GRAPH TOPOLOGY FOR FUNCTION SPACES[1]

BY

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1. **Introduction.** Most of the work in function space topologies concerns continuous functions. In this connection see a remark by Kelley [3, p. 217]. As soon as we begin to consider function spaces of noncontinuous functions we come face to face with some extremely difficult problems. So in order to make a beginning, it is advisable to consider first a subfamily of noncontinuous functions which, in a certain sense, can be approximated by continuous functions. One such subfamily consists of almost continuous functions which were introduced by Stallings [6]. An almost continuous function is one whose graph can be approximated by graphs of continuous functions (see 2.3). The need to introduce a suitable topology for the function space of almost continuous functions arose when the author was investigating the essential fixed points of such functions in his doctoral thesis [4]. The introduction of a new function space topology, called "the graph topology", enabled him to tackle almost continuous functions.

Let F denote an arbitrary subfamily of functions on a topological space X to a topological space Y and let F be given some topology. Most problems concerning F center round the following question, "what conditions on X and Y are sufficient to ensure that F has a desired property?" In this paper a few problems of the above nature are discussed.

This paper has a nonempty intersection with the author's doctoral thesis written under the supervision of Professor J. G. Hocking of Michigan State University. The author is grateful to his former colleague Professor D. E. Sanderson for valuable suggestions and comments. The referee suggested several improvements, supplied Example 5.1 and the references [1] and [5].

2. **Graph topology.**

2.1. Let X and Y be topological spaces and let F denote the set of all functions on X to Y. Let C denote the subset of F consisting of all continuous functions.

2.2. For $f \in F$, the graph of f, denoted by $G(f)$, is the set

$$\{(x, f(x)) \mid x \in X\} \subset X \times Y.$$

Let $X \times Y$ be assigned the usual product topology.

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2.3. A function $f \in F$ is called *almost continuous* iff for each open set U in $X \times Y$ containing $G(f)$, there exists a $g \in C$ such that $G(g) \subset U$ (Stallings [6]).

2.4. Let A be the subset of F consisting of all almost continuous functions. Whereas every continuous function is almost continuous, there exist almost continuous functions which are not continuous. If $X = Y =$ the set of all real numbers with the usual topology, then the function $f \in F$ defined by

$$f(x) = \sin\frac{1}{x} \quad \text{for } x \neq 0$$

$$= 0 \qquad \text{for } x = 0,$$

is almost continuous but not continuous.

2.5. Corresponding to each open set U in $X \times Y$, let $F_U = \{f \in F \mid G(f) \subset U\}$. The topology induced on F by a basis consisting of sets of the form $\{F_U\}$ for each open set U in $X \times Y$ is called the *graph topology* Γ for F. Let $A_U = F_U \cap A$ and $C_U = F_U \cap C$.

Even a casual glance at Chapter 7 on Function Spaces in Kelley [3] shows that the properties of F under the usual function space topologies, lean rather heavily on the topology of Y and that the topology of X does not enter into the definitions or the theorems. In contrast to this it will be seen that, in general, the properties of F under Γ depend on the topologies of both X and Y, and so in a certain sense Γ is a "natural" topology for F.

When Y is a uniform space the uniform convergence topology for F is such that the family C of functions continuous relative to a topology for X is closed in F. Compare this result with the following theorem, the proof of which is obvious.

2.6. THEOREM. *The family of almost continuous functions on X to Y is closed in (F, Γ); in fact, it is the closure of the set C of continuous functions.*

3. Separation properties.

3.1. EXAMPLE. Let X consist of two points a, b with the topology consisting of $\varnothing$, $\{a\}$, $\{a, b\}$. Let Y be the discrete topological space formed by two points p, q. Let $f, g \in F$ such that $f(a) = p$, $f(b) = g(a) = g(b) = q$. Any open set in $X \times Y$ containing $G(f)$ also contains $G(g)$ and so (F, Γ) is not even T_1 although Y is a metrizable space. This is in contrast to the usual function space topologies (Kelley [3, pp. 217–237]), whose separation properties T_1, T_2 are inherited from the corresponding ones for Y irrespective of the topology for X. Incidentally f is almost continuous, but is not even a connected function.

