SOME THEOREMS ON DIOPHANTINE APPROXIMATION(1)

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Introduction. The study of the values at rational points of transcendental functions defined by linear differential equations with coefficients in $Q[z]$ (2) can be traced back to Hurwitz [1] who showed that if

$$y(z) = 1 + \frac{1}{b} \cdot \frac{z}{1!} + \frac{1}{b(b+a)} \frac{z^2}{2!} + \cdots$$

where a is a positive integer, b is an integer, and b/a is not a negative integer, then for all nonzero z in $Q((-1)^{1/2})$ the number $y'(z)/y(z)$ is not in $Q((-1)^{1/2})$. Ratner [2] proved further results. Then Hurwitz [3] generalized his previous results to show that if

$$y(z) = 1 + \frac{f(0)}{g(0)} \frac{z}{1!} + \frac{f(0) \cdot f(1)}{g(0) \cdot g(1)} \frac{z^2}{2!} + \cdots$$

where $f(z)$ and $g(z)$ are in $Q[z]$, neither $f(z)$ nor $g(z)$ has a nonnegative integral zero, and degree $(f(z)) <$ degree $(g(z)) = r$, then for all nonzero z in the imaginary quadratic field $Q((-n)^{1/2})$ two of the numbers $y(z), y^{(1)}(z), \cdots, y^{(r)}(z)$ have a ratio which is not in $Q((-n)^{1/2})$. Perron [4], Popken [5], C. L. Siegel [6], and K. Mahler [7] have obtained important results in this area.

In this paper we shall use a generalization of the method which was developed by Mahler [7] to study the approximation of the logarithms of algebraic numbers by rational and algebraic numbers.

DEFINITION. Let K denote the field $Q((-n)^{1/2})$ for some nonnegative integer n.

DEFINITION. For any monic $\theta(z)$ in $K[z]$ of degree $k > 0$ and such that $\theta(z)$ has no positive integral zeros we define the entire function

$$f(z) = \sum_{d=0}^{\infty} \frac{z^d}{\prod_{e=1}^{d} \theta(e)}.$$

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(1) This paper formed part of the author's doctoral thesis written under the direction of Professor R. S. Lehman at the University of California, Berkeley.

(2) We denote the rational numbers by Q and the integers by Z.

(Note that $f(z)$ satisfies a linear differential equation of order k with coefficients in $Q[z]$.)

Let $\omega_0, \cdots, \omega_{t-1}$ be t distinct nonconjugate nonzero elements of K. We assume that the ω_j for $0 \leqq j \leqq t_1 - 1$ are in K but not in Q and that the ω_j for $t_1 \leqq j \leqq t_1 + t_2 - 1 = t - 1$ are in Q. Set $J = \{a + b(-n)^{1/2}; a, b \in Z\}$, and define $t_3 = 2t_1 + t_2$.

THEOREM I. *If $\theta(z)$ has only rational roots, then for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that for all positive integers q*

$$\max \left| D^r f(\omega_j) - \frac{p_{rj}}{q} \right| > c(\varepsilon) q^{-(1+1/kt_3+\varepsilon)} ,$$

where the maximum is taken over $0 \leqq r \leqq k-1$, $0 \leqq j \leqq t-1$, and the p_{rj} are in J.

THEOREM II. *If $\theta(z)$ has only rational roots, then for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*

$$\left| \sum_{r=0}^{k-1} \sum_{j=0}^{t-1} A_{rj} D^r f(\omega_j) + A \right| \geqq c(\varepsilon) H^{-(kt_3+\varepsilon)}$$

for all nonzero $(A, A_{00}, \cdots, A_{k-1,t-1})$ in J^{kt+1} with $|A| \leqq H$ and $|A_{rj}| \leqq H$ for $0 \leqq r \leqq k-1$, $0 \leqq j \leqq t-1$.

Let $\theta_1(z)$ be a polynomial of degree $t > 0$ with integral coefficients and distinct roots. We assume also that $\theta_1(0) \neq 0$. The operator E is defined on the elements of a sequence S_d by $ES_d = S_{d+1}$. Let M be a nonsingular kt by kt matrix with entries in Q. For $0 \leqq i \leqq kt-1$ define S_d^i by $(\theta_1(E))^k \cdot S_d^i = 0$ and $S_0^i, \cdots, S_{kt-1}^i)$ is the ith row of M. Finally define μ_i for $0 \leqq i \leqq kt-1$ by

$$\mu_i = \sum_{d=0}^{\infty} \frac{S_d^i}{\prod_{e=1}^{d} \theta(e)} .$$

THEOREM III. *If $\theta(z)$ has only rational roots, then:*

(i) *for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*

$$\max_{0 \leqq i \leqq kt-1} \left| \mu_i - \frac{p_i}{q} \right| > c(\varepsilon) q^{-(1+1/kt+\varepsilon)}$$

for all positive integers q where the p_i are integers.

(ii) *for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*

$$\left| \sum_{i=0}^{kt-1} A_i \mu_i + A \right| > c(\varepsilon) H^{-(kt+\varepsilon)}$$

where $(A, A_0, \cdots, A_{kt-1})$ *is a nonzero element of* Z^{kt+1}, $|A| \leqq H$, *and* $|A_i| \leqq H$ *for* $0 \leqq i \leqq kt-1$.

We remark that it can be shown that Theorem III implies Theorems I and II.

EXAMPLE. Let $T_d(x)$ denote the dth Chebyshev polynomial. Now

$$(E^2 - 2xE + 1)\,T_d(x) = 0,$$

$T_0(x) = 1$, and $T_1(x) = x$. Assume $x \neq \pm 1$, then $z^2 - 2xz + 1$ has distinct roots. Thus if $x \neq \pm 1$ we see by Theorem III that for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$\left| \sum_{d=0}^{\infty} \frac{T_d(x)}{\prod_{e=1}^{d} \theta(e)} - \frac{p}{q} \right| > c(\varepsilon) q^{-(2k+1+\varepsilon)} .$$

Let the r_j for $0 \leqq j \leqq t-1$ denote t distinct nonzero rational numbers. We choose (i_1, j_1) belonging to $Z \times Z$ such that $0 \leqq i_1 \leqq t-1$ and $0 \leqq j_1 \leqq t-1$.

THEOREM IV. *If* $\theta(0) = 0$ *and* $k > 1$, *then for every* $\varepsilon > 0$ *there exists* $c(\varepsilon) > 0$ *such that*

$$\max_{ij} \left| \frac{D^i f(r_j)}{D^{i_1} f(r_{j_1})} - \frac{p_{ij}}{q} \right| > c(\varepsilon) q^{-(1+1/(kt-1)+\varepsilon)}$$

where q *is a positive integer and the* p_{ij} *for* $0 \leqq i \leqq k-1$ *and* $0 \leqq j \leqq t-1$ *are integers.*

THEOREM V. *If* $\theta(0) = 0$ *and* $k > 1$, *then for every* $\varepsilon > 0$ *there exists* $c(\varepsilon) > 0$ *such that*

$$\left| \sum_{i=0}^{k-1} \sum_{j=0}^{t-1} A_{ij} D^i f(r_j) \right| > c(\varepsilon) H^{-(kt-1+\varepsilon)}$$

where $(A_{00}, \cdots, A_{k-1,t-1})$ *is a nonzero element of* Z^{kt} *and* $|A_{ij}| \leqq H$ *for* $0 \leqq i \leqq k-1$, $0 \leqq j \leqq t-1$.

We define the S_d^i and the μ_i as before Theorem III. Choose μ_{i_1} arbitrarily from the μ_i for $0 \leqq i \leqq kt-1$.

THEOREM VI. *If* $\theta(0) = 0$ *and* $k > 1$, *then*:

(i) *for every* $\varepsilon > 0$ *there exists* $c(\varepsilon) > 0$ *such that*

$$\max_i \left| \frac{\mu_i}{\mu_{i_1}} - \frac{p_i}{q} \right| \geqq c(\varepsilon) q^{-(1+1/(kt-1)+\varepsilon)}$$

where q *is a nonzero integer and the* p_i *for* $0 \leqq i \leqq kt$ *are integers.*

(ii) *for every* $\varepsilon > 0$ *there exists* $c(\varepsilon) > 0$ *such that*

$$\left| \sum_{i=0}^{kt-1} A_i \mu_i \right| \geqq c(\varepsilon) H^{-(kt-1+\varepsilon)}$$

for all nonzero $(A_0, \cdots, A_{kt-1})$ *in* Z^{kt} *with* $|A_i| \leqq H$ *for* $0 \leqq i \leqq kt-1$.

