THE SEMI-SIMPLICITY MANIFOLD
OF ARBITRARY OPERATORS

BY

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Introduction. In finite dimensional complex Euclidian space, any linear operator has a unique maximal invariant subspace on which it is semi-simple. It is easily described by means of the Jordan decomposition theorem. The main result of this paper (Theorem 2.1) is a generalization of this fact to infinite dimensional reflexive Banach space, for arbitrary bounded operators T with real spectrum. Our description of the "semi-simplicity manifold" for T is entirely analytic and basis-free, and seems therefore to be a quite natural candidate for such generalizations to infinite dimension.

A similar generalization to infinite dimension of the concepts of Jordan cells and Weyr characteristic will be presented elsewhere.

The theory is motivated and illustrated by examples in §3.

NOTATIONS. The following notations are fixed throughout this paper, and will be used without further explanation.

R: the field of real numbers.

C: the field of complex numbers.

$\mathscr{B}$: the sigma-algebra of all Borel subsets of R.

$C(R)$: the space of all continuous complex valued functions on R.

$L^p(R)$: the usual Lebesgue spaces on R $(p \geqq 1)$.

$\int$: integration over R.

$\hat{f}$: the Fourier transform of $f \in L^1(R)$.

X: an arbitrary complex Banach space.

X^*: the conjugate of X.

$B(X)$: the Banach algebra of all bounded linear operators acting on X.

I: the identity operator on X.

$|\cdot|$: the original norms on X, X^* and $B(X)$ (new norms, to be defined, will be denoted by double bars).

$|\cdot|_1, |\cdot|_\infty$: the $L^1(R)$ and $L^\infty(R)$ norms (respectively).

$\sigma(T)$: the spectrum of $T \in B(X)$;

$\rho(T)$: the resolvent set of T;

$R(\lambda; T)$: the resolvent of T;

$T|W$: the restriction of T to an invariant linear manifold W.

1. **The semi-simplicity manifold.** Let $T \in B(X)$ be a fixed operator with *real* spectrum.

DEFINITION 1.1. For $x \in X$, we let

$$\| x \| = \sup \left\{ \left| \left| \int f(t) e^{it\,T} x \, dt \right| \right| ; f \in L^1(R), \, |\hat{f}|_\infty \leq 1 \right\}.$$

and

$$W = W(T) = \{ x \in X; \| x \| < \infty \}.$$

If the integral in the definition of $\| x \|$ does not converge (strongly) for some $f \in L^1(R)$, we set $\| x \| = \infty$. Clearly, W contains all the eigenvectors of T.

THEOREM 1.1. *For any* $x \in X$,

$$\| x \| = \sup \left| \sum_{j=1}^n c_j \exp(it_j T)x \right| ,$$

where the sup *is taken over all* $n = 1, 2, \cdots, c_j \in C$ *and* $t_j \in R$ $(1 \leq j \leq n)$ *for which* $\left| \sum_{j=1}^n c_j \exp(it_j s) \right|_\infty \leq 1$.

Proof. Denote the supremum above by $\| x \|'$. Suppose $\| x \| < \infty$. For any $x^* \in X^*$ and $f \in L^1(R)$, we have

$$\left| \int f(t) x^* e^{it\,T} x \, dt \right| \leq \| x \| \, |x^*| \, |\hat{f}|_\infty.$$

By Schoenberg's Criterion [9], there exists a finite regular Borel measure $\mu = \mu(\cdot \,| x, x^*)$ on R such that

(1.1) $$\| \mu \| \leq \| x \| \, |x^*|$$

and

(1.2) $$x^* e^{it\,T} x = \int e^{its} \, d\mu(s) \qquad \text{(all } t \in R).$$

Therefore [1]

$$\left| \sum_{j=1}^n c_j x^* \exp(it_j T)x \right| \leq \| x \| \, |x^*| \left| \sum_{j=1}^n c_j \exp(it_j s) \right|_\infty$$

for all $c_j \in C$, $t_j \in R$ and $n = 1, 2, \cdots$. Hence $\| x \|' \leq \| x \|$.

Next, suppose $\| x \|' < \infty$.

By Bochner's Criterion [1], there is a representation (1.2) with $\| \mu \| \leq \| x \|' |x^*|$. In particular (or directly from the definition of $\| x \|'$), $|e^{t\,T} x| \leq \| x \|'$. Therefore the integral $\int f(t) e^{it\,T} x \, dt$ converges strongly for each $f \in L^1(R)$, and so

$$\left| \int f(t) e^{it\,T} x \, dt \right| = \sup_{|x^*| = 1} \left| \int f(t) x^* e^{it\,T} x \, dt \right| \leq \| \mu \| |\hat{f}|_\infty \leq \| x \|' |\hat{f}|_\infty$$

i.e., $\| x \| \leq \| x \|'$. Q.E.D.

