cap \alpha = q$, an endpoint of α, and $\Sigma_i \cup \alpha$ is locally flat at $q, i = 1,2$, then there is a homeomorphism $h: S^3 \twoheadrightarrow S^3$ such that $h(\Sigma_1) = \Sigma_2$ and $h(\alpha) = \alpha$.*

Proof. Since Σ_1 and Σ_2 are flat and α is locally flat at q, there is a flat 2-sphere Σ_3 in S^3 such that $\Sigma_1 \cup \Sigma_2 \subset E$, an open complementary domain of Σ_3, $\Sigma_3 \cap \alpha = p$, an interior point of α, $\Sigma_3 \cup \alpha$ is locally flat at p and $\beta = \alpha \cap [\Sigma_3, \Sigma_1]$ is locally flat. By Lemma 6 there is a homeomorphism $h: ([\Sigma_3, \Sigma_1], \beta) \twoheadrightarrow ([\Sigma_3, \Sigma_2], \beta)$. Without loss of generality $h|\Sigma_3 = 1$, so we can extend h to $S^3 - E$ by the identity. Since Σ_1 and Σ_2 are flat, their closed complementary domains are closed 3-cells and we can extend h to a homeomorphism of S^3 onto itself with the desired properties.

3. **Duality theorems.** Let $\mathscr{I}$ be the set of pairs (α, p) where α is an arc in S^3 and p is an endpoint of α such that α is locally flat except possibly at p and let $\mathscr{S}$ be the set of pairs (Σ, p) where Σ is a 2-sphere in S^3 and p is a point of Σ such that Σ is locally flat except possibly at p. Two sets or pairs of sets embedded in a manifold are *equivalent* (denoted by $\Leftrightarrow$) if there is a global homeomorphism carrying one set or pair of sets onto the other. Let $\mathscr{I}_*$ and $\mathscr{S}_*$ be the sets of equivalence classes of $\mathscr{I}$ and $\mathscr{S}$ in S^3, respectively.

Let $(\alpha, p) \in \mathscr{I}$. Let Σ be any 2-sphere in S^3 such that Σ intersects α only at the endpoint which is not p and such that $\Sigma \cup \alpha$ is locally flat at every point of Σ. Let ϕ be a map of S^3 onto itself such that $\phi(\alpha) = p$ and $\phi|(S^3 - \alpha)$ is a homeomorphism onto $S^3 - p$. Such a map exists by Theorem 1. Define $\Psi: \mathscr{I} \to \mathscr{S}$ by $\Psi(\alpha, p) = (\phi(\Sigma), p)$.

Let $(\Sigma, p) \in \mathscr{S}$. We noticed in §2 that one of the closed complementary domains G of Σ is a 3-cell. Let g be a homeomorphism of B^3 onto G such that $g(1) = p$. Define $\Gamma: \mathscr{S} \to \mathscr{I}$ by $\Gamma(\Sigma, p) = (g(I), p)$.

THEOREM 3. *Ψ and Γ are well defined up to equivalence class and Ψ induces a one-to-one correspondence $\Psi_*: \mathscr{I}_* \twoheadrightarrow \mathscr{S}_*$ such that its inverse is Γ^*, the function induced by Γ.*

We will generalize this result to 2-manifolds in a 3-manifold in Theorem 4. The proof of Theorem 3 will then follow from Theorem 4.

Now let N be some fixed 3-manifold. Let $\mathscr{A}$ be the collection of sets in N each of which is the union of a locally flat 2-manifold K and a set of disjoint arcs α_i, $i = 1, \cdots, m$, such that α_i intersects K at one endpoint and $K \cup \alpha_i$ is locally flat except at the other endpoint for each i. Let $\mathscr{M}$ be the collection of nearly flat 2-manifolds M in N (i.e., M is wild at a finite number of points). Let $\mathscr{A}_*$ and $\mathscr{M}_*$ be the sets of equivalence classes of $\mathscr{A}$ and $\mathscr{M}$ in N, respectively.

Let $K \cup (\bigcup_1^m \alpha_i) \in \mathscr{A}$ and let p_i be the wild endpoint of $\alpha_i, i = 1, \cdots, m$. Let ϕ be a map of N onto itself such that $\phi(\alpha_i) = p_i, i = 1, \cdots, m$, and $\phi \mid (N - \bigcup_1^m \alpha_i)$ is a homeomorphism onto $N - \bigcup_1^m p_i$. The existence of such a map follows from Theorem 1. Define $\Psi: \mathscr{A} \to \mathscr{M}$ by $\Psi(K \cup (\bigcup_1^m \alpha_i)) = \phi(K)$.

Let $M \in \mathscr{M}$ and let p_i be the wild points of M, $i = 1, \cdots, m$. Let $g_i: B^3 \to N$ be homeomorphisms with disjoint images such that:

(1) $g_i(S^2_+) \subset M$,

(2) $g_i(a) = p_i$,

(3) $g_i(B^3 - S^2_+) \subset N - M$,

(4) $g_i(S^2)$ is locally flat except at p,

(5) $K = (M - \bigcup_1^m g_i(P_0)) \cup (\bigcup_1^m g_i(D_0))$ is locally flat.