I. DEFINITION. Let K denote throughout §I the field $Q((-n)^{1/2})$ for some nonnegative integer n.

DEFINITION. For any monic $\theta(z)$ in $K[z]$ of degree $k>0$ and such that $\theta(z)$ has no positive integral zeros we define the entire function

$$f(z) = \sum_{d=0}^{\infty} \frac{z^d}{\prod_{e=1}^{d} \theta(e)}. \tag{1}$$

DEFINITION. Given $\phi(z)$ a monic polynomial with rational coefficients of degree $l > 0$ we define for each positive integer N,

$$P_N = \frac{1}{2\pi i} \int_C \frac{f(z) z^h dz}{(\phi(z))^m}, \tag{2}$$

where $N = ml - h$, $m \geqq 1$, $0 \leqq h < l$, and C winds about the zeros of $\phi(z)$ once in the positive direction.

A few words before proceeding. We shall begin by establishing five lemmas. Lemma V contains the mathematical machinery which makes all of the proofs go through. It asserts the existence of certain linear forms $g_N(\delta)$ $(N = 1, 2, \cdots)$ over $Z[(-n)^{1/2}]$ having some very special properties. Each form $g_N(\delta)$ is defined to equal an appropriate positive integer $v(m)$ times $P_N(\delta)$ where $P_N(\delta)$ is the integral P_N defined above with some additional assumptions on $\phi(z)$. Lemma I, which proves the existence of many linear relations among the P_N, is needed to help show parts (i) and (ii) of Lemma V. Lemmas II through IV involve evaluating each P_N by the residue theorem as a linear form over $Q((-n)^{1/2})$, estimating the absolute values of each P_N and its coefficients, and determining $v(m)$ so that $g_N(\delta) = v(m) P_N(\delta)$ has coefficients in $Z[(-n)^{1/2}]$. Thus we are able to establish parts (iii) through (v) of Lemma V.

It is obvious that all linear forms with coefficients in $Z[(-n)^{1/2}]$ of *fixed numbers in* $Q((-n)^{1/2})$ must either equal zero or have their absolute values uniformly bounded away from zero. The $D^i f(w_j)$ of Theorems I, II, IV, and V are not all zero. Thus by (i) of Lemma V one of the $g_{N+\alpha}(\delta)$ is not zero $(\alpha = 0, 1, \cdots, (k+2)l)$. But by (v) of Lemma V $|g_{N+\alpha}(\delta)| \to 0$ as $N \to \infty$. Therefore at least one of the $D^r f(w_j)$ is *not* in $Q((-n)^{1/2})$. Theorems I, II, IV, and V are proven by a more sophisticated use of Lemma V. They may be viewed as merely making quantitative the result shown above. (The proofs of Theorems III and VI depend upon the *proof* of Lemma V, as well as its statement.)

LEMMA 1. *There exist α_i depending on N in K such that*

$$P_N = \sum_{i=1}^{(k+2)l} \alpha_i P_{N+i}.$$

Proof. We shall show first that $f(z)$ satisfies a linear differential equation. From (1) it follows immediately that where

$$Df(z) = \frac{df(z)}{dz}$$

we have

$$[\theta(zD) - z]f(z) = \theta(0). \tag{3}$$

Then we may simplify (3) and write

$$\left(\sum_{j=0}^{k} B_j z^j D^j - z\right) f(z) = \theta(0), \tag{4}$$

where the B_j are in K and B_k is nonzero. Multiplying (4) through by z^{l-1} yields

$$\left(\sum_{j=0}^{k} B_j z^{j+l-1} D^j - z^l\right) f(z) = \theta(0) z^{l-1}.$$

This can be put in the form,

$$\phi(z)f(z) = \left[\sum_{j=0}^{k} B_j z^{j+l-1} D^j + (\phi(z) - z^l)\right] f(z) - \theta(0) z^{l-1}. \tag{5}$$

Since $\phi(z)$ is monic of degree l it follows that $\gamma(z) = \phi(z) - z^l$ belongs to $K[z]$ and has degree less than l. Then we have

$$f(z) = (\phi(z))^{-1}\left(\left[\sum_{j=0}^{k} B_j z^{j+l-1} D^j + z^h \gamma(z)\right] f(z) - \theta(0) z^{l-1}\right). \tag{6}$$

Now substituting (6) for $f(z)$ in (2) we obtain

$$P_N = \frac{1}{2\pi i} \int_C \frac{\left(\left[\sum_{j=0}^{k} B_j z^{j+l+h-1} D^j + z^h \gamma(z)\right] f(z) - \theta(0) z^{h+l-1}\right) dz}{(\phi(z))^{m+1}} \tag{7}$$

DEFINITION. By the order (henceforth denoted as ord) of a rational function we mean the order of the zero at infinity. Ord of the zero function is $+\infty$.

If $\theta(0)$ is nonzero then, recalling that $m \geqq 1$, we see that

$$\operatorname{ord}\left[\theta(0) \cdot z^{h+l-1} (\phi(z))^{-(m+1)}\right] \geqq 2.$$

Hence we conclude that

$$P_n = \sum_{j=0}^{k} \left(B_j \frac{1}{2\pi i} \int_C \frac{z^{j+l+h-1} D^j f(z) dz}{(\phi(z))^{m+1}} \right) + \frac{1}{2\pi i} \int_C \frac{z^h \gamma(z) f(z) dz}{(\phi(z))^{m+1}}. \tag{8}$$

The proof will now consist of showing that both

$$\frac{1}{2\pi i} \int_C z^h \phi(z)^{-(m+1)} f(z) dz$$

and

$$\frac{1}{2\pi i} \int_C z^{j+l+h-1} (\phi(z))^{-(m+1)} D^j f(z) dz \qquad (0 \leqq j \leqq k)$$

can be expressed as linear combinations of the P_{ml-h+i} for $i = 1, 2, \cdots (k+2)l$ with coefficients in K. We have in the latter case

$$\frac{1}{2\pi i} \int_C \frac{z^{j+l+h-1} D^j f(z) dz}{(\phi(z))^{m+1}} = \frac{(-1)^j}{2\pi i} \int_C f(z) D^j \left(\frac{z^{j+l+h-1}}{(\phi(z))^{m+1}} \right) dz, \tag{9}$$

using integration by parts j times. Next we observe that

$$\text{ord } [z^{j+l+h-1} (\phi(z))^{-(m+1)}] = N + 1 - j,$$

hence

$$\text{ord } [D^j (z^{j+l+h-1} (\phi(z))^{-(m+1)})] \geqq N + 1,$$

and

$$\text{ord } [z^h \gamma(z) (\phi(z))^{-(m+1)}] \geqq N + 1.$$

Now we shall show that

$$\frac{1}{2\pi i} \int_C z^h \gamma(z) (\phi(z))^{-(m+1)} f(z) dz$$

and

$$\frac{(-1)^j}{2\pi i} \int_C f(z) D^j [z^{j+l+h-1} (\phi(z))^{-(m+1)}] dz$$

can each be written in the form

$$\frac{1}{2\pi i} \int_C g(z) (\phi(z))^{-(m+1+k)} f(z) dz$$

where $g(z)$ belongs to $K[z]$ and is of degree less than or equal to $(m+1+k)l$

$-(N+1)$. Recalling that $j \leqq k$, the representation of each rational function in the integrands as $g(z) \cdot (\phi(z))^{-(m+1+k)}$ for some $g(z)$ in $K[z]$ is obvious. Setting

$$N+1 \leqq \operatorname{ord}[z^h \gamma(z)(\phi(z))^{-(m+1)}] = \operatorname{ord}[g(z) \cdot (\phi(z))^{-(m+1+k)}]$$

we obtain where $d = \text{degree } (g(z)) \equiv -\operatorname{ord}(g(z))$

$$d \leqq (m+1+k)l - (N+1). \tag{10}$$

Also, setting

$$N+1 \leqq \operatorname{ord}[D^j z^{j+l+h-1}(\phi(z))^{-(m+1)}] = \operatorname{ord}[g(z)(\phi(z))^{-(m+1+k)}]$$

we again obtain (10).