THEOREM 1.2. (a) $(W, \| \cdot \|)$ *is a normed linear space, and* $\| x \| \geqq \| x \|$ *(for all* $x \in X$*).* (b) W *is an invariant linear manifold for any* $S \in B(X)$ *which commutes with* T*, and* S *is continuous on* $(W, \| \cdot \|)$ *with bound* $\leqq |S|$*.*

Proof. (a) Clearly, $\| x + y \| \leqq \| x \| + \| y \|$ and $\| \lambda x \| = |\lambda| \| x \|$ $(x, y \in W;$ $\lambda \in C)$. Therefore W is a linear manifold. Taking $n = 1$, $c_1 = 1$ and $t_1 = 0$, we have $\left| \sum_{j=1}^{n} c_j \exp(it_j s) \right|_{\infty} = 1$ and $\left| \sum_{j=1}^{n} c_j \exp(it_j T)x \right| = |x|$, hence $|x| \leqq \| x \|$ (by Theorem 1.1). In particular, $\| \cdot \|$ is a norm on W, and (a) is proved.

(b) Let $S \in B(X)$ commute with T, and let $x \in W$. Suppose $c_j \in C$ and $t_j \in R$ $(j = 1, \cdots, n)$ are such that $\left| \sum_{j=1}^{n} c_j \exp(it_j s) \right|_{\infty} \leqq 1$. Then, by Theorem 1.1,

$$\left| \sum_{j=1}^{n} c_j \exp(it_j T) S x \right| = \left| S \sum_{j=1}^{n} c_j \exp(it_j T)x \right| \leqq |S| \| x \|,$$

i.e., $Sx \in W$ and $\| Sx \| \leqq |S| \| x \|$, proving (b).

DEFINITION 1.2. The *semi-simplicity manifold* for T is the linear manifold $W = W(T)$.

REMARK. The norms $\| \cdot \|$ and $| \cdot |$ are equivalent (or, what amounts to the same thing, $W = X$) if and only if T is of class C (i.e., if and only if T is scalar for X weakly complete, or T is similar to a hermitian operator if X is a Hilbert space; cf. [7]). In particular, if X is a Hilbert space, $\| \cdot \| = | \cdot |$ if and only if T is hermitian (cf. Theorem 3.4 in [7]).

An equivalent definition of W follows from our next expression for the norm $\| x \|$.

THEOREM 1.3. *For any* $x \in X$,

$$\| x \| = \sup \left\{ \frac{1}{2\pi} \int \left| x^*[R(s - i\varepsilon; T) - R(s + i\varepsilon; T)]x \right| \, ds \right\}$$

where the sup *is taken over all* $\varepsilon > 0$ *and all unit vectors* $x^* \in X^*$*.*

Proof. Let $\varepsilon > 0$ and

$$F_\varepsilon(s) = \frac{1}{2\pi i} [R(s - i\varepsilon; T) - R(s + i\varepsilon; T)], \qquad s \in R.$$

Since $\sigma(T)$ is real, F_ε is well defined and $x^*F_\varepsilon(\cdot)x \in L^1(R) \cap L^2(R)$ for any $x \in X$ and $x^* \in X^*$. By [6, p. 164, equation (2)], the Fourier transform of $x^*F_\varepsilon(\cdot)x$ is equal to $\exp(-2\pi\varepsilon|t|)x^* e^{-2\pi it\, T}x$. But $\exp(-2\pi\varepsilon|t|)$ is the Fourier transform of $f_\varepsilon(s) = \varepsilon(s^2 + \varepsilon^2)^{-1}/\pi$, and if $x \in W$, $x^*\exp(-2\pi it\, T)x$ is the Fourier-Stieltjes transform of the measure μ defined in (1.1) and (1.2). Thus $x^*F_\varepsilon(\cdot)x$ is the convolution of $f_\varepsilon \in L^1(R)$ and μ, and therefore, using (1.1):

$$\left| x^*F_\varepsilon(\cdot)x \right|_1 \leqq |f_\varepsilon|_1 \| \mu \| = \| \mu \| \leqq \| x \| |x^*|.$$

Denoting the supremum in the statement of the theorem by $\|x\|''$, we have proved that $\|x\|'' \leqq \|x\|$ whenever $\|x\| < \infty$.