Let $\alpha_i = g_i(J)$, $i = 1, \cdots, m$. Define $\Gamma: \mathscr{M} \to \mathscr{A}$ by $\Gamma(M) = K \cup (\bigcup_1^m \alpha_i)$.

THEOREM 4. *Ψ and Γ are well defined up to equivalence class and Ψ induces a one-to-one correspondence $\Psi_*: \mathscr{A}_* \twoheadrightarrow \mathscr{M}_*$ such that its inverse is Γ_*, the function induced by Γ.*

Proof. (i) Ψ is well defined up to equivalence class and induces a function $\Psi_*: \mathscr{A}_* \to \mathscr{M}_*$. Indeed, given the diagram with the solid arrows:

$$\begin{tikzcd}
K_1 \cup (\bigcup_1^m \alpha_i^1) \arrow[r, Leftrightarrow] \arrow[d, "\Psi"'] & K_2 \cup (\bigcup_1^m \alpha_i^2) \arrow[d, "\Psi"] \\
M_1 \arrow[r, Leftrightarrow, dashed] & M_2
\end{tikzcd}$$

where $K_1 \cup (\bigcup_1^m \alpha_i^1)$, $K_2 \cup (\bigcup_1^m \alpha_i^2) \in \mathscr{A}$ and $M_1, M_2 \in \mathscr{M}$, we will show that we can fill in the dotted arrow.

Let p_i^j be the wild endpoint of α_i^1, $i=1,\cdots,m$, $j=1,2$. There is a homeomorphism

$$f:(N,K_1\cup(\bigcup_1^m\alpha_i^1),\ \bigcup_1^m p_i^1)\twoheadrightarrow(N,K_2\cup(\bigcup_1^m\alpha_i^2),\ \bigcup_1^m p_i^2).$$

Without loss of generality $f(\alpha_i^1)=\alpha_i$, $i=1,\cdots,m$. By the definition of Ψ, for each $j=1,2$, there is a map ϕ_j of N onto itself such that $\phi_j(\alpha_i^j)=p_i^j$, $i=1,\cdots,m$, $\phi_j|(N-\bigcup_1^m\alpha_i^j)$is a homeomorphism onto $N-\bigcup_1^m p_i^j$ and $\phi_j(K_j)=M_j$.

Define $g:N\twoheadrightarrow N$ by

$$g(x)=\begin{cases}\phi_2 f\phi_1^{-1}(x), & x\in N-\bigcup_1^m p_i^1,\\ p_i^2, & x=p_i^1,\ i=1,\cdots,m.\end{cases}$$

Evidently g is a homeomorphism and

$$\begin{aligned}g(M_1-\bigcup_1^m p_i^1) &= \phi_2 f\phi_1^{-1}(M_1-\bigcup_1^m p_i^1)\\ &= \phi_2 f(K_1-\bigcup_1^m\alpha_i^1)\\ &= \phi_2(K_2-\bigcup_1^m\alpha_i^2)\\ &= M_2-\bigcup_1^m p_i^2.\end{aligned}$$

Thus $g(M_1)=M_2$ and so M_1 is equivalent to M_2.

Now define $\Psi_*:\mathscr{A}_*\to\mathscr{M}_*$ by

$$\Psi_*[K\cup(\bigcup_1^m\alpha_i)]=[\Psi(K\cup(\bigcup_1^m\alpha_i))].$$

(ii) Γ is well defined up to equivalence class and induces a function

$$\Gamma_*:\mathscr{M}_*\to\mathscr{A}_*.$$

Indeed, given the diagram with the solid arrows:

$$\begin{tikzcd} M_1 \arrow[r, Leftrightarrow, dashed] \arrow[d, "\Gamma"'] & M_2 \arrow[d, "\Gamma"] \\ K_1\cup(\bigcup_1^m\alpha_i^1) \arrow[r, Leftrightarrow, dashed] & K_2\cup(\bigcup_1^m\alpha_i^2) \end{tikzcd}$$

where $M_1,M_2\in\mathscr{M}$ and $K_1\cup(\bigcup_1^m\alpha_i^1)$, $K_2\cup(\bigcup_1^m\alpha_i^2)\in\mathscr{A}$, we will show that we can fill in the dotted arrow.

Let p_i^j, $i=1,\cdots,m$, be the wild points of M_j, $j=1,2$. There is a homeomorphism

$$h:(N,M_1,\bigcup_1^m p_i^1)\twoheadrightarrow(N,M_2,\bigcup_1^m p_i^2).$$

Without loss of generality $h(p_i^1)=p_i^2$, $i=1,\cdots,m$. By the definition of Γ, for each $j=1,2$, there are homeomorphisms $g_i^j:B^3\to N$ with disjoint images such that

(1) $g^j(S_+^2)\subset M_j$,
(2) $g_i^j(a)=p_i^j$,
(3) $g_i(B^3-S_+^2)\subset N-M_j$,

(4) $g_i^j(S^2)$ is locally flat except at p_i^j,

(5) $K_j = (M_j - \bigcup_1^m g_i^j(P_0)) \cup (\bigcup_1^m g_i^j(D_0))$ is locally flat,

(6) $\alpha_i^j = g_i^j(J)$.

Let $q_i^j = g_i^j(0,0,\frac{1}{2})$, $i = 1,\cdots,m$, $j = 1,2$.