We can write

$$g(z) = \sum_{j=0}^{[d/l]} g_j(z)(\phi(z))^j$$

where the $g_j(z)$ are in $K[z]$ and degree $g_j(z) \leqq l-1$. It follows then that

$$\frac{1}{2\pi i}\int_C \frac{g(z)f(z)\,dz}{(\phi(z))^{m+1+k}} = \sum_{j=0}^{[d/l]} \frac{1}{2\pi i}\int_C \frac{g_j(z)f(z)\,dz}{(\phi(z))^{m+1+k-j}}. \tag{11}$$

Now each

$$\frac{1}{2\pi i}\int_C g_j(z)(\phi(z))^{-(m+1+k-j)} f(z)dz$$

is certainly a linear combination of the

$$\{P_\gamma;\, l(m+1+k-j-1)+1 \leqq \gamma \leqq l(m+1+k-j\}$$

with coefficients in K. Hence

$$\frac{1}{2\pi i}\int_C g(z)(\phi(z))^{-(m+1+k)} f(z)dz$$

may be written as a linear combination of the P_γ for

$$\gamma \leqq l(m+1+k) = N + l(k+1) + h < N + (k+2)l.$$

This is the proper upper bound. Suppose that γ_0 is the least integer such that P_{γ_0} appears with a nonzero coefficient in the above representation of P_N. Then

$$\operatorname{ord}\ (g(z)((z))^{-(m+1+k)}) = \gamma_0.$$

But we know that

$$\operatorname{ord}\ (g(z)(\phi(z))^{-(m+1+k)}) \geqq N_0 + 1,$$

thus $\gamma_0 \geqq N+1$ and Lemma 1 is proven.

DEFINITION. Let $\phi(z) = z^{\delta}(\psi(z))^k$ where δ is 0 or 1 and $\delta = 0$ if $\phi(z)$ has an irrational root.

We assume $\psi(z)$ has no repeated roots and $\psi(0) \neq 0$. Let $\omega_0, \cdots, \omega_{t-1}$ be the roots of $\psi(z)$. (Then $l = kt + \delta$.) We define the set

$$S = \{D^r f(\omega_j); 0 \leqq r \leqq k-1, 0 \leqq j \leqq t-1\} \cup \{1\}.$$

Let $P_N(\delta) \equiv P_N$ to reflect the dependence on δ.

LEMMA 2. (i) *Each $P_N(\delta)$ is a linear combination of the elements of S over the field $K(\omega_0, \cdots, \omega_{t-1})$.* (ii) *In one such representation of $P_N(\delta)$ as a linear combination of the elements of S the coefficients have absolute values less than K_1^m for a constant K_1 independent of m, $(N = ml - h)$.*

Proof. (i) From (2) we have

$$P_N(\delta) = \frac{1}{2\pi i} \int_C z^h(\phi(z))^{-m} f(z) dz.$$

By the residue theorem applied to $P_N(\delta)$, combined with (1) the power series expansion of $f(z)$ about the origin and (4) the differential equation, which may be used to express all derivatives of $f(z)$ as a linear combination of the elements of $\{1, f(z), \cdots, D^{k-1}f(z)\}$ if $z \neq 0$, we see that

$$P_N(\delta) = \varepsilon + \sum_{r=0}^{km-1} \sum_{j=0}^{t-1} \varepsilon_{rj} D^r f(\omega_j) \tag{12}$$

where ε and the ε_{rj} are in $K(\omega_0, \cdots, \omega_{t-1})$.

(ii) Recalling (4) we see that for $r > k$ we have

$$D^{r-k}\left(\sum_{i=0}^{k} B_i z^i D^i - z\right) f(z) = 0. \tag{13}$$

We define $\binom{a}{y}$ where a is a nonnegative integer and y is a real number, to be the coefficient of x^y in the binomial expansion of $(1+x)^a$. Expanding (13) yields

$$\begin{aligned} &\left\{\left(\sum_{i=0}^{k} B_i \left[\sum_{0 \leqq j \leqq i} \binom{r-k}{j} \frac{i!}{(i-j)!} z^{i-j} D^{i+(r-k)-j}\right]\right)\right. \\ &\qquad \left. - zD^{r-k} - (r-k)D^{r-k-1}\right\} f(z) = 0. \end{aligned} \tag{14}$$

Setting $j = p + i - k$, rearranging the double sum, and solving for $D^r f(z)$ we obtain if z is nonzero,

$$\begin{aligned} D^r f(z) = \frac{-1}{B_k} &\left(\left\{\sum_{p=1}^{k}\left[\sum_{i=0}^{k} B_i \binom{r-k}{p+i-k}\binom{i}{k-p}(|p+i-k|!) z^{-p} D^{r-p}\right]\right\}\right. \\ &\left. - z^{-k+1} D^{r-k} - (r-k) z^{-k} D^{r-k-1}\right) f(z). \end{aligned} \tag{15}$$

Using (15) and (4) we may write for $r = 0, 1, \cdots$ that

$$D^r f(z) = A^r_{k-1} D^{k-1} f(z) + \cdots + A^r_{k-q} D^{k-q} f(z) + \cdots + A^r_0 f(z) + A^r_{-1}$$

where the A_{k-q} are in $K(z)$ $(1 \leqq q \leqq k+1)$.

We shall use (15) to show that

$$\left| A^r_{k-q} \right| \leqq (K_2 r)^r \tag{16}$$

where $1 \leqq q \leqq k+1, K_2$ depends on z, and z is assumed to be nonzero. From (15) we see that

$$D^r f(z) = \gamma_{r-1} D^{r-1} f(z) + \cdots + \gamma_{r-k} D^{r-k-1} f(z)$$

if $r > k$, where

$$\begin{aligned} \left| \gamma_{r-p} \right| \leqq \frac{(k+1)^2}{\left| B_k \right|} 2^k r^p \max(1, \left| z \right|^{-k}) + \frac{\max(1, \left| z \right|^{-k})}{\left| B_k \right|} \\ + (r-1)\max(1, \left| z \right|^{-k}) \leqq (K_3 r)^p. \end{aligned} \tag{17}$$

(The inequality

$$\left| \binom{i}{k-p} \right| \leqq 2^i \leqq 2^k$$

is useful in (17).) Choose K_4 such that $\left| A^r_{k-q} \right| \leqq (K_4 r)^r$ for $0 \leqq r \leqq k, 1 \leqq q \leqq k+1$. Then choose $K_2 \geqq k \cdot \max(K_3, K_4)$. Using induction on r we assume that for $r - 1 \geqq k$, $\left| A^{r-1}_{k-q} \right| \leqq \left| K_2(r-1) \right|^{r-1}$ where $1 \leqq q \leqq k+1$. We see

$$\begin{aligned} \left| A^r_{k-q} \right| &\leqq \sum_{p=1}^{k+1} \left| \gamma_{r-p} \right| \left| A^{r-p}_{k-q} \right| \\ &\leqq \sum_{p=1}^{k+1} (K_3 r)^p (K_2(r-p))^{r-p} \\ &\leqq \sum_{p=1}^{k+1} (K_2 r)^r \cdot k^{-p} \leqq (K_2 r)^r. \end{aligned}$$

Hence (16) has been established.

Let Γ_j be the coefficient of $D^r f(\omega_j)$, $0 \leqq r \leqq mk - 1$, $0 \leqq j \leqq t$, in the evaluation of $P_N(\delta)$ by the residue theorem. We shall show that $\left| \Gamma_j \right| \leqq r^{-r} K_6^m$. Choose u_j such that $\left| \omega_j \right| > 2u_j > 0$ and $\left| \omega_j - \omega'_j \right| > 2u_j$ for all $j' \neq j$. Let C_j be a circular path about ω_j in the positive direction with radius u_j. Then

$$\frac{1}{2\pi i} \int_{C_j} \frac{f(z) z^h dz}{(\phi(z))^m} = \sum_{r=0}^{km-1} \frac{D^r f(\omega_j)}{r!} \left(\frac{1}{2\pi i} \int_{C_j} \frac{(z-\omega_j)^r z^h dz}{(\phi(z))^m} \right).$$

Hence,

$$(18)\qquad |\Gamma_j^r| \leqq \frac{1}{r!}\,\frac{u_j^r(|\omega_j| + u_j)^{h+1}}{u_j^{(kt+\delta)m}} \leqq r^{-r}K_6^m,$$

for $0 \leqq j \leqq t-1$. If we choose K_2 such that (16) holds for $z = \omega_0, \cdots, \omega_{t-1}$ simultaneously, then

$$\max|\varepsilon_{r_j}| \leqq \sum_{r=0}^{km-1} (K_2 r)^r \cdot (r^{-r}K_6^m) \leqq K_7^m.$$

We shall conclude the proof of (ii) by demonstrating that $|\varepsilon| \leqq K_8^m$.