Next, suppose $\|x\|'' < \infty$ for some $x \in X$. Then, for $\varepsilon > 0$ and $x^* \in X^*$ with unit norm,

$$\left| \exp(-2\pi\varepsilon|t|)x^* \exp(-2\pi it\, T)x \right| = \left| [x^* F_\varepsilon(\cdot)x]^\wedge \right| \leqq \left| x^* F_\varepsilon(\cdot)x \right|_1$$

$$\leqq \|x\|''$$

i.e., $\exp(-2\pi\varepsilon|t|)|\exp(-2\pi it\, T)x| \leqq \|x\|''$ (for all $t \in \mathbf{R}$). Therefore the integral $\int f(t)\exp(-2\pi\varepsilon|t|)\exp(-2\pi it\, T)x\, dt$ converges strongly for any $f \in L^1(\mathbf{R})$, and the dominated convergence theorem implies that

$$\int f(t)\exp(-2\pi it\, T)x\, dt$$

(converges strongly and) is equal to

$$\lim_{\varepsilon \to 0+} \int f(t)\exp(-2\pi\varepsilon|t|)\exp(-2\pi it\, T)x\, dt$$

(strongly). Since $\exp(-2\pi\varepsilon|t|)x^* \exp(-2\pi it\, T)x$ is the Fourier transform of $x^* F_\varepsilon(\cdot)x$ and $|x^* F_\varepsilon(\cdot)x|_1 \leqq \|x\|''$, the latter integral has norm $\leqq \|x\|'' |\hat{f}|_\infty$ (for all $\varepsilon > 0$ and $f \in L^1(\mathbf{R})$). Hence $\|x\| \leqq \|x\|''$. Q.E.D.

2. **Spectral decomposition of T on W.** If Z is any linear manifold in X, we note by $T(Z)$ the algebra of all linear transformations of X with domain Z and range contained in Z.

DEFINITION 2.1. Let Z be a linear manifold in X. A *spectral measure on Z* is a map $\delta \to E(\delta)$ of $\mathscr{B}$ into $T(Z)$ such that

(i) $E(\mathbf{R}) = I\,|\,Z$;

(ii) $E(\delta)E(\varepsilon) = E(\delta \cap \varepsilon)$ for all $\delta, \varepsilon \in \mathscr{B}$; and

(iii) for each $x \in Z$, $E(\cdot)x$ is a regular strongly countably additive vector measure on $\mathscr{B}$.

By Corollary III.4.5 in [3], $E(\cdot)x$ is necessarily bounded (with a bound depending of course on $x \in Z$).

If Z is of the second category in X and if each $E(\delta)$ is closed, the closed graph and the uniform boundedness theorems imply that $E(\delta) \in B(X)$ (all δ) and $\sup_\delta |E(\delta)| < \infty$, so that our definition coincides in this case with the usual one.

We can state now the main result of this paper.

THEOREM 2.1. *Let X be reflexive, and $T \in B(X)$ have real spectrum. Let W be the semi-simplicity manifold for T. Then, for all $x \in W$ and all polynomials p,*

$$(2.1) \qquad\qquad p(T)x = \int p(t)\, dE(t)x,$$

where E is a spectral measure on W supported by $\sigma(T)$ and commuting with each $S \in B(X)$ which commutes with T. The representation (2.1) is "maximal-unique" in the following sense: if W' is an invariant linear manifold for T, and (2.1) is valid with W' and E' replacing W and E respectively, then $W' \subset W$ and $E'(\delta) = E(\delta) \,\big|\, W'$ for all $\delta \in \mathscr{B}$.

Proof. Assuming that representations (2.1) exist with spectral measures E and E' on W and W' respectively, both supported by $\sigma(T)$, we show first that $W' \subset W$ and $E'(\delta) = E(\delta) \,\big|\, W'$ for all $\delta \in \mathscr{B}$. Fix $x \in W'$. Since $p(T)x = \int_{\sigma(T)} p(t) dE'(t)x$ for any polynomial p, and since $\sigma(T)$ is compact, we obtain

$$(2.2) \qquad e^{it\,T}x = \int_{\sigma(T)} e^{its} d\, E'(s)x \qquad \text{(all } t \in R\text{)}.$$

Let $M(x)$ be a uniform bound for the vector measure $E'(\,\cdot\,)x$. Then, for $c_j \in C$ and $t_j \in R$ $(j = 1, \cdots, n)$, we have

$$\left| \sum_{j=1}^{n} c_j \exp(it_j\, T)x \right| = \left| \int_{\sigma(T)} \left(\sum_{j=1}^{n} c_j \exp(it_j s) \right) d\, E'(s)x \right|$$

$$\leq 4M(x) \left| \sum_{j=1}^{n} \, \iota c_j \exp(it_j s) \right|_{\infty},$$

i.e., $\| x \| \leq 4M(x)$ (by Theorem 1.1) and $x \in W$. Thus $W' \subset W$. For $x \in W'$, equation (2.2) is valid with both E and E'; i.e., $\int e^{its} d\, E(t)x = \int e^{its} dE'(t)x$ for all $t \in R$. Hence $E(\delta)x = E'(\delta)x$ for all $\delta \in \mathscr{B}$, by the uniqueness of the Fourier transform. This proves the maximality-uniqueness assertion.