Without loss of generality $h(g_i^1(D_0)) \subset g_i^2(G_0 - D_0)$. Let $f_i \colon B^3 \twoheadrightarrow g_i^2(B^3)$ be a homeomorphism such that:

(7) $f_i(a) = p_i^2$,

(8) $f_i(D_0) = h(g_i^1(D_0))$,

(9) $f_i(B^2) = g_i^2(D_0)$,

(10) $f_i(0) = q_i^2$.

Since $g_i^2(S^2)$ is locally flat except at p_i^2, we can extend f_i to a homeomorphism $f_i \colon B_2(b) \to N$ such that the images are disjoint and

$$f_i^{-1}(\mathrm{Cl}(M_2 - g_i^2(S_+^2))) \subset ((\mathrm{Int}\, R_-^3) \cap B_2(b)).$$

Choose $\varepsilon > 0$ such that

$$f_i^{-1}(\mathrm{Cl}(M_2 - g_i^2(S_+^2))) \subset \{(x,y,z) \in B_2(b) \mid z < -\varepsilon\}.$$

It follows from Lemma 3 that there is a homeomorphism r_i of N onto itself such that:

(11) $r_i \mid (N - f_i(B_2(b))) = 1$,

(12) $r_i(M_2) = M_2$,

(13) $r_i(h(K_1) \cap f_i(B_2(b)) = K_2 \cap f_i(B_2(b))$,

(14) $r_i(h(q_i^1)) = q_i^2$.

Now $r_i(h(\alpha_i^1))$ may not be equal to α_i^2. However, $r_i h g_i^1$ and g_i^2 are homeomorphisms of G_0 onto $g_i^2(G_0)$ such that $r_i h g_i^1(a) = g_i^2(a) = p_i^2$ and $r_i h g_i^1(0,0,\frac{1}{2}) = g_i^2(0,0,\frac{1}{2}) = q_i^2$. Thus it follows from Lemma 5 that there is a homeomorphism s_i of N onto itself such that:

(15) $s_i \mid (N - g_i^2(G_0)) = 1$,

(16) $s_i r_i h g_i^1 \mid J = g_i^2 \mid J$.

Define $h_1 \colon N \twoheadrightarrow N$ by $h_1 = s_m r_m \cdots s_1 r_1 h$. Then $h_1(K_1 \cup (\bigcup_1^m \alpha_i^1)) = K_2 \cup (\bigcup_1^m \alpha_i^2)$ so that $K_1 \cup (\bigcup_1^m \alpha_i^1)$ is equivalent to $K_2 \cup (\bigcup_1^m \alpha_i^2)$.

Now define $\Gamma_* \colon \mathscr{M}_* \to \mathscr{A}_*$ by $\Gamma_*[M] = [\Gamma(M)]$.

(iii) $\Gamma_* \Psi_* = 1$. Indeed, given the diagram with the solid arrows:

$$\begin{tikzcd} K_1 \cup (\bigcup_1^m \alpha_i^1) \arrow[r, "\Psi"] \arrow[dr, dashed, leftrightarrow] & M \arrow[d, "\Gamma"] \\ & K_2 \cup (\bigcup_1^m \alpha_i^2 \end{tikzcd}$$

where $K_1 \cup (\bigcup_1^m \alpha_i^1)$, $K_2 \cup (\bigcup_1^m \alpha_i^2) \in \mathscr{A}$ and $M \in \mathscr{M}$, we will show that we can fill in the dotted arrow.

Let p_i be the wild endpoint of α_i^1 and let q_i be the other endpoint, $i = 1, \cdots, m$. Since $K_1 \cup \alpha_i^1$ is locally flat at q_i, there are disjoint neighborhoods W_i of the q_i in N and homeomorphisms

$$h_i: (R^3, (R^2 \times 0) \cup (0 \times R_+^1)) \twoheadrightarrow (W_i, W_i \cap (K_1 \cup \alpha_i^1)),$$

$i = 1, \cdots, m$. Let $\beta_i = h_i(0 \times B_-^1)$. There is a homeomorphism f_i of N onto itself such that:

(1) $f_i(\alpha_i^1 \cup \beta_i) = \alpha_i^1$,
(2) $f_i h_i(S_-^2) = h_i(B^2)$,
(3) $f_i h_i(B^2) = h_i(S_+^2)$,
(4) $f_i \,|\, ((N - W_i) \cup (K_1 - h_i(B^2))) = 1$.

Choose $\varepsilon > 0$ sufficiently small that ε-neighborhoods of the $f_i(\alpha_i^1)$'s are disjoint and do not intersect K_1. By Theorem 1 there is a map ϕ_i of N onto itself such that:

(5) $\phi_i f_i(\alpha_i^1) = p_i$,
(6) $\phi_i f_i(\beta_i) = \alpha_i^1$,
(7) $\phi_i \,|\, (N - V_\varepsilon f_i(\alpha_i^1)) = 1$,
(8) $\phi_i \,|\, (N - f_i(\alpha_i^1))$ is a homeomorphism onto $N - p_i$.

Then $\phi = \phi_m f_m \cdots \phi_1 f_1$ is a map of N onto itself such that:

(9) $\phi(\alpha_i^1) = p_i$, $i = 1, \cdots, m$,
(10) $\phi \,|\, (N - \bigcup_1^m \alpha_i^1)$ is a homeomorphism onto $N - \bigcup_1^m p_i$.