Let ε' be the contribution to ε from the evaluation of the residues at $\omega_0, \cdots, \omega_{t-1}$ (due to the inhomogeneous term $\theta(0)$ in the differential equation (4)). The inequality (18) applies to $|\varepsilon'|$ as well as $|\Gamma_j^r|$. Set $\varepsilon'' = \varepsilon - \varepsilon'$. Let $u_t = \min\{|u_j|; 0 \leqq j \leqq k-1\}$ and let C_t be a circular path about the origin in the positive direction and with radius u_t. Then

$$(19)\qquad \varepsilon'' = \frac{1}{2\pi i}\int_{C_t} \frac{f(z)z^h dz}{(\phi(z))^m} = \sum_{d=0}^{m-1} \frac{1}{\prod_{e=1}^{d} \theta(e)} \left(\frac{1}{2\pi i}\int_{C_t} \frac{z^{d+h}dz}{(\phi(z))^m}\right),$$

hence

$$|\varepsilon''| \leqq \sum_{d=0}^{m-1} \frac{u^{d+h+1}}{u^{m(kt+\delta)}} \leqq K_8^m.$$

This proves Lemma 2.

LEMMA 3. *Let $\phi(z)$ be a monic polynomial with rational coefficients of degree $k > 1$ which has only rational roots, none of which is a positive integer. Then there exists $K_9 > 0$ such that the least common denominator of the fractions $(km)!\left|\prod_{s=1}^{r}(\phi(s))^{-1}\right|$ for $r = 1, 2, \cdots m$ is less that K_9^m.*

Proof. Set $\phi(s) = \prod_{i=1}^{k}(s - \eta_i)$. Let b_i be the least positive integer such that $b_i\eta_i$ is integral. We see that

$$\frac{(km)!}{\left|\prod_{s=1}^{r} \phi(s)\right|} = \frac{(km)!}{(m!)^k}\cdot\left(\frac{m!}{r!}\right)^k \cdot \left(\prod_{i=1}^{k} b_i\right)^r \cdot \prod_{i=1}^{k}\left(\frac{r!}{\left|\prod_{s=1}^{r}(b_i s - b_i\eta_i)\right|}\right).$$

The factor $(km)!(m!)^{-k}$ is a multinomial coefficient, hence an integer. If we can show for each $1 \leqq i \leqq k$ that there is a K_{10} such that the least common denominator of the fractions $r!\left|\prod_{s=1}^{r}(b_i s - b_i\eta_i)\right|^{-1}$ for $r = 1, 2, \cdots, m$ is less than K_{10}^m, we will have demonstrated the lemma.

We divide the set of all primes into three classes. Class 1 consists of all primes which divide b_i. Class 2 contains all primes less than $m+1$ which do not divide

b_i. Class 3 contains all primes not in classes 1 and 2. We note that no primes in class 1 divide $\prod_{s=1}^{r}(b_i s - b_i\eta_i)$ for $r = 1, 2, \cdots, m$. For primes p not in class 1 the congruence

$$b_i s - b_i\eta_i \equiv 0 \bmod p^d \tag{20}$$

has a unique solution s_0 where $1 \leqq s_0 \leqq p^d$, for $d = 1, 2, \cdots$. Hence the number of solutions of (20) for $1 \leqq s \leqq r$ is between $[rp^{-d}]$ and $[rp^{-d}] + 1$, where $[\ \ ]$ denotes the greatest integer function. The number of solutions of $s \equiv 0 \bmod p^d$, $1 \leqq s \leqq r$ is exactly $[rp^{-d}]$. We reduce $r!\left|\prod_{s=1}^{r}(b_i s - b_i\eta_i)\right|^{-1}$ to lowest terms. Then for all primes p in classes 2 and 3, we see p does not divide the numerator and p divides the denominator at most $[(\log(b_i r + |b_i\eta_i|))/\log p]$ times. Let $T_2(m)$ be the least common multiple of $1, 2, \cdots, b_i m + |b_i\eta_i|$. (By Chebyshev's theorem, [8, p. 340], $|T_2(m)| < K_{11}^m$.) Let $T_1(m)$ be the factor in $\prod_{s=1}^{r}(b_i s - b_i\eta_i)$ due to primes of class 1.

Let $T_3(r)$ be the total factor in $\prod_{s=1}^{r}(b_i s - b_i\eta_i)$ due to primes of Class 3. Note that $T_3(r)$ divides $T_3(r+1)$. Thus the least common denominator of the $r!\left|\prod_{s=1}^{r}(b_i s - b_i\eta_i)^{-1}\right|$ for $r = 1, \cdots, m$ divides $T_2(m) \cdot T_3(m)$. There are only a finite number of primes p in class 1 and each such prime p divides $m!$ exactly $\sum_{d=1}^{m}[mp^{-d}] \leqq m/(p-1)$ times. Hence $T_1(m) \leqq K_{12}^m$. We note that

$$\frac{T_3(m)}{T_1(m)} \cdot \frac{m!}{\left|\prod_{s=1}^{m}(b_i s - b_i\eta_i)\right|} \leqq 1.$$

Hence

$$T_3(m) \leqq T_1(m)\left(\frac{\left|\prod_{s=1}^{m}(b_i s - b_i\eta_i)\right|}{m!}\right) \leqq K_{13}^m.$$

This proves Lemma 3, since we have seen

$$T_2(m) \cdot T_3(m) \leqq (K_{11} \cdot K_{13})^m.$$

LEMMA 4. *As before we write* $P_N(\delta) = \sum_{r,j}\varepsilon_{r,j}D^r f(\omega_j) + \varepsilon$ *for* $0 \leqq i \leqq k-1$, $0 \leqq j \leqq t-1$. (i) *There is a positive integer* $v(m) < m^{km}K_{14}^m$ *such that* $v(m) \cdot \varepsilon_{rj}$ *and* $v(m)\varepsilon$ *are algebraic integers.* (ii) $|P_{N+a}(\delta)| \leqq (K_{15}m^{-k})^m$ *for* $0 \leqq a \leqq (k+2)l$ *where* $N = ml - h$ *and* $l = kt + \delta$.

Proof. (i) Let E be a positive integer such that $E \cdot \omega_j^{-k}$, $E \cdot \theta(0)$, $E \cdot B_i$, and $E \cdot B_k^{-1}$ are all algebraic integers for $0 \leqq j \leqq t-1$, $0 \leqq i \leqq k-1$. From (4) and (15) we see that for $r \geqq k$

$$ED^r f(\omega_j) = \sum_{p=1}^{k} G_p^r D^{r-p} f(\omega_j) + G_0^r$$

where the G_p^r, $0 \leqq p \leqq k$, are algebraic integers. Hence

$$E^{m(k-1)}D^r f(\omega_j) = \sum_{p=0}^{k-1} H_p^r D^{r-p} f(\omega_j) + H_k^r$$

where the H_p^r are algebraic integers. Recall that Γ_j^r was defined to be the coefficient of $D^r f(\omega_j)$, $0 \leqq r \leqq km-1$, in the evaluation of $P_N(\delta)$ by the residue theorem. We shall set $v_1(m) = E^{m(k-1)} \cdot v_2(m)$ where $v_2(m)$ is a positive integer less than $m^{km}K_{16}^m$ and $v_2(m) \cdot \Gamma_j^r$ is an algebraic integer for $0 \leqq r \leqq km-1$, $0 \leqq j \leqq t-1$. Then $v_1(m)\varepsilon_{rj}$ will be an algebraic integer for $0 \leqq r \leqq k-1$, $0 \leqq j \leqq t-1$. Also $v_1(m) \cdot \varepsilon'$ will be an algebraic integer where ε' is the contribution to ε from the residues at $\omega_0, \cdots, \omega_{t-1}$. Finally we shall determine a positive integer $v_3(m) < K_{17}^m$ such that $v_2(m) \cdot v_3(m) \cdot \varepsilon''$ is an algebraic integer, where $\varepsilon' + \varepsilon'' = \varepsilon$. Then we may set $v(m) = v_1(m) \cdot v_3(m)$.