The existence of the spectral decomposition (2.1) on the semi-simplicity manifold W will be proved in the following sequence of eight lemmas.

LEMMA 2.1. *There exists a family $\{E(\delta); \delta \in \mathscr{B}\}$ of linear transformations of X with domain W with the following properties:*

(i) $E(R) = I \,\big|\, W$;

(ii) *for each $x \in W, E(\,\cdot\,)x$ is a regular strongly countably additive vector measure on $\mathscr{B}$; and*

(iii) $e^{it\,T}x = \int e^{its} d E(s)x$ *for all $t \in R$ and $x \in W$.*

Proof. Fix $x \in W$ and $x^* \in X^*$. Let $\mu = \mu(\,\cdot\,\big| x, x^*)$ be as in (1.1) and (1.2). The uniqueness of the representation (1.2) implies that $\mu(\delta \big| x, x^*)$ is a continuous linear functional on X^* (with bound $\leq \| x \|$) for each $\delta \in \mathscr{B}$ and $x \in W$ fixed. Since X is reflexive, there exists a unique element of X, which we denote by $E(\delta)x$, such that $\big| E(\delta)x \big| \leq \| x \|$ and $\mu(\delta \big| x, x^*) = x^* E(\delta)x$ for all $x^* \in X^*$. Similarly, for δ and x^* fixed, $\mu(\delta \big| x, x^*)$ is a linear function of x on W; therefore the map $x \to E(\delta)x$ is a linear transformation of X with domain W. Now, (i) follows from (1.2) with $t = 0$; (ii) follows from the equation $x^* E(\,\cdot\,)x = \mu(\,\cdot\,\big| x, x^*)$ and Theorem IV.10.1 in [3]; (iii) is a rewriting of (1.2), which is justified by (ii). Q.E.D.

LEMMA 2.2. *Let $\{E(\delta)\}$ be as in Lemma 2.1. Then $E(\delta)$ commutes with each $S \in B(X)$ which commutes with T (for all $\delta \in \mathscr{B}$).*

Proof. If $S \in B(X)$ commutes with T, then $SW \subset W$ by Theorem 1.2. Therefore, for each $x \in W$,

$$e^{it\,T}Sx = \int e^{its}\,dE(s)Sx \qquad \text{(Lemma 2.1)}.$$

The left-hand side is equal to $Se^{it\,T}x$, which, by Lemma 2.1 (iii) and (ii), is given by $\int e^{its}\,dS\,E(s)x$. The lemma follows now from the uniqueness of the Fourier transform.

Next, we show that the vector measures $E(\cdot)x$ $(x \in W)$ have compact support (this will be refined in Lemma 2.6).

LEMMA 2.3. *The supports of the vector measures $E(\cdot)x$ are contained in $[-|T|, |T|]$ (for all $x \in W$).*

Proof. Fix $x \in W$ and $x^* \in X^*$, and let $\mu = x^*E(\cdot)x$. The function $f(z) = x^*e^{iz\,T}x$ of the complex variable z is entire of exponential type $\leq |T|$. For $t \in R$, $f(t) = O(1)$ (in fact, $|f(t)| \leq \|x\|\,|x^*|$ by (1.1) and (1.2)). Consider f as a distribution on the Schwartz space $\mathscr{S}$ of rapidly decreasing functions on R. By Schwartz' generalization of the Paley-Wiener theorem (cf. [10] or [5, Theorem 5, p. 145]), the Fourier transform $\hat{f}$ of the distribution f has its support in the interval $|t| \leq |T|$. Since $\hat{f} = 2\pi\mu$ (by 1.2), the lemma is proved.

NOTATION. Let $B(\Delta)$ denote the space of all complex Borel functions f on R, which are bounded on $\Delta = [-|T|, |T|]$, with the pseudo-norm $|f|_\Delta = \sup_\Delta |f|$.

LEMMA 2.4. *The map*

$$\pi : (f, x) \rightarrow \int f(s)\,dE(s)x$$

of $B(\Delta) \times W$ into X is continuous, has range in W, and is also continuous as a map of $B(\Delta) \times W$ into $(W, \|\cdot\|)$. In fact,

$$(2.3) \qquad \left\| \int f(s)\,dE(s)x \right\| \leq \|x\|\,|f|_\Delta$$

for all $f \in B(\Delta)$ and $x \in W$.