Let $M_1 = \phi(K_1)$. Then $M_1 \in \Psi_*[K_1 \cup (\bigcup_1^m \alpha_i^1)]$ and so by (i) M_1 is equivalent to M.

Define $e: B^3 \twoheadrightarrow B_-^3$ by a canonical push down so that $e(0 \times B^1) = 0 \times B_-^1$. Define $g_i: B^3 \to N$ by $g_i = \phi_i f_i h_i e$. Then it is easy to see that the images of the g_i are disjoint and that g_i satisfies the properties:

(11) $g_i(S_+^2) \subset M_1$,
(12) $g_i(B^3 - S_+^2) \subset N - M_1$,
(13) $g_i(S^2)$ is locally flat except at p_i,
(14) $K_1 = (M_1 - \bigcup_1^m g_i(S_+^2)) \cup (\bigcup_1^m g_i(S_-^2))$ is locally flat,
(15) $\alpha_i^1 = g_i(0 \times B^1)$.

Thus $K_1 \cup (\bigcup_1^m \alpha_i^1) \in \Gamma[M_1]$ and so by (ii) $K_1 \cup (\bigcup_1^m \alpha_i^1)$ is equivalent to $K_2 \cup (\bigcup_1^m \alpha_i^2)$. Hence $\Gamma_* \Psi_* = 1$.

(iv) $\Psi_* \Gamma_* = 1$. Indeed, given the diagram with the solid arrows:

$$\begin{tikzcd} M_1 \arrow[rr, "\Gamma"] \arrow[drr, dashed, leftrightarrow] & & K \cup (\bigcup_1^m \alpha_i) \arrow[d, dash, "\Psi"] \\ & & M_2 \end{tikzcd}$$

where $M_1, M_2 \in \mathscr{M}$ and $K \cup (\bigcup_1^m \alpha_i) \in \mathscr{A}$, we will show that we can fill in the dotted arrow.

Let p_i be the wild points of M_1, $i = 1, \cdots, m$. By the definition of Γ there are homeomorphisms g_i: $B^3 \to N^3$ with disjoint images such that:

(1) $g_i(S_+^2) \subset M_1$,
(2) $g_i(a) = p_i$,
(3) $g_i(B^3 - S_+^2) \subset N - M_1$,
(4) $g_i(S^2)$ is locally flat except at p_i,
(5) $K = (M_1 - \bigcup_1^m g_i(P_0)) \cup (\bigcup_1^m g_i(D_0))$ is locally flat,
(6) $\alpha_i = g_i(J)$.

Let $h_i: B^3 \twoheadrightarrow g_i(B^3)$ be a homeomorphism such that:
(7) $h_i(0 \times B_+^1) = \alpha_i$,
(8) $h_i(S_+^2) = g_i(P_0)$,
(9) $h_i(B^2) = g_i(D_0)$.

Since $g_i(S^2)$ is locally flat except at p_i, we can extend h_i to a homeomorphism $h_i: B_2(b) \to N$ such that the images are disjoint and

$$h_i^{-1}(M_1 - g_i(P_0)) \subset R_-^3 \cap (B_2(b) - V_1(0)).$$

It follows from Lemma 4 that there is a map ϕ_i of N onto itself such that:
(10) $\phi_i | (N - h_i(B_2(b))) = 1$,
(11) $\phi_i(\alpha_i) = p_i$,
(12) $\phi_i(K \cap h_i(B_2(b))) = M_1 \cap h_i(B_2(b))$,
(13) $\phi_i | (N - \alpha_i)$ is a homeomorphism onto $N - p_i$.

Then $\phi = \phi_m \cdots \phi_1$ is a map of N onto itself such that:
(14) $\phi(\alpha_i) = p_i$, $i = 1, \cdots, m$,
(15) $\phi | (N - \bigcup_1^m \alpha_i)$ is a homeomorphism onto $N - \bigcup_1^m p_i$,
(16) $\phi(K) = M_1$.

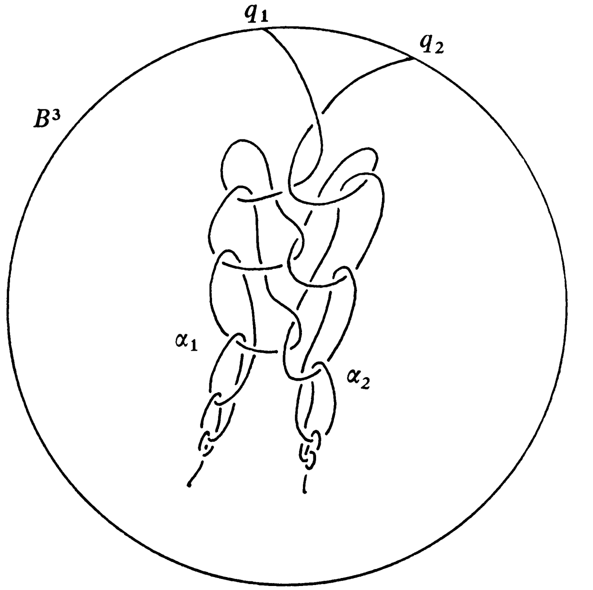
Thus $M_1 \in \Psi_*[K \cup (\bigcup_1^m \alpha_i)]$ and so by (i) M_1 is equivalent to M_2. Hence $\Psi_* \Gamma_* = 1$.