Let L be a positive integer such that $L \cdot (\omega_i - \omega_j)^{-1}$, $L \cdot (\omega_j)^{-1}$, and $L \cdot \omega_j$ are algebraic integers, where $i \neq j$, $0 \leqq i \leqq t-1$, and $0 \leqq j \leqq t-1$. We set

$$v_2(m) = (km)! \cdot L^{ktm+km+kt} < m^{km}K_{16}^m.$$

From (18) we can conclude as before that

$$\Gamma_j^r = \frac{1}{r!}\frac{1}{2\pi i}\int_{C_j} \frac{(z-\omega_j)^r z^h dz}{\phi(z)},$$

for $0 \leqq r \leqq mk-1$, $0 \leqq j \leqq t-1$. Define the polynomial $\mu(x)$ by $\mu(z-\omega_j) = L^{kt} \cdot z^h$ and note that since $h \leqq kt$ that $\mu(x)$ has algebraic integral coefficients. We see that

$$v_2(m) \cdot \Gamma_j^r = \left[\frac{(km)!}{r!}\left(\prod_{i \neq j} \frac{L}{(\omega_j - \omega_i)}\right)^{km} \cdot \left(\frac{L}{\omega_j}\right)^{\delta m} \cdot L^{m(k-\delta)}\right]$$
$$\cdot \frac{1}{2\pi i}\int_{C_j}\left[L^{mk} \cdot (z-\omega_j)^{-km+r} \cdot \mu(z-\omega_j) \cdot \left(\prod_{i \neq j}\left(1 + \frac{z-\omega_j}{\omega_j - \omega_i}\right)\right)^{-km}\right.$$
$$\left. \cdot \left(1 + \frac{z-\omega_j}{\omega_j}\right)^{-\delta m}\right] dz.$$

The factor in square brackets before the integral is an algebraic integer as is seen by using the definition of L; the integral is seen to be an algebraic integer by use of the residue theorem and the definition of L. Set $v_3(m)$ equal to the least common denominator of the fractions $(km-1)!(\prod_{e=1}^{d}\theta(e))^{-1}$ for $0 \leqq d \leqq (km-1)$. From Lemma 3 we know that $v_3(m) < K_{17}^m$. Referring to (19) we see that

$$v_2(m) \cdot v_3(m) \cdot \varepsilon'' = L^{k(t+m)-m} \cdot (km) \cdot \sum_{d=0}^{m-1}\left(\frac{(km-1)!v_3(m)}{\prod_{e=1}^{d}\theta(e)}\right)$$
$$\cdot \frac{L^{ktm}}{2\pi i}\int_{C_t}\frac{L^m \cdot z^{r+h}}{(\phi(z))^m}\, dz.$$

Also we have

$$\frac{L^{ktm}}{2\pi i}\int_{C_t}\frac{L^m\cdot z^{r+h}}{(\phi(z))^m}dz=\left(\prod_{j=0}^{t-1}\left(\frac{L}{\omega_j}\right)\right)^{km}\cdot\frac{1}{2\pi i}\int_{C_t}L^m\cdot z^{r+h-m}$$
$$\cdot\left(\prod_{j=0}^{t-1}\left(1-\frac{z}{\omega_j}\right)\right)^{-km}dz.$$

Using the definition of L and the residue theorem, we see that $v_2(m)\cdot v_3(m)\varepsilon''$ is an algebraic integer.

(ii) We must estimate $|P_{ml-h}(\delta)|$. Without loss of generality we may assume $|\omega_j|<m^k/2$ for $0\leqq j\leqq t-1$. Let C be a circle of radius m^k. From (1) we see that $|f(z^k)|\leqq K_{18}|z|$. Hence

$$|P_{m-l}(\delta)|=\left|\frac{1}{2\pi i}\int_C\frac{f(z)z^h dz}{(\phi(z))^m}\right|\leqq\frac{m^{k(h+1)}\cdot K_{18}^m}{(m/2)^{kml}}\leqq(K_{19}m^{-k})^{ml}.$$

If m is sufficiently large $(K_{19}m^{-k})^{ml}$ is monotonically decreasing. Thus $|P_{N+a}|\leqq(K_{15}m^{-k})^{ml}$ for $0\leqq a\leqq(k+2)l$.

Recall $S=\{D^rf(\omega_j);\ 0\leqq j\leqq t-1,\ 0\leqq r\leqq k-1\}\cup\{1\}$. Also recall $\delta=0$ or 1 and $l=kt+\delta$. We define the height of a linear form to be the maximum of the absolute values of the coefficients of the form.

LEMMA 5. *There exist linear forms $g_N(\delta)$ for $N=1,2,\cdots$ in the elements of S with coefficients in $K(\omega_0,\cdots,\omega_{t-1})$ which possess the following properties:*

(i) $$D^rf(\omega_j)=\sum_{\alpha=0}^{(k+2)l}\gamma_{rj\alpha}\,g_{N+\alpha}(\delta)$$

where $0\leqq r\leqq k-1$ and each $\gamma_{rj\alpha}$ belongs to $K(\omega_0,\cdots,\omega_{t-1})$.

(ii) *If either $\delta\neq 0$ or $\theta(0)\neq 0$ then*

$$1=\sum_{\alpha=0}^{(k+2)l}\gamma_\alpha g_{N+\alpha}(\delta)$$

for numbers γ_α in $K(\omega_0,\cdots,\omega_{t-1})$.

(iii) *If $\delta=0$ and $\theta(0)=0$ then each $g_{N+\alpha}(0)$ for $1\leqq N<\infty$ and $0\leqq\alpha\leqq(k+2)l$ has no constant term.*

(iv) *The height of $g_{N+\alpha}(\delta)$ is less than $(K_{20}m)^{mk}$ for $0\leqq\alpha\leqq(k+2)l$. The coefficients of $g_N(\delta)$ are algebraic integers for $1\leqq N<\infty$.*

(v) *For every $\varepsilon>0$ there exists an N_0 such that if $N\geqq N_0$ then*

$$|g_{N+\alpha}(\delta)|\leqq[(K_{20}m)^{mk}]^{-(l-1)+\varepsilon}.$$

Proof. Let $L=K(\omega_0,\cdots,\omega_{t-1})$. Set $g_N(\delta)=v(m)P_N(\delta)$. (By Lemma 2, part (i) the number $P_N(\delta)$ was shown to equal a linear form in the elements of S over the field L.) We shall demonstrate (i), (ii), and (iii) for $P_N(\delta)$, hence also for the $g_N(\delta)$.

(i) We need only prove (i) for $N = 1$ since we may obtain part (i) for $N = N' > 1$ by applying Lemma 1 $N' - 1$ times. We see that

$$\frac{D^r f(\omega_j)}{r!} = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-\omega_j)^{r+1}} = \frac{1}{2\pi i}\int_C \frac{Q(z)f(z)dz}{(\phi(z))^{r+1}}, \tag{21}$$

where $Q(z)$ belongs to $L[z]$ and the degree of $Q(z)$ is $(r+1)(l-1)$. Then $Q(z) = \sum_{\mu=0}^{r} Q_\mu(z)\cdot(\phi(z))^\mu$ for $Q_\mu(z)$ in $L[z]$ where $0 \leqq$ degree $(Q_\mu(z)) < l$ for $0 \leqq \mu \leqq r$. Hence

$$\frac{D^r f(\omega_j)}{r!} = \sum_{\mu=0}^{r} \frac{1}{2\pi i}\int_C \frac{Q_\mu(z)\cdot f(z)dz}{(\phi(z))^{r+1-\mu}}.$$

Each

$$\frac{1}{2\pi i}\int_C \frac{f(z)Q_\mu(z)dz}{(\phi(z))^{r+1-\mu}}$$

is a linear combination of the elements of $\{P_N(\delta); (r-\mu)l < N \leqq (r-\mu+1)l\}$.