Proof. Fix $x \in W$. Let

$$h(s) = \sum_{j=1}^{n} c_j \exp(it_j s) \qquad (c_j \in C;\ t_j \in R;\ j = 1, \cdots, n),$$

and write

$$H = \sum_{j=1}^{n} c_j \exp(it_j T).$$

Since $H \in B(X)$ commutes with T, we have $Hx \in W$ (Theorem 1.2). Let us estimate $\|Hx\|$. In what follows, all suprema are taken over all $g \in L^1(R)$ with $|\hat{g}|_\infty \le 1$. We have:

$$\|Hx\| = \sup \left| \int g(t) e^{it\,T} Hx\, dt \right|$$

$$= \sup \left| \sum_{j=1}^{n} c_j \int g(t) \exp[i(t+t_j)T]x\, dt \right|$$

$$= \sup \left| \int \left[\sum_{j=1}^{n} c_j g(t-t_j) \right] e^{it\,T}x\, dt \right|.$$

Since $\sum_{j=1}^{n} c_j g(t-t_j) = g^*(t) \in L^1(R)$ and $x \in W$, we obtain

$$\|Hx\| \le \sup |\hat{g}^*|_\infty \|x\| \qquad \text{(sup over } g\text{, as before).}$$

But $\hat{g}^*(s) = \sum_{j=1}^{n} c_j \exp(it_j s)\hat{g}(s) = h(s)\hat{g}(s)$, so that $|\hat{g}^*|_\infty \le |h|_\infty |\hat{g}|_\infty \le |h|_\infty$; hence

(2.4) $$\|Hx\| \le |h|_\infty \|x\|.$$

Now, for $x \in W$ and $f \in B(\Delta)$ fixed, let $y = \pi(f,x) = \int f(s) d\,E(s)x$. By Lemma 2.1 (ii), this is a well-defined element of X, and for H as above and $x^* \in X^*$ with unit norm, we have (using Lemmas 2.2 and 2.3):

$$|x^*Hy| = \left| \int_\Delta f(s) d\, x^*HE(s)x \right| = \left| \int_\Delta f(s) d\, x^*E(s)Hx \right|$$

$$= \left| \int_\Delta f(s) d\, \mu(s|Hx, x^*) \right|$$

$$\le |f|_\Delta \|\mu(\cdot|Hx, x^*)\| \le |f|_\Delta \|Hx\|$$

$$\le |f|_\Delta |h|_\infty \|x\|,$$

where we have written $\mu(\cdot|x, x^*) = x^*E(\cdot)x$ and applied (1.1) and (2.4). Hence $|Hy| \le |f|_\Delta \|x\| |h|_\infty$, and therefore, by Theorem 1.1, $\|y\| \le |f|_\Delta \|x\|$. This proves (2.3) and the lemma. Taking in particular $f = c_\delta$ (the characteristic function of $\delta \in \mathscr{B}$), we obtain

COROLLARY 2.1. *For each $\delta \in \mathscr{B}$, $E(\delta) \in T(W)$ and $\|E(\delta)x\| \le \|x\|$ (for all $x \in W$).*

NOTATION. We write $T(f)x = \pi(f,x) = \int f(s) d\, E(s)x$.
By Lemma 2.4, $T(f) \in T(W)$ and

(2.5) $\| T(f)x \| \leqq |f|_\Delta \| x \|$ for all $f \in B(\Delta)$ and $x \in W$.

By Lemma 2.2, $T(f)$ commutes with each $S \in B(X)$ which commutes with T.

The next lemma follows from a standard density argument, which we reproduce because of the special precautions needed in dealing with our unbounded operators.

LEMMA 2.5. *The map $f \to T(f)$ of $C(\mathbf{R})$ into $T(W)$ is multiplicative, and $T(p) = p(T) \,|\, W$ for all polynomials p.*

Proof. By Lemmas 2.1 and 2.3,

$$e^{itT}x = \int_\Delta e^{its}\, d\, E(s)x \qquad\qquad (\text{all } x \in W \text{ and } t \in \mathbf{R}).$$

Expanding both sides in powers of t, we obtain for $x \in W$, $T^n x = \int_\Delta s^n d\, E(s)x$ $(n = 0, 1, 2, \cdots)$, and therefore $T(p) = p(T) \,|\, W$ for all polynomials p. It follows in particular that the map $f \to T(f)$ of $B(\Delta)$ into $T(W)$ is multiplicative when restricted to polynomials. If $f, g \in C(\mathbf{R})$, choose polynomials p_n, q_n $(n = 1, 2, \cdots)$ such that $p_n \to f$ and $q_n \to g$ uniformly on Δ. Fix $x \in W$. By Theorem 1.2,

$$\left|[T(f)T(g) - T(fg)]x\right| \leqq \| T(f - p_n)T(g)x \|$$
$$+ \| T(p_n)T(g - q_n)x \| + \| T(p_n q_n - fg)x \|$$
$$= A + B + C.$$