3.2. THEOREM. *If Y contains at least two points, then the following are equivalent:* (i) *X and Y are T_1-spaces,* (ii) (F, Γ) *is T_1,* (iii) (A, Γ) *is T_1.*

Proof. First let X and Y be T_1-spaces and let $f, g \in F$ and $f \neq g$. Then there exists an $a \in X$ such that $f(a) \neq g(a)$. Since Y is T_1 there exists an open set U in Y such that $f(a) \in U$ but $g(a) \notin U$. Also since X is T_1, $(X - \{a\}) \times Y$ is open in $X \times Y$. Consequently, $X \times U \cup (X - \{a\}) \times Y$ contains $G(f)$ but not $G(g)$. So (F, Γ) is T_1.

Next let (F, Γ) be T_1. If Y is not T_1, there exist distinct points $p, q \in Y$ such that every open set which contains p also contains q. Let $f, g \in F$ be such that $f(x) = p, g(x) = q$ for all $x \in X$. Then every open set in $X \times Y$ which contains $G(f)$ also contains $G(g)$ and so F is not T_1, a contradiction. Thus Y is T_1. To show that X is T_1, assume this is false. Then there exist $a, b \in X$ such that each open set in X containing a also contains b. Let p, q be two distinct points of Y. Define $f, g \in F$ such that $f(b) = q$, $g(b) = p$, $f(x) = g(x) = p$ for all $x \in X$, $x \neq b$. Then every open set in $X \times Y$ which contains $G(f)$ also contains $G(g)$ and so (F, Γ) is not T_1, a contradiction.

This shows the equivalence of (i) and (ii). Their equivalence with (iii) follows by noting that f's and g's constructed are in fact almost continuous.

We can similarly prove the following theorem.

3.3. THEOREM. *If Y has at least two points, then (F, Γ) is T_2 if and only if X is T_1 and Y is T_2.*

4. **Graph topology and other function space topologies.** In this section we compare the graph topology with some of the other function space topologies such as the pointwise convergence (p.c.) topology, the compact-open or k-topology, the uniform convergence (u.c.) topology. Recall that in Example 3.1 (F, Γ) is not T_1 whereas F is discrete in all the other topologies mentioned above. This together with Theorem 3.2 shows that in order to get meaningful results it would be desirable to have X a T_1-space.

4.1. THEOREM. *If X is a T_1-space then the p.c. topology is contained in the graph topology.*

Proof. For $a \in X$ and U an open subset of Y, let $W(a, U) = \{f \in F \mid f(a) \in U\}$. The sets $\{W(a, U)\}$ for each $a \in X$ and U an open subset of Y form a subbasis for the p.c. topology of F. Since X is T_1, the set $V = (X \times U) \cup (X - \{a\}) \times Y$ is open in $X \times Y$ and clearly $F_V = W(a, U)$. Thus $W(a, U)$ is open in Γ.

4.2. THEOREM. *If X is a T_2-space then the graph topology contains the k-topology. If further X is compact, the graph topology coincides with the k-topology.*

Proof. A subbasis for the k-topology for F consists of sets of the form $W(K, U) = \{f \in F \mid f(K) \subset U\}$ for each compact set K in X and each open set U in Y. Since X is T_2 the set $V = (X \times U) \cup (X - K) \times Y$ is open in $X \times Y$ and clearly $F_V = W(K, U)$. This proves the first statement and the proof of the second statement is trivial.

4.3. EXAMPLE. We now give an example to show that, in general, the k-topology is properly contained in the graph topology. Let $X = Y =$ the set of all real numbers with the usual topology. Let $f \in F$ be such that $f(x) = x$ for all $x \in X$. Let U be the open set in $X \times Y$, which is the union of all open discs with centers at the points of $G(f)$ and radius $\varepsilon > 0$. Then $G(f) \subset U$. If $K_1, K_2, \cdots, K_n$ are compact subsets of X and $U_1, U_2, \cdots, U_n$ are open subsets of Y such that $f(K_i) \subset U_i$, $i = 1, 2, \cdots, n$, then $f \in \bigcap_{i=1}^n W(K_i, U_i)$. Let $p \in X$, be such that $p \notin \bigcup_{i=1}^n K_i$. Define $g \in F$ such that $g(x) = f(x)$ for all $x \in X$, $x \neq p$ and $g(p) = p + 2\varepsilon$. Then $g \in \bigcap_{i=1}^n W(K_i, U_i)$ but $G(g) \not\subset U$. Thus F_U is not open in the k-topology. The above construction can easily be altered to make g continuous, showing that the k-topology $\neq$ the graph topology, even on the subspace C.