(ii) Again we need only demonstrate the result for $N = 1$. Assume $\theta(0) \neq 0$. Then by (4) we have

$$1 = \frac{1}{\theta(0)}\cdot\frac{1}{2\pi i}\int_C \frac{\theta(0)dz}{(z-\omega_0)} = \frac{1}{\theta(0)}\cdot\frac{1}{2\pi i}\int_C \frac{\left(\sum_{j=0}^{k} B_j z^j D^j - z\right)f(z)dz}{(z-\omega_0)},$$

$$= \frac{1}{\theta(0)}\cdot\frac{1}{2\pi i}\int_C \frac{\left(\sum_{j=0}^{k} B_j z^j D^j + \omega_0\right)f(z)dz}{(z-\omega_0)},$$

$$= \frac{1}{\theta(0)}\sum_{j=0}^{k}\frac{B_j}{2\pi i}\int_C \frac{z^j\cdot\dfrac{\phi(z)}{(z-\omega_0)}\cdot D^j f(z)dz}{\phi(z)} + \frac{\omega_0}{\theta(0)}\cdot\frac{1}{2\pi i}\int_C \frac{\dfrac{\phi(z)}{(z-\omega_0)}f(z)dz}{\phi(z)}.$$

Using repeated integration by parts and collecting terms we may write

$$1 = \frac{1}{2\pi i}\int_C \frac{Q_1(z)f(z)dz}{(\phi(z))^{k+l}},$$

where $Q_1(z)$ belongs to $L[z]$ and degree $(Q_1(z)) \leqq (k+1)\cdot l - 1$. By the proof of part (i) then we can express 1 as a linear combination of $\{P_N; 1 \leqq N \leqq (k+2)\cdot l\}$.

Assume $\delta = 1$. Then

$$1 = \frac{1}{2\pi i}\int_C \frac{f(z)}{z}dz = \frac{1}{2\pi i}\int_C \frac{\left(\dfrac{\phi(z)}{z}\right)f(z)dz}{\phi(z)} \tag{22}$$

from line (1). The extreme right side of (22) can be expressed as a linear combination of $\{P_N; 1 \leqq N \leqq l-1\}$ over L.

(iii) In writing

$$\frac{1}{2\pi i}\int_C f(z)z^h(\phi(z))^{-m}dz$$

as a linear combination of the elements of S we first evaluated the integral by the residue theorem. If $\delta = 0$ there was no pole at 0. About the points ω_j, $0 \leqq j \leqq t-1$, we used (4) to express $D^r f(\omega_j)$, $0 \leqq r \leqq km-1$, in terms of the elements of $\{D^r f(\omega_j), \theta(0); 0 \leqq r \leqq k-1\}$. If $\delta = \theta(0) = 0$ then $P_N(0)$ has a zero constant term for $N = 1, 2, \cdots$.

(iv) By Lemma 2, part (ii) and Lemma 4, part (i) we see that the height of $g_N(\delta)$ is less than or equal to $(m^{km} \cdot K_{14}^m) \cdot (K_1^m) \leqq (K_{21}m)^{mk}$. Hence

$$\text{height}(g_{N+a}(\delta)) \leqq (K_{20}m)^{mk}$$

for $0 \leqq a \leqq (k+2)l$. The coefficients are algebraic integers by Lemma 4, part (i).

(v) $|g_{N+a}(\delta)| \leqq (K_{15}m^{-k})^{ml} \cdot (K_{20}m)^{mk}$ by Lemma 4, part (ii) and Lemma 5, part (iv). Hence, for $0 \leqq a \leqq (k+2)l$,

$$|g_{N+a}(\delta)| \leqq [(K_{20}m)^{mk}]^{-(l-1)+\varepsilon}$$

if $N \geqq N_0$.

II. **Proof of Theorem I.** Set $2t_1 + t_2 = t_3$. Let

$$\{v_j; 0 \leqq j \leqq t_3 - 1\} = \{\omega_0, \cdots, \omega_{t_1-1}, \bar{\omega}_0, \cdots, \bar{\omega}_{t_1-1}, \omega_{t_1}, \cdots, \omega_{t_1+t_2-1}\},$$

where the bar denotes complex conjugation. We apply Lemma 5 to see that there exist linear forms $g_N(1)$, $N = 1, 2, \cdots$ in the $D^r f(v_j)$ and 1 with algebraic integral coefficients in K, $(0 \leqq r \leqq k-1, 0 \leqq j \leqq t_3 - 1)$. Let V be the vector space over K generated by the $Df(v_j)$, $0 \leqq r \leqq k-1$, $0 \leqq j \leqq t_3 - 1$, and 1. As $\delta = 1$ here, Lemma 5 implies that for every positive integer N the $g_N, g_{N+1}, \cdots, g_{N+(k+2)(kt_3+1)}$ span V. Let $\Gamma = [V:K]$. Then $\Gamma \geqq 1$. Choose a basis for V of the form $e_0 = 1, e_1, \cdots e_{\Gamma-1}$ where e_i $(1 \leqq i \leqq \Gamma - 1)$ belongs to $\{D^r f(v_j); 0 \leqq r \leqq k-1, 0 \leqq j \leqq t_3 - 1\}$. Now there exist positive integers K_{22} and K_{23} such that for every positive integer N, $K_{22} \cdot g_N(1)$ is a linear form in $\{e_0 = 1, e_1, \cdots, e_{\Gamma-1}\}$ where the coefficients are algebraic integers and

$$\text{height}\ (K_{22}g_N(1)) \leqq K_{22}(K_{20}m)^{km} \leqq (K_{23}m)^{km}. \tag{23}$$

To see (23) one need only use the dependence relations which exist among the $D^r f(v_j)$ and 1 to express each g_N in terms of $\{e_0 = 1, e_1, \cdots, e_{\Gamma-1}\}$ and then observe how the coefficient of e_i for $0 \leqq i \leqq \Gamma - 1$ is generated. For every $\varepsilon_0 > 0$, if N is sufficiently large, then

$$\left| K_{22} g_N(1) \right| \leqq [(K_{23} m)^{km}]^{-(k\varepsilon t_3)+\varepsilon_0}. \tag{24}$$

Let $p_1, \cdots, p_{\Gamma-1}$ be numbers of the form $a + b(-n)^{1/2}$ for a and b integers. Let V^* be the dual space of V. We define e_i^*, $0 \leqq i \leqq \Gamma - 1$ to be the element of V^* defined by $e_i^*(e_j) = \delta_i^j$. Then $e_0^* + \sum_{i=1}^{\Gamma-1}(p_i/q)e_i^*$ is a nonzero vector in V^* for any choice of the p_i. Hence there exists α_0 such that $0 \leqq \alpha_0 \leqq (k+2)(kt_3+1)$ and

$$\left(e_0^* + \sum_{i=1}^{\Gamma-1} \frac{p_i}{q} e_1^* \right) \cdot (K_{22} \cdot g_{N+\alpha_0}(1)) \neq 0. \tag{25}$$

If we set $K_{22} \cdot g_{N+\alpha_0}(1) = \sum_{j=1}^{\Gamma-1} \gamma_j e_j + \gamma_0 e_0$ then (25) may be rewritten as

$$\sum_{i=1}^{\Gamma-1} \frac{\gamma_i p_i}{q} + \gamma_0 \neq 0.$$

This implies that

$$\left| \sum_{i=1}^{\Gamma-1} \frac{\gamma_i p_i}{q} + \gamma_0 \right| \geqq \frac{1}{q},$$

since here $L = K = Q((-n)^{1/2})$. Hence if we know that $\left| K_{22} \cdot g_{N+\alpha_0}(1) \right| \leqq 1/2q$ then

$$\left| K_{22} g_{N+\alpha_0}(1) - \sum_{i=1}^{\Gamma-1} \frac{\gamma_i p_i}{q} - \gamma_0 \right| \geqq \frac{1}{2q}. \tag{26}$$

Thus

$$\left| \sum_{i=1}^{\Gamma-1} \gamma_i \left(e_i - \frac{p_i}{q} \right) \right| \geqq \frac{1}{2q}. \tag{27}$$

We let $\gamma = \max_i |\gamma_i|$ for $0 \leqq i \leqq \Gamma - 1$. Observe that line (27) implies that $\Gamma \geqq 2$. Then

$$\max_i \left| e_i - \frac{p_i}{q} \right| \geqq \frac{1}{2q(\Gamma-1)\gamma}.$$

If $e_i = D^r f(\omega_j)$ let $p_i = p_{rj}$; if $e_i = D^r f(\overline{\omega}_j) = (D^r f(\omega_j))^-$ let $p_i = \bar{p}_{rj}$. Then

$$\max_{r,j} \left| D^r f(\omega_j) - \frac{p_{rj}}{q} \right| \geqq \max_i \left| e_i - \frac{p_i}{q} \right| \geqq \frac{1}{2(\Gamma-1)q\gamma} \geqq \frac{1}{2kt_3 q\gamma} \tag{28}$$

where $0 \leqq r \leqq k-1$, $0 \leqq j \leqq t-1$, and $0 \leqq i \leqq \Gamma - 1$.