Proof of Theorem 3. Let $\mathscr{A}$ be the set of pairs (Σ, α) where Σ is a flat 2-sphere in S^3, α is an arc which intersects Σ at one end point and $\Sigma \cup \alpha$ is locally flat except at the other endpoint p and let $\mathscr{A}_*$ be the sets of equivalence classes of $\mathscr{A}$ in S^3. By Lemma 7 the mapping $(\Sigma, \alpha) \to (\alpha, p)$ induces a one-to-one correspondence between $\mathscr{A}_*$ and $\mathscr{I}_*$. By Theorem 4 there is a one-to-one correspondence between $\mathscr{S}_*$ and $\mathscr{A}_*$ and the composition of these two is the desired one-to-one correspondence between $\mathscr{S}_*$ and $\mathscr{I}_*$.

4. **Uniqueness of a decomposition space.**

Theorem 5. *Let $\alpha_i^1, i = 1, \cdots, m$, be disjoint arcs in B^3 and let $\alpha_i^2, i = 1, \cdots, m$, be disjoint arcs in B^3 such that a_i^j intersects S^2 at one endpoint and $S^2 \cup \alpha_i^j$ is locally flat in R^3 except at the other endpoint p_i^j, $i = 1, \cdots, m, j = 1, 2$. For each $j = 1, 2$, let H_j be the decomposition space of B^3 whose nondegenerate elements are the arcs α_i, $i = 1, \cdots, m$. If H_1 is homeomorphic to H_2, then, with a suitable ordering of the α_i^j's, there is a homeomorphism of B^3 onto itself carrying α_i^1 onto α_i^2, $i = 1, \cdots, m$.*

Before proving the theorem, let us consider the example illustrated in the following figure:



It is impossible to find disjoint closed 3-cells G_i in B_3 such that $\alpha_i - q_i \subset \operatorname{Int} G_i$, $i = 1,2$, so that it looks like the theorem may require a global proof. However, these arcs are cellular in R^3 by Theorem 1 so that there are disjoint Euclidean neighborhoods of the arcs in R^3. Thus we see that there may be a local proof for the theorem. In fact, the proof follows from Theorem 4 which has a local proof.

Proof. Without loss of generality, for each $j = 1,2, H_j$ may be considered as a subset of R^3 which is the image of B^3 under a decomposition map of R^3, i.e., there is a map ϕ_j of R^3 onto itself such that:

(1) $\phi_j(\alpha_i^j) = p_i^j, \quad i = 1,\cdots,m$,

(2) $\phi_j \big| (R^3 - \bigcup_1^m \alpha_i^j)$ is a homeomorphism onto $R^3 - \bigcup_1^m p_i^j$,

(3) $\phi_j(B^3) = H_j$.

Since $\mathrm{Cl}(R^3 - H_j) \approx \mathrm{Cl}(R^3 - B^3)$, $j = 1,2$, we can extend the homeomorphism of H_1 onto H_2 to a homeomorphism of R^3 onto itself. Thus $\phi_1(S^2)$ is equivalent to $\phi_2(S^2)$ in R^3. Since $\phi_j(S^2) = \Psi(S^2 \cup (\bigcup_1^m \alpha_i^j))$, $j = 1,2$, it follows from Theorem 4 that $S^2 \cup (\bigcup_1^m \alpha_i^1)$ is equivalent to $S^2 \cup (\bigcup_1^m \alpha_i^2)$. Thus there is a homeomorphism $h: R^3 \twoheadrightarrow R^3$ such that $h(S^2 \cup (\bigcup_1^m \alpha_i^1)) = S^2 \cup (\bigcup_1^m \alpha_i^2)$. Without loss of generality $h(\alpha_i^1) = \alpha_i^2, i = 1,\cdots,m$. Then $h \big| B^3$ is the required homeomorphism.

5. **Characterization of a class of crumpled cubes.** A *crumpled n-cube* is a topological space which is homeomorphic to a closed complementary domain of an $(n-1)$-sphere embedded in the n-sphere S^n.

THEOREM 6. *Let H be a crumpled n-cube in S^n such that $G = \mathrm{Cl}(S^n - H)$ is a* *closed n-cell. Then H is homeomorphic to a decomposition space of B^n whose nondegenerate elements are arcs which interesct S^{n-1} at one endpoint and are locally flat except possibly at the other endpoint. Moreover, these arcs correspond to the singular points of $\Sigma = \mathrm{Bd}H$ (i.e., the points at which Σ is not locally flat).*

Notice that for $n \geqq 4$ it follows from [19], [10] that the arcs are locally flat at every point. However, for $n = 3$ the arcs may or may not be locally flat at the endpoint. Also for $n = 3$ it follows from [20], [22] that any crumpled 3-cube can be embedded in S^3 so that the closure of the complement is a closed 3-cell. Thus the conclusion of Theorem 6 holds for any crumpled 3-cube.