By (2.5),

$$A \leqq |f - p_n|_\Delta \| T(g)x \| \leqq |f - p_n|_\Delta |g|_\Delta \| x \|.$$

Since $T(p_n) = p_n(T) \,|\, W$ commutes with $T(h)$ for any $h \in B(\Delta)$, we have by (2.5):

$$B = \| T(g - q_n)\, T(p_n)x \| \leqq |g - q_n|_\Delta \| T(p_n)x \|$$
$$\leqq |g - q_n|_\Delta |p_n|_\Delta \| x \|.$$

Finally,

$$C \leqq |p_n q_n - fg|_\Delta \| x \|.$$

Letting $n \to \infty$, we obtain $T(f)T(g) = T(fg)$. Q.E.D.

LEMMA 2.6. *The support of $E(\,\cdot\,)x$ lies in $\sigma(T)$ (for each $x \in W$).*

Proof. (Foias [4], proof of Proposition 1.) Suppose $f \in C(\mathbf{R})$ has compact support disjoint from $\sigma(T)$. From $z \in C \backslash \operatorname{supp} f$, define $f_z(t) = (z - t)^{-1} f(t)$. Clearly, $f_z \in C(\mathbf{R})$ and $T(f_z)x$ is an analytic X-valued function of z on $C \backslash \operatorname{supp} f$, for each $x \in W$. By Lemma 2.5,

$$(zI - T)T(f_z)x = T(z - t)T(f_z)x = T((z - t)f_z)x = T(f)x \qquad (\text{all } x \in W).$$

Thus, for $z \in (C \backslash \operatorname{supp} f) \cap \rho(T)$,

(2.6) $T(f_z)x = R(z; T)T(f)x$ (all $x \in W$),

and the right-hand side provides an analytic continuation of $T(f_z)x$ to $\rho(T) \supset \mathrm{supp}\, f$. Hence $T(f_z)x$ is entire, and since $T(f_z)x \to 0$ for $z \to \infty$ (by (2.6)), we have $T(f_z)x = 0$ and hence $T(f)x = 0$ (by (2.6)) for all $x \in W$. This proves that for each $x \in W$ and $x^* \in X^*$, the support of the linear functional on $C(R)$ defined by $f \to \int f(t) d\, x^*E(t)x$ is contained in $\sigma(T)$. Equivalently, the supports of the complex measures $x^*E(\,\cdot\,)x$ (and so, of the vector measures $E(\,\cdot\,)x$) are contained in $\sigma(T)$ (for all $x \in W$ and $x^* \in X^*$).

Lemma 2.7. *If* $f \in C(R)$ *and* $g \in B(\Delta)$, *then* $T(f)$ *and* $T(g)$ *commute (as elements of* $T(W)$). *In particular,* $E(\delta)$ *commutes with* $T(f)$ *for all* $\delta \in \mathscr{B}$ *and* $f \in C(R)$.

Proof. Choose polynomials p_n which converge uniformly to f on Δ. Since $T(g)$ commutes with $p_n(T) = T(p_n)$ (by Lemma 2.2), we obtain (by (2.5) and Theorem 1.2):

$$\left| [T(g)T(f) - T(f)T(g)]x \right| \le \| T(g)T(f - p_n)x \| + \| T(f - p_n)T(g)x \|$$

$$\le |g|_\Delta \| T(f - p_n)x \| + |f - p_n|_\Delta \| T(g)x \|$$

$$\le 2|f - p_n|_\Delta |g|_\Delta \| x \| \underset{n \to \infty}{\to} 0$$

for all $x \in W$, Q.E.D.

Lemma 2.8. $E(\delta \cap \varepsilon) = E(\delta)E(\varepsilon)$ *for all* δ, ε *in* $\mathscr{B}$.

Proof. Fix $\delta \in \mathscr{B}$, $x \in W$ and $g \in C(R)$. Recalling that $T(g)x \in W$ (Lemma 2.4), we may choose a finite positive measure λ such that the vector measures $E(\,\cdot\,)x$ and $E(\,\cdot\,)T(g)x$ are both λ-continuous (cf. [3, p. 321]). Let c_δ denote the characteristic function of δ, and choose $f_n \in C(R)$ such that $|f_n| \le 1$ and $f_n \to c_\delta$ a.e. $[\lambda]$ on Δ. Then $|f_n g| \le g$ and $f_n g \to c_\delta g$ a.e. $[\lambda]$ on Δ. Applying the dominated convergence theorem for vector measures (cf. [3, p. 328]), we obtain that $\lim_{n \to \infty} T(f_n)T(g)x = E(\delta)T(g)x$ and $\lim_{n \to \infty} T(f_n g)x = T(c_\delta g)x$. However, by Lemma 2.5, $T(f_n)T(g)x = T(f_n g)x$. Hence $T(c_\delta g) = E(\delta)T(g)$, and thus, by Lemma 2.7, $T(c_\delta g) = T(g)E(\delta)$. Equivalently,