We take $\varepsilon_0 < kt_3$ and choose $\varepsilon_1 > 0$. Then it will be shown that if q is sufficiently large we can always find an m such that

$$(2q)^{(1+\varepsilon_1)/(kt_3-\varepsilon_0)} \geqq (K_{23} m)^{km} \geqq (2q)^{1/(kt_3-\varepsilon_0)}. \tag{29}$$

We assume that $(2q)^{1/(kt_3-\varepsilon_0)} > (K_{23}^k)$ and choose $m > 1$ to be the largest integer such that

$$(2q)^{1/(kt_3-\varepsilon_0)} \geqq (K_{23}(m-1))^{k(m-1)}.$$

We note that if m is sufficiently large, say $m \geqq m_0$, then

$$[(K_{23}(m-1))^{k(m-1)}]^{1+\varepsilon_1} \geqq (K_{23}m)^{km}.$$

If q is sufficiently large then we will have $m \geqq m_0$. Then it follows that

$$(K_{23}m)^{km} \geqq (2q)^{1/(kt_3-\varepsilon_0)}$$

and

$$(2q)^{(1+\varepsilon_1)/(kt_3-\varepsilon_0)} \geqq [(K_{23}(m-1))^{k(m-1)}]^{1+\varepsilon_1} \geqq (K_{23}m)^{km}.$$

This demonstrates (29). From (24) and (29) it follows that for sufficiently large q we can find an N with

$$\text{height}(K_{22}g_{N+\alpha}(1)) \leqq (2q)^{(1+\varepsilon_1)/(kt_3-\varepsilon_0)} \tag{30}$$

while

$$|K_{22}g_{N+\alpha}(1)| \leqq \frac{1}{2q}$$

for all $0 \leqq \alpha \leqq (k+2)(kt_3+1)$. (We note that q large implies that m is large which implies that N is large.) Then in (28) we may assume that

$$\gamma \leqq (2q)^{-(1+(1+\varepsilon_1)/(kt_3-\varepsilon_0))}. \tag{31}$$

Thus

$$\max_{r,j}\left| D^r f(\omega_j) - \frac{p_{rj}}{q} \right| \geqq \frac{1}{kt_3}(2q)^{-(1+(1+\varepsilon_1)/(kt_3-\varepsilon_0))} \tag{32}$$

if q is sufficiently large. We choose ε_0 and ε_1 such that

$$\frac{1+\varepsilon_1}{kt_3-\varepsilon_0} < \frac{1}{kt_3} + \frac{\varepsilon}{2}$$

Then if q is sufficiently large

$$\max_{r,j}\left| D^r f(\omega_j) - \frac{p_{rj}}{q} \right| \geqq q^{-(1+1/kt_3+\varepsilon)}. \tag{33}$$

From (33) it follows that there exists $c(\varepsilon) > 0$ such that

$$\max_{r,j}\left| D^r f(\omega_j) - \frac{p_{rj}}{q} \right| \geqq c(\varepsilon)q^{-(1+1/kt_3+\varepsilon)}$$

for all q.

Proof of Theorem II. Let $\|x\| = x - [x]$. Khintchine's transference principle [9, p. 80] states that if $\theta_1, \cdots, \theta_n$ are any irrational numbers and $\omega_1 \geqq 0$ and $\omega_2 \geqq 0$ are the respective upper bounds of the values ω, ω' such that

$$\|u_1\theta_1 + \cdots + u_n\theta_n\| \leqq \left(\max_j |u_j|\right)^{-n-\omega},$$

$$\max_i \|x\theta_j\| \leqq |x|^{-((1+\omega')/n)}$$

have infinitely many integral solutions, then

$$\omega_1 \geqq \omega_2 \geqq \frac{\omega_1}{n^2+(n-1)\omega_1}. \tag{34}$$

I claim that because of Theorem I we may apply Khintchine's transference principle to the numbers,

$$\begin{aligned}\{\theta_j; 0 \leqq j \leqq t_3 - 1) = \{&\operatorname{Re} D^r f(\omega_j);\ 0 \leqq j \leqq t_1 - 1, 0 \leqq r \leqq k-1\} \\ &\cup \left\{\frac{\operatorname{Im} D f(\omega_j)}{n^{1/2}};\ 0 \leqq j \leqq t_1 - 1,\ 0 \leqq r \leqq k-1\right\} \\ &\cup \{D^r f(\omega_j);\ t_1 \leqq j \leqq t-1,\ 0 \leqq r \leqq k-1\}.\end{aligned}$$

Theorem I implies that for x a nonzero integer

$$\max_j \| x\theta_j \| \geqq c(\varepsilon) x^{-((1+\varepsilon/(kt_3))}. \tag{35}$$

Elementary theorems on diophantine approximation tell us that the θ_j for $0 \leqq j \leqq kt_3 - 1$ are irrational because of (35). Using (35) we see that $\omega_2 \leqq 0$. But $\omega_2 \geqq 0$ always. Hence $\omega_2 = 0$. Then (34) yields

$$\omega_1 \geqq 0 \geqq \frac{\omega_1}{(kt_3)^2 + (kt_3 - 1)\omega_1}. \tag{36}$$

Hence $\omega_1 = 0$.

Thus we see that for every $\varepsilon > 0$

$$\| u_0\theta_0 + \cdots + u_{kt_3-1} \cdot \theta_{kt_3-1} \| \leqq (\max |u_j|)^{-kt_3-\varepsilon} \tag{37}$$

has only finitely many solutions where $u_0, \cdots, u_{kt_3-1}$ are integers. Let the A_{rj} be in J. Define

$$\| x + y(-n)^{1/2} \| = \max(\| x \|, \| y \| n^{1/2}).$$

Then from (37) we see that

$$\Big\| \sum_r \sum_j A_{rj} D^r f(\omega_j) \Big\| \leqq (\max |A_{rj}|)^{-kt_3-\varepsilon} \tag{38}$$

has only finitely many solutions. (Separate line (38) into two inequalities and use (37).) Theorem II follows easily from (38).

Proof of Theorem IV. From Lemma 5 we know that there exist the linear forms $g_N(0)$ in the $D^i f(r_j)$ with rational integral coefficients and zero constant term for $N = 1, 2, \cdots$. Also

$$\operatorname{height}(g_{N+\alpha}(0)) \leqq (K_{20} m)^{km}$$

and for every $\varepsilon > 0$ there exists N_0 such that if $N \geqq N$

$$\left| g_{N+\alpha}(0) \right| \leqq [(K_{20}m)^{km}]^{(kt-1+\varepsilon)} .$$

Let us pick i_2 such that $0 \leqq i_2 \leqq k-1$ and $D^{i_2}f(r_{j_1}) \neq 0$. (Since equation (4) is homogeneous and $f(z) \not\equiv 0$ this is possible.) Then $\{g_N(0)/D^{i_2}f(r_{j_1});\ N = 1,2,\cdots\}$ is a set of linear forms in 1 and $\{D^if(r_j)/D^{i_2}f(r_{j_1});\ 0 \leqq i \leqq k-1,\ 0 \leqq j \leqq t-1,\ (i,j) \neq (i_2,j_1)\}$. If V is the vector space over Q spanned by 1 and the $D^if(r_j)/D^{i_2}f(r_{j_1})$ then for every $N \geqq 1$ the $\{g_{N+\alpha}(0)/D^{i_2}f(r_{j_1});\ 0 \leqq \alpha \leqq (k+2)(kt)\}$ spans V. Applying the method of proof used in Theorem I to the linear forms $g_N(0)/D^{i_2}f(r_{j_1})$ yields the result: For every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$\max_{i,j} \left| \frac{D^if(r_j)}{D^{i_2}f(r_{j_1})} - \frac{p_{ij}}{q} \right| > c(\varepsilon)q^{-1(+1/(kt-1)+\varepsilon)} \tag{39}$$

where $0 \leqq i \leqq k-1$, $0 \leqq j \leqq t-1$, $(i,j) \neq (i_2,i_1)$. Since any r real numbers can always be approximated infinitely often better than $q^{-(1+1/r-\varepsilon)}$ we see that (39) is impossible if any of the numbers being approximated are rational. Hence $D^{i_1}f(r_{j_1}) \neq 0$. Then we may take $i_2 = i_1$. This proves Theorem IV.