Proof. Let g be a homeomorphism of B^n onto G. Let a point $p \in B^n$ be represented by the coordinates (u, x) where u is the distance from p to the origin and x is the point of S^{n-1} which lies on the ray from the origin through p. Let X be the set of singular points of Σ, let $X' = g^{-1}(X)$ and let μ be a map from S^{n-1} into I such that $\mu(X') = 1$ and $\mu(S^{n-1} - X') \subset I'$. Define a map $\theta\colon B^n \twoheadrightarrow B^n$ as follows:

$$\theta(1/2, x) = (1/2 + 1/2\mu(x), x),$$

$$\theta \text{ maps } [0, (1/2, x)] \text{ linearly onto } [0, \theta(1/2, x)],$$

$$\theta \text{ maps } [(1/2, x), (1, x)] \text{ linearly onto } [\theta(1/2, x), (1, x)].$$

Define a map $\phi\colon S^n \twoheadrightarrow S^n$ by

$$\phi(p) = \begin{cases} g\theta g^{-1}(p), & p \in G, \\ p, & p \in H. \end{cases}$$

Now $H - X$ is a manifold with boundary $\Sigma - X$ and $\phi g([1/2, 1] \times (S^{\ -1} - X'))$ is a closed collar attached to $H - X$. Thus by Theorem 2 there is a homeomorphism

$$h_1\colon (H - X) \cup \phi g([1/2, 1] \times (S^{n-1} - X')) \twoheadrightarrow H - X.$$

Let H_1 be the closed complementary domain of $g(\mathrm{Bd}B_{1/2}(0))$ which contains Σ. Then we can extend h_1 to a homeomorphism $h_1\colon \phi(H_1) \twoheadrightarrow H$ *via* the identity on X. Let h_2 be a homeomorphism from B^n onto H_1. Then $h = h_1 \phi h_2$ is the required map of B^n onto H. For, if $x \in X$, then $h^{-1}(x) = h_2^{-1} g([1/2, 1] \times g^{-1}(x))$, an arc which intersects S^{n-1} at one endpoint and is locally flat except possibly at the other endpoint, and if $x \in H - X$, then $h^{-1}(x)$ is a single point.

6. **Characterization of pseudo-half spaces.** In this section we will characterize pseudo-half spaces. First we state a lemma.

LEMMA 8. *If $(X, Y) \approx (R^n_+, R^{n-1})$ and $X \cup p$ is the one-point compactification of X, then $(X \cup p, Y \cup p) \approx (B^n, S^{n-1})$.*

THEOREM 7. *M is an n-pseudo-half space if and only if $M \approx B^n - \alpha$ where α is an arc in B^n such that α intersects S^{n-1} at one endpoint and $S^{n-1} \cup \alpha$ is locally flat except possibly at the other endpoint.*

Proof. Assume M is an n-pseudo-half space. By Theorem 2 we can add an open collar $\mathrm{Bd}M \times [0,1)$ to M by identifying $(x,0)$ with x for $x \in \mathrm{Bd}M$, so that $M \cup (\mathrm{Bd}M \times [0,1)) \approx \mathrm{Int}M$. Without loss of generality the one-point compactification $M \cup (\mathrm{Bd}M \times [0,1)) \cup p$ is equal to S^n. By Lemma 8 there is a homeomorphism:

$$f\colon ((\mathrm{Bd}M \times [0,1)) \cup p, (\mathrm{Bd}M \times 0) \cup p) \twoheadrightarrow (B^n, S^{n-1})$$

such that $f(p) = 1$. Let $B' = B_{1/4}(-1/4,0,0)$ and let $S' = \mathrm{Bd}B'$. Now $f^{-1}(\mathrm{Cl}\,|\,(B^n - B') - I)$ is a closed collar attached to M. By Theorem 2 there is a homeomorphism

$$\begin{aligned} h\colon M \twoheadrightarrow M &\cup f^{-1}(\mathrm{Cl}(B^n - B') - I) \\ &= \mathrm{Cl}(S^n - f^{-1}(B')) - f^{-1}(I). \end{aligned}$$

Now $f^{-1}(S')$ is bi-collared in S^n and hence flat. Thus there is a homeomorphism $g\colon S^n \twoheadrightarrow S^n$ such that $g(\mathrm{Cl}(S^n - f^{-1}(B'))) = B^n$. Let $\alpha = gf^{-1}(I)$. Then we have:

$$\begin{aligned} gh(M) &= g(\mathrm{Cl}(S^n - f^{-1}(B')) - f^{-1}(I)) \\ &= g(\mathrm{Cl}(S^n - f^{-1}(B'))) - gf^{-1}(I) \\ &= B^n - \alpha. \end{aligned}$$

It is evident that α has the required properties.

Assume $M \approx B^n - \alpha$ where α is an arc in B^n which intersects S^{n-1} at one endpoint q and is locally flat except at the other endpoint p. We can identify S^n with the one-point compactification of R^n. It is easy to show that $\mathrm{Int}(B^n - \alpha) \approx S^n - \alpha$ by shrinking $\mathrm{Cl}(S^n - B^n)$ to q. By Theorem 1 there is a map $g\colon S^n \twoheadrightarrow S^n$ such that $g(\alpha) = p$ and $g\,|\,(S^n - \alpha)$ is a homeomorphism onto $S^n - p$. Thus

$$\mathrm{Int}M \approx \mathrm{Int}(B^n - \alpha) \approx S^n - \alpha \approx S^n - p \approx R^n,$$

$$\mathrm{Bd}M \approx \mathrm{Bd}(B^n - \alpha) = S^{n-1} - q \approx R^{n-1}.$$

Hence M is an n-pseudo-half space.