The first part of the following theorem is obvious and the proof of the second part is similar to that of Example 4.3.

4.4. THEOREM. *The p.c. topology $\subset$ the k-topology $\subset$ the graph topology when X is T_2. Moreover, if X is not compact and the topology of Y is not trivial (indiscrete), the last inclusion is not reversible.*

4.5. THEOREM. *Let X and Y be uniform spaces with uniformities $\mathscr{U}$ and $\mathscr{V}$ respectively. Let $F' \subset F$ consist of functions which are uniformly continuous relative to $\mathscr{U}$ and $\mathscr{V}$. Then the u. c. topology for F' is contained in the graph topology.*

Proof. A basis for the u.c. uniformity for F' consists of sets of the form $W(V) = \{(f, g) \in F' \times F' \mid (f(x), g(x)) \in V \in \mathscr{V} \text{ for all } x \in X\}$. Consider $W(V)[f]$ where $f \in F'$, $V \in \mathscr{V}$. Let $V_2 \in \mathscr{V}$ be such that $V_2 \circ V_2 \subset V$. Since $f \in F'$, corresponding to V_2 there exists a $V_1 \in \mathscr{U}$ such that for all $p, q \in X$, $(p, q) \in V_1$ implies $(f(p), f(q)) \in V_2$. We may, without any loss of generality, suppose that V_1, V_2 are symmetric. Then $U = \bigcup_{x \in X} \{V_1[x] \times V_2[f(x)]\}$ is an open set in $X \times Y$ containing $G(f)$. Let $g \in F'_U = F' \cap F_U$ i.e. $G(g) \subset U$. For an arbitrary $p \in X$, there exists a $q \in X$ such that $(p, g(p)) \in V_1[q] \times V_2[f(q)]$. But then $f(q) \in V_2[f(p)]$ and so $g(p) \in V_2 \circ V_2[f(p)] \subset V[f(p)]$. This shows that $F'_U \subset W(V)[f]$ i.e. $W(V)[f]$ is open in (F', Γ).

4.6. THEOREM. *Let X and Y be uniform spaces with uniformities $\mathscr{U}$ and $\mathscr{V}$ respectively. If X is compact then the u.c. topology is equivalent to the graph topology for C.*

Proof. Each $f \in C$ is uniformly continuous since X is compact and so in view of Theorem 4.6, it is sufficient to prove that the graph topology is contained in the u.c. topology for C. Let $f \in C$ and let U be an open set in $X \times Y$ containing

$G(f)$. Since $f \in C$, $G(f)$ is homeomorphic to X and so $G(f)$ is compact. Hence there exists a $V_1 \in \mathscr{U}$ and a $V_2 \in \mathscr{V}$ such that $\bigcup_{x \in X} \{V_1[x] \times V_2[f(x)]\} \subset U$ (see Kelley [3, p. 199]). Now if $g \in C$ and $g \in W(V_2)\,[f]$ then $g(x) \in V_2\,[f(x)]$ for all $x \in X$ and so $G(g) \subset U$. This shows that C is open in the u.c. topology for C.

Finally we consider metric spaces. Let X and Y be metric spaces with bounded metrics d, d' respectively and let $X \times Y$ be assigned the product metric D defined by $D((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d'(y_1, y_2)$ where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Let H be the Hausdorff metric (see Kelley [3, p. 131]) on the hyperspace of all nonempty closed subsets of $X \times Y$. We introduce a metric ρ in F as follows, $\rho(f, g) = H\,(\overline{G(f)}, \overline{G(g)})$ where $f, g \in F$ and the bar denotes the closure operator. Clearly ρ is a pseudometric for F but we can make F a metric space by agreeing that $f \sim g$ iff $\overline{G(f)} = \overline{G(g)}$ for $f, g \in F$ and then passing on to the quotient space with respect to this equivalence relation.