$$\int g(t)c_\delta(t)\, d\, E(t)x = \int g(t)\, d\, E(t)E(\delta)x$$

for all $g \in C(R)$ and $x \in W$ (since g and x were arbitrary). The uniqueness of the Riesz representation implies that $\int_\varepsilon c_\delta(t)\, d\, E(t)x = \int_\varepsilon d\, E(t)E(\delta)x$ for each ε in $\mathscr{B}$ and x in W. Equivalently, $E(\varepsilon \cap \delta)x = E(\varepsilon)E(\delta)x$ for all $x \in W$ and $\varepsilon, \delta \in \mathscr{B}$. Q.E.D.

This completes the proof of Theorem 2.1. If we replace W by its completion with respect to $\| \cdot \|$, the structure of T can be described in previously established terminology. This has however the inconvenience of taking us out from the original space X (in general).

DEFINITION 2.2. The *semi-simplicity space* for T is the completion Y of the normed linear space $(W, \| \cdot \|)$.

By Theorem 1.2, if $S \in B(X)$ commutes with T, then $S | W$ has a unique extension S_Y as a bounded linear operator on Y, and $\| S_Y \| \leq | S |$ (norms in Y, Y^* and $B(Y)$ are denoted by double bars). In the following corollary, we use the the terminology of [2] and [7].

COROLLARY 2.2. *Let X be an arbitrary Banach space (not necessarily reflexive), and let $T \in B(X)$ have real spectrum. Let Y be the semi-simplicity space for T. Then T_Y is of class C. (Thus T_Y^* is spectral of class Y and scalar type, and if Y is weakly complete, T_Y is spectral of scalar type.)*

Proof. By (2.5) and Lemma 2.5, $\| p(T)x \| \leq | p |_\Delta \| x \|$ for all polynomials p and $x \in W$. Hence $\| p(T)_Y \| \leq | p |_\Delta$, i.e., $\| p(T_Y) \| \leq \sup_\Delta | p |$ for all polynomials p. Q.E.D.

Another description of the semi-simplicity manifold of an operator with real spectrum follows from Theorem 2.1. Suppose, without loss of generality, that $\sigma(T) \subset [0, 1]$. Then

COROLLARY 2.3.

$$\| x \| = \sup \left\{ \sum_{j=0}^{n} \binom{n}{j} | x^* T^j (I - T)^{n-j} x |; | x^* | = 1, \quad n = 0, 1, 2, \cdots \right\}.$$

Proof. Denote the right-hand side by $\| x \|'$. Suppose $\| x \| < \infty$. Then, by Theorem 2.1 (Equation (2.1)):

$$\sum_{j=0}^{n} \binom{n}{j} | x^* T^j (I - T)^{n-j} x | = \sum_{j=0}^{n} \binom{n}{j} \left| \int_0^1 s^j (1-s)^{n-j} dx^* E(s) x \right|$$

$$\leq \sum_{j=0}^{n} \int_0^1 \binom{n}{j} s^j (1-s)^{n-j} du(s | x, x^*)$$

$$= \int_0^1 du(s | x, x^*) = \| \mu(\cdot | x, x^*) \| \leq \| x \| | x^* |$$

where u is the variation of the complex measure $\mu = x^* E(\cdot)x$ (cf. (1.1)). Thus $\| x \|' \leq \| x \|$. The reversed inequality follows at once from the Hausdorff moments theorem, as in the proof of Theorem 2.18 in [8].

3. Examples.

3.1. Suppose X is finite dimensional. Let $x_1, \cdots, x_n$ be a basis of X with respect to which T has a Jordan canonical matrix representation; i.e., for suitable integers $0 = n_0 < n_1 < n_2 < \cdots < n_k = n$ and a suitable enumeration $\lambda_1, \cdots, \lambda_k$ of $\sigma(T)$ (with possible repetitions), $Tx_q = \lambda_j x_q + x_{q+1}$ for $n_{j-1} < q < n_j$ and $Tx_{n_j} = \lambda_j x_{n_j}$ $(j = 1, \cdots, k)$. Since $e^{itT} x_{n_j} = \exp(it\lambda_j)x_{n_j}$ and λ_j are real (our standing assumption