REMARK. We have actually proved that $B^n - \alpha$ is an n-pseudo-half space even if $S^{n-1} \cup \alpha$ is not locally flat at $S^{n-1} \cap \alpha$.

COROLLARY [CANTRELL, DOYLE]. *For $n \neq 3, M \approx R^n_+$.*

Proof. The proof is essentially that of Cantrell [7] as pointed out by Doyle [11] which we include for completeness. It follows from Theorem 2.1 of [16], a generalization of a theorem of Homma [19], that for $n > 3$ an arc in R^n which is locally flat except at one endpoint is equivalent to an arc which is locally polyhedral except at one endpoint. By [10] the arc is locally flat at every point. For $n < 3$, this is true for every arc. So by Theorem 1 there is a map $g: B^n \twoheadrightarrow B^n$ such that $g \mid S^{n-1} = 1$, $g(\alpha) = q$, and $g \mid (B^n - \alpha)$ is a homeomorphism onto $B^n - q$. Hence for $n \neq 3$, $M \approx B^n - \alpha \approx B^n - q \approx R^n_+$.

THEOREM 8. *If α_1 and α_2 are two arcs in B^3 which are not equivalent in R^3 such that α_1 intersects S^2 at one endpoint q_i and $\alpha_i \cup S^2$ is locally flat in S^3 except possibly at the other endpoint p_i, $i = 1, 2$, then $B^3 - \alpha_1$ and $B^3 - \alpha_2$ are topologically different.*

Proof. Suppose we have $h: B^3 - \alpha_1 \approx B^3 - \alpha_2$. We can identify S^3 with the one-point compactification of R^3 and extend h to $h: S^3 - \alpha_1 \approx S^3 - \alpha_2$. By Theorem 1 there is a map $g_i: S^3 \twoheadrightarrow S^3$ such that $g_i(\alpha_i) = p_i$ and $g_i \mid (S^3 - \alpha_i)$ is a homeomorphism onto $S^3 - p_i$, $i = 1,2$. Let $\Sigma_i = g_i(S^2)$, $i = 1,2$, and define $f: S^3 \twoheadrightarrow S^3$ by:

$$f(x) = \begin{cases} g_2 h g_1^{-1}(x), & x \in S^3 - p_1, \\ p_2, & x = p_1. \end{cases}$$

Evidently f is a homeomorphism. Now

$$f(\Sigma_1 - p_1) = g_2 h g_1^{-1}(\Sigma_1 - p_1) = g_2 h(S^2 - q_1) = g_2(S^2 - q_2) = \Sigma_2 - p_2$$

so that $f(\Sigma_1) = \Sigma_2$. Thus Σ_1 is equivalent to Σ_2. By Theorem 3, α_1 is equivalent to α_2, a contradiction. Hence $B^3 - \alpha_1 \not\approx B^3 - \alpha_2$.

COROLLARY. *There are uncountably many topologically different 3-pseudo-half spaces.*

Proof. By [14] there are uncountably many inequivalent arcs in R^3 which are locally flat except at one endpoint.

THEOREM 9. *Let M_1 and M_2 be 3-pseudo-half spaces with common boundary F and disjoint interiors such that $M_1 \not\approx R^3_+$. Then $M_1 \cup M_2 \approx R^3$ if and only if $M_2 \approx R^3_+$.*

Proof. Assume $M_1 \cup M_2 = R^3$. We can identify S^3 with the one-point compactification $R^3 \cup p$ of R^3. Then $F \cup p$ is a 2-sphere in S^3 which is locally flat except at p. Since $M_1 \cup p \not\approx B^3$, by [18] $M_2 \cup p \approx B^3$. Thus $M_2 \approx R^3_+$.

The converse follows immediately from Theorem 2.

COROLLARY. *If M is a 3-pseudo-half space such that $M \not\approx R^3_+$, then $M \times I \not\approx R^4_+$.*

Proof. $\mathrm{Bd}(M \times I) = (\mathrm{Bd}\, M \times I) \cup (M \times \mathrm{Bd}\, I) \not\approx R^3$.

7. Cellularity of arcs in S^3.

THEOREM 10. *If α is an arc in S^3 such that α contains a subarc β both of whose endpoints are isolated wild points of β, then α is not cellular.*

Proof. Suppose α is cellular.

Case 1. α is only wild at its endpoints a and b. Since α is cellular, there is a homeomorphism $h: S^3 - \alpha \twoheadrightarrow S^3 - p$ for some point $p \in S^3$. Let $q \in \operatorname{Int} \alpha$. Since α is locally flat at q, there is an open 2-cell D in S^3 such that $D \cap \alpha = q$ and $D \cup \alpha$ is locally flat at every point of D. Then $D_1 = h(D - q) \cup p$ is an open 2-cell in S^3 which is locally flat except at p. It follows from Lemma 2 that there is an open 2-cell $D_2 \subset D_1$ such that $p \in D_2$ and D_2 is contained in a 2-sphere Σ_2 which is locally flat except at p. Then $\Sigma = h^{-1}(\Sigma_2 - p) \cup q$ is locally flat at every point and hence flat in S^3 and $\Sigma \cap \alpha = q$.