Under the metric ρ the elements of C still retain their identity since for each $f \in C$, $G(f)$ is closed in $X \times Y$. In fact, we show below that ρ is equivalent to the "supremum metric" ρ_1 which is usually introduced C where,

$$\rho_1(f, g) = \sup_{x \in X} d'(f(x), g(x)), \quad f, g \in C.$$

4.7. Theorem. *If X is compact then the metrics ρ and ρ_1 are equivalent for C.*

Proof. Let $\varepsilon > 0$ and let for $f \in C$, $U(f, \varepsilon) = \{g \in C \mid \rho(f, g) < \varepsilon\}$ and $U_1(f, \varepsilon) = \{g \in C \mid \rho_1(f, g) < \varepsilon\}$. Clearly $U_1(f, \varepsilon) \subset U(f, \varepsilon)$. We now show that there exists a $\delta > 0$ such that $U(f, \delta) \subset U_1(f, \varepsilon)$. Since $f \in C$ and X is compact, there exists a $\delta > 0$ $(\delta < \varepsilon/3)$ such that for all $x, y \in X$, $d(x, y) < \delta$ implies $d'(f(x), f(y)) < \varepsilon/3$. If $g \in U(f, \delta)$ and $a \in X$ then there exists a $b \in X$ such that $d(a, b) + d'(g(a), f(b)) < \delta$. Thus $d'(f(a), g(a)) \leqq d'(f(a), f(b)) + d'(f(b), g(a)) < \varepsilon/3 + \delta$. This shows that $U(f, \delta) \subset U_1(f, \varepsilon)$.

Similarly we can prove the following theorem.

4.8. Theorem. *Under the hypothesis of Theorem 4.7, the graph topology is equivalent to the ρ-metric topology for C.*

Analogs of Theorems 4.7 and 4.8 for A would be false even when X, Y are closed linear intervals, no two of ρ, ρ_1, Γ need agree on A. This can be shown by using examples similar to 2.4.

5. **Connectedness.** An interesting question is "if X and Y are connected is (A, Γ) connected?" For noncompact spaces the following counter example was supplied by the referee.

5.1. Example. De Groot [1] has shown that there exist nondegenerate connected subsets S, T of the plane such that evety continuous image of either in the other is a single point. Define a plane set X as follows. Let P_n denote the point (n, o), where $n \in N$, the set of all integers. For each $n \in N$, join P_{2n} to P_{2n+1} by a copy S_n of S (lying, except for P_{2n} and P_{2n+1}, in the slab $\{(x, y) \mid 2n < x < 2_{n+1}\}$), and P_{2n-1} to P_{2n} by a copy T_n of T (lying likewise between these two points). Let $X = \bigcup \{S_n \cup T_n \mid n \in N\}$. Let E = set of all almost continuous maps f of X in X such that $f(P_n) = P_n$ for all $n \in N$. It is easy see that E is closed in (A, Γ). Also it is open. For let V_n denote the open neighborhood $\{z \mid z \in X, \rho(P_n, z) < 1\}$ of P_n in X, put $U = \{(X - N) \times X\} \cup \{V_n \times V_n \mid n \in N\}$, an open subset of $X \times X$, and $W = \{g \mid g \in F, G(g) \subset U\}$; we show that $W \cap A = E$. Clearly $E \subset W \cap A$, so it suffices (since W is open and E is closed and $A = \bar{C}$) to prove $W \cap C \subset E$. Suppose $f \in W \cap C$; then $f(P_n) \in V_n$, for all $n \in N$. Fix n for the present, and suppose first that n is even, $= 2m$. If $f(P_n)$ is to the left of P_n, it is in T_m. There is a retraction r of X onto T_m (mapping all points to the right of P_n onto P_n, and all points to the left of P_{n-1} onto P_{n-1}); $r(f(S_m))$ is a continuous image of S_m in T_m, hence a single point. But it contains both $r(f(P_n))$ and $r(f(P_{n+1})) = P_n$; so $f(P_n) = P_n$. Similar arguments apply if $f(P_n)$ is to the right of P_n, or if n is odd. Thus $f(P_n) = P_n$ for all $n \in N$, and $f \in E$, as required. Clearly $\varnothing \neq E \neq A$, so A is not connected.

The question is still open for compact spaces.(2)

Two remarks are in order here. From Theorem 2.6 it follows that if (C, Γ) is connected then so is (A, Γ). It is well known (see Hocking and Young [2, p. 155], for example) that if Y is a contractible T_2 space then C is arcwise connected. So if X is compact T_2 and Y a contractible T_2 space then (A, Γ) is connected. (A, Γ) is also connected if X is contractible and Y is arcwise connected, since every continuous $f: X \to Y$ is homotopic to a constant map.

References

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(2) Professor D. E. Sanderson has communicated a solution of this problem.