about $\sigma(T)$!), it follows trivially that $x_{n_j} \in W(T)$ $(j = 1, \cdots, k)$. For $n_{j-1} < q < n_j$, we have $(T - \lambda_j I)^{n_j - q + 1} x_q = 0$ and $(T - \lambda_j I)^{n_j - q} x_q \neq 0$. Therefore

$$
\begin{aligned}
\left| e^{itT} x_q \right| &= \left| e^{it(T - \lambda_j I)} x_q \right| \\
&= \left| \sum_{m=0}^{n_j - q} \frac{(it)^m}{m!} (T - \lambda_j I)^m x_q \right| \neq O(1)
\end{aligned}
$$

when $|t| \to \infty$, because $n_j - q \geqq 1$. Thus $x_q \notin W(T)$ for $n_{j-1} < q < n_j$.

This shows that $W(T)$ is the linear span of $x_{n_1}, \cdots, x_{n_k}$; its dimension is exactly k, the number of Jordan cells. Of course, our definition of $W(T)$ does not depend on a choice of basis.

3.2. Let $X = L^p[0,1]$ $(1 \leqq p < \infty)$ and

$$
(1) \qquad (T\phi)(x) = x\phi(x) + \int_x^1 \phi(s)\, ds, \qquad \phi \in L^p[0,1], \ x \in [0,1].
$$

Then, for $t \in R$,

$$
(2) \qquad (e^{itT} \phi)(x) = e^{itx} \phi(x) + it \int_x^1 e^{its} \phi(s)\, ds.
$$

The characteristic function c_s of the interval $[0,s]$ $(0 < s \leqq 1)$ is an eigenvector corresponding to the eigenvalue s. Since the functions c_s $(0 \leqq s \leqq 1)$ are trivially in $W(T)$ and are dense in X, $W(T)$ is dense in X. A spectral decomposition of T on the linear span W_0 of c_s $(0 \leqq s \leqq 1)$ is very easily described (cf. last example in [8]). However $W(T)$ is much wider than W_0. Indeed, suppose ϕ is a function of bounded variation in $[0,1]$. Assume, without loss of generality, that ϕ is continuous from the left. An integration by parts shows that

$$
(3) \qquad (e^{itT} \phi)(x) = e^{it} \phi(1) - \int_x^1 e^{its}\, d\phi(s).
$$

It is clear from (3) that $\phi \in W(T)$, and so $W(T) \supset BV[0,1]$ (functions and equivalence classes in $L^p[0,1]$ are confused as usual).

On the other hand, $W(T) \neq L^p[0,1]$. In fact, there are continuous functions which are *not* in $W(T)$. This follows from (2) and the well-known fact that there exist 2π-periodic continuous functions with Fourier coefficients which are *not* $O(n^{-1})$. The spectral decomposition of T on $BV[0,1]$ follows easily from (3). The spectral measure on $BV[0,1]$ is given by

$$
(E(\delta)\phi)(x) = c_\delta(1)\phi(1) - \int_{\delta \cap [x,1]} d\phi \quad (\phi \in BV[0,1];\ x \in [0,1];\ \delta \in \mathscr{B}).
$$

Note that if we consider T as an operator in $BV[0,1]$ (normed as usual by the total variation norm), then T is of class C (i.e., $W(T)$ is the whole space).

If we consider T as an operator on $C[0,1]$, it has no eigenvectors, but still

$W(T)$ is dense in $C[0,1]$ (and properly contained in it), because $W(T)$ contains all the continuous functions of bounded variation on $[0,1]$.

3.3. The theory has an obvious generalization to closed densely defined operators T such that iT generates a strongly continuous group of operators $T(\cdot)$ on the real line. The norm $\|\cdot\|$ and the manifold W are defined as in Definition 1.1, with $T(t)$ replacing e^{itT}. Consider for example the translation group in $L^p(R)$ $(1 \leqq p < \infty)$, i.e.,

$$[T(t)x](s) = x(s-t), \qquad x \in L^p(R).$$

Then

$$\|x\| = \sup\left\{\left|\int f(t)x(s-t)\,dt\right|_p ; f \in L^1(R), |\hat{f}|_\infty \leqq 1\right\}$$

$$= \sup\{|f*x|_p ; f \in L^1, |\hat{f}|_\infty \leqq 1\}.$$

This is just the norm $\|\cdot\|_0$ considered in [11], and W is the space $(L^p)_0$ discussed in this paper in the context of the multipliers problem for Fourier transforms. It is proved there that $W = \{0\}$ for $p < 2$, is the whole space for $p = 2$, and is dense in L^p for $p > 2$ (W contains all the Fourier Transforms of elements of L^q, $p^{-1} + q^{-1} = 1$). The connection of our work with [1] was noticed by the referee.

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