Let G_1 and G_2 be the closed complementary domains of Σ in S^3 and let $M_i = G_i - \alpha$, $i = 1,2$. By Theorems 7 and 8, M_i is a 3-pseudo-half space but not R^3_+, $i = 1,2$. But $M_1 \cup M_2 = S^3 - \alpha \approx R^3$, which contradicts Theorem 9.

Case 2. α is wild at both endpoints a and b and at one interior point d. If $x, y \in \alpha$, let $\langle x, y\rangle$ denote the subarc of α from x to y. By [23], for $n \neq 4$, every subarc of a cellular arc is cellular. Thus $\langle a, d\rangle$ and $\langle d, b\rangle$ are both cellular and if either one is wild at both endpoints, we get a contradiction by Case 1. Hence suppose both $\langle a, d\rangle$ and $\langle d, b\rangle$ are locally flat at d. By [25] there is a neighborhood U of $\alpha - a$ such that every arc in $U \cup a$ with a as an endpoint is wild. By Theorem 1 there is a map $\phi: S^3 \twoheadrightarrow S^3$ such that $\phi\langle a, d\rangle = a$, $\phi \mid (S^3 - U) = 1$ and $\phi \mid (S^3 - \langle a, d\rangle)$ is a homeomorphism onto $S^3 - a$. Thus $\phi\langle d, b\rangle$ is cellular and wild at both endpoints. Again we get a contradiction by Case 1.

General Case. Let γ be a subarc of β such that γ contains all the wild points of β except its endpoints. Then β and γ are both cellular. Thus there is a map $\phi: S^3 \twoheadrightarrow S^3$ such that $\phi(\gamma)$ is a point and $\phi \mid (S^3 - \gamma)$ is a homeomorphism. Then $\phi(\beta)$ reduces to either Case 1 or Case 2 and we get a contradiction. Hence α is not cellular.

The following theorem is a special case of Theorem 1 of [12]. However, the proof here does not use the axiom of choice.

THEOREM 11 (DOYLE). *If α is an arc in S^3 such that α contains no subarc both of whose endpoints are wild, then α is cellular.*

Proof. Let p and r be the endpoints of α. There is a natural ordering, denoted by $<$, of the points of α from p to r. If β and γ are subarcs of α, we will say that $\beta < \gamma$ if $x < y$ for arbitrary $x \in \operatorname{Int} \beta$ and $y \in \operatorname{Int} \gamma$.

Let X be the set of wild points of α. Then X is countable since it has the same order as the set of components of $\alpha - X$. There is at most one point q of X such that q does not lie on some flat subarc of α and with no loss of generality such a q exists.

Let $\varepsilon > 0$. Since X is a countable compact set, there is a flat closed 3-cell $E \subset V_\varepsilon(\alpha)$ such that $X \subset \operatorname{Int} E$.

Let $\mathscr{U}_0$ be the collection of closed complementary domains of $X \cap \langle p, q\rangle$ in $\langle p, q\rangle$ which are not contained in $\operatorname{Int} E$ and let $\beta_1 = \langle y_1, z_1\rangle$ be the last such arc in $\mathscr{U}_0$. Let $y_1' \in \operatorname{Int} \beta_1$ such that $\langle y_1', z_1\rangle \subset \operatorname{Int} E$. Let h_1 be a homeomorphism of S^3 onto itself such that:

(1) $h_1 \,|\, ((S^3 - V_\varepsilon(\alpha)) \cup \langle z_1, r\rangle) = 1$,

(2) $h_1(\beta_1) = \langle y_1', z_1\rangle$,

(3) $h_1(X) \subset \operatorname{Int} E$.

Then $F_1 = h_1^{-1}(E)$ is a flat closed 3-cell in $V_\varepsilon(\alpha)$ such that $X \cup \langle y_1, q\rangle \subset \operatorname{Int} F_1$.

Let $\mathscr{U}_1$ be the collection of closed complementary domains of $X \cap \langle p, q\rangle$ in $\langle p, q\rangle$ which are not contained in $\operatorname{Int} F_1$ and let $\beta_2 = \langle y_2, z_2\rangle$ be the last such arc in $\mathscr{U}_1$. As before we construct a flat closed 3-cell F_2 in $V_\varepsilon(\alpha)$ such that $X \cup \langle y_2, q\rangle \subset \operatorname{Int} F_2$.

If this process continued indefinitely, we would get a sequence of points $y_1 \in X$ with $y_{i+1} < y_i$ and a sequence of flat closed 3-cells F_i such that

$$X \cup \langle y_i, q\rangle \subset \operatorname{Int} F_i, \qquad i = 1, 2, \cdots.$$

Then $y = \lim_{i\to\infty} y_i$ would be an element of X such that $y \neq q$ and y is contained in no flat subarc of α, a contradiction. Hence the process must end, i.e., there is a flat closed 3-cell $F \subset V_\varepsilon(\alpha)$ such that $X \cup \langle p, q\rangle \subset \operatorname{Int} F$.

Similarly, we can start at the other end of α and construct a flat closed 3-cell $F' \subset V_\varepsilon(\alpha)$ such that $\alpha \subset \operatorname{Int} F'$. Since ε is arbitrary, α is cellular.

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