Proof of Theorem V. As in the proof of Theorem II from Theorem I we can apply Khintchine's transference principle to the irrational numbers $D^if(r_j)/D^{i_1}f(r_{j_1})$ where $(i,j) \neq (i_1,j_1)$. By Theorem III we see that $\omega_2 = 0$, hence it follows $\omega_1 = 0$ and we have: For every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$\left| \sum_{i=0}^{k-1} \sum_{j=0}^{t-1} A_{ij} \frac{D^if(r_j)}{D^{i_1}f(r_{j_1})} \right| \geqq c(\varepsilon)H^{-(kt-1+\varepsilon)} \tag{40}$$

where the A_{ij} are integers which are not all zero and $\left| A_{ij} \right| \leqq H$. Multiplication of (40) by $D^{i_1}f(r_{j_1})$ gives Theorem V.

Proof of Theorem III. Let $\omega_0,\cdots,\omega_{t-1}$ be the roots of $\theta_1(z)$. Then from the calculus of finite differences we conclude that

$$S_d^i = \sum_{q=0}^{k-1} \sum_{j=0}^{t-1} \gamma_{qj} d^q (\omega_j)^d , \tag{41}$$

where the coefficients γ_{qj} are unique and belong to the field $L = Q(\omega_0,\cdots,\omega_{t-1})$. Now

$$\mu_i = \sum_{d=0}^{\infty} \frac{S_d^i}{\prod_{e=1}^{d} \theta(e)} = \sum_{d=0}^{\infty} \frac{\sum_{q=0}^{k-1} \sum_{j=0}^{t-1} \gamma_{qj} d^q (\omega_j)^d}{\prod_{e=1}^{d} \theta(e)} = \sum_{q=0}^{k-1} \sum_{j=0}^{t-1} \gamma_{qj} \left(\sum_{d=0}^{\infty} \frac{d^q (\omega_j)^d}{\prod_{e=1}^{d} \theta(e)} \right).$$

Thus if

$$f(z) = \sum_{d=0}^{\infty} \frac{z^d}{\prod_{e=1}^{d} \theta(e)},$$

then

$$\sum_{d=0}^{\infty} \frac{d^q z^d}{\prod_{e=1}^{d} \theta(e)} = (zd)^q f(z) = \sum_{r=0}^{k-1} \alpha_r^q(z) D^r f(z).$$

Hence where the $\alpha_r^q(x)$ are polynomials with rational coefficients,

$$\mu_i = \sum_{r=0}^{k-1} \sum_{j=0}^{t-1} \left(\sum_{q=0}^{k-1} \gamma_{qj} \alpha_r^q(\omega_j) \right) D^r f(\omega_j)$$

(42)

$$= \sum_{r=0}^{k-1} \sum_{j=0}^{t-1} B_{jr} D^r f(\omega_j)$$

where

$$B_{jr} = \sum_{q=0}^{k-1} \gamma_{qj} \alpha_r^q(\omega_j).$$

Let ω_{j_1} denote throughout the rest of this proof $\sigma \cdot \omega_j$ where σ is some element of $G(L/Q)$. We shall show that $B_{j_1 r} = \sigma \cdot B_{jr}$. From (41)

$$S_d^i = \sigma S_d^i = \sum_{q=0}^{k-1} \sum_{j=0}^{t-1} \sigma \cdot \gamma_{qj} d^q (\omega_{j_1})^d.$$

But the coefficients in (41) are unique. Hence $\sigma \gamma_{qj} = \gamma_{qj_1}$. Clearly $\sigma \alpha_r^q(\omega_j) = \alpha_r^q(\omega_{j_1})$. This shows that $\sigma \cdot B_{jr} = B_{j_1 r}$.

Let the degree of the minimal polynomial with integral coefficients satisfied by ω_j be λ_j. Define δ_{jp}^d in Q by

$$(\omega_j)^d = \sum_{p=0}^{\lambda_j - 1} \delta_{jp}^d (\omega_j)^p.$$

Then there exists K_{24} such that

$$\left| \delta_{jp}^d \right| \leqq K_{24}^d.$$

Suppose that $g(z) = \sum_{d=0}^{\infty} C_d z^d$ is an entire function. Then

$$g(\omega_j) = \sum_{d=0}^{\infty} C_d \left(\sum_{p=0}^{\lambda - 1} \delta_{jp}^d (\omega_j)^p \right) = \sum_{p=0}^{\lambda_j - 1} \left(\sum_{d=0}^{\infty} C_d \delta_{jp}^d \right) (\omega_j)^p.$$

Where $g(z)$ is taken to be $D^r f(z)$ we define

$$\Delta_{jp}^r = \sum_{d=0}^{\infty} C_d \delta_{jp}^d.$$

Then

$$D^r f(\omega_j) = \sum_{p=0}^{\lambda_j - 1} \Delta^r_{jp}(\omega_j)^p.$$

Hence from (42) we obtain

$$\mu_i = \sum_{p=0}^{\lambda_j - 1} \sum_{r=0}^{k-1} \sum_{j=0}^{t-1} B_{jr}(\omega_j)^p \Delta^r_{jp}. \tag{43}$$

We note that $\Delta^r_{j_1 p} = \Delta^r_{jp}$. Let L_j be the least normal extension of Q which contains ω_j. Suppose $[L_j : Q] = l_j$. Then

$$\mu_i = \sum_j{}' \left[\sum_{p=0}^{\lambda_j - 1} \sum_{r=0}^{k-1} \frac{\lambda_j}{l_j} \cdot \operatorname{trace} L_j/_Q(B_{jr} \cdot \omega_j^p)\Delta^r_{jp} \right], \tag{44}$$

where the summation on j is over a set of representatives of the conjugacy classes of the $\omega_0, \cdots, \omega_{t-1}$ over the rationals. There are $k \cdot t$ distinct numbers (at most) among the Δ^r_{jp}, and we note that the coefficients of the Δ^r_{jp} in (44) are rational.

It is now possible to outline the course which the remainder of the proof will take. (a) We shall show that for each Δ^r_{jp}

$$\Delta^r_{jp} = \sum_{d=0}^{\infty} \frac{S_d}{\prod_{e=1}^{d} \theta(e)},$$

where S_d is an appropriately chosen rational valued sequence satisfying $(\theta_1(E))^k S_d = 0$. From this it follows that each Δ^r_{jp} is a linear combination of the μ_i with rational coefficients. (We recall that the S^i_d for $0 \leqq i \leqq kt - 1$ are linearly independent.) (b) We shall show the statement of Theorem III, part (ii) for the distinct Δ^r_{jp} (not the μ_i) from which, by (44) and (a), it will follow for the μ_i. Using Khintchine's transference principle with $\omega_1 = 0$ we may conclude $\omega_2 = 0$, i.e., part (i) of Theorem III holds for the μ_i. Now to show (a). We see that

$$\Delta^r_{jp} = \sum_{d=0}^{\infty} \frac{d \cdots (d - r + 1)\delta^{d-r}_{jp}}{\prod_{e=1}^{d} \theta(e)}.$$

We observe that δ^{d-r}_{jp} takes values in Q. From the calculus of finite differences we know that if

$$\theta_1(E)\delta^{d-r}_{jp} = 0$$

then

$$(\theta_1(E))^k[d \cdots (d - r + 1)\delta^{d-r}_{jp}] = 0$$

for $0 \leqq r \leqq k - 1$.

Consider the equation

$$\omega_j^{d-r} = \sum_p \delta^{d-r}_{jp}(\omega_j)^p \tag{45}$$

and the equations obtained from (45) by replacing ω_j by its conjugates. Let us define the matrix A by

$$A = (\sigma_i \omega_j^p) \tag{46}$$

where p is the column index $(0 \leqq p < \lambda_j - 1)$, i is the row index, and the $\sigma_i \omega_j$ $(0 \leqq i \leqq \lambda_j - 1)$ are a complete set of conjugates of ω_j. Then by the nonvanishing of the Vandermonde determinant we see that A is nonsingular—hence we may write δ_{jp}^{d-r} as a linear combination of the $(\sigma_i \omega_j)^{d-r}$. Thus

$$(\theta_1(E))\delta_{jp}^{d-r} = 0.$$

We note that up to this point we have not used the assumption that the roots of $\theta(z)$ are rational.

To show (b), that the distinct Δ_{jp}^r satisfy part (ii) of Theorem III, we need to produce linear forms $h_N(\delta)$ in the Δ_{jp}^r (and 1 if $\delta = 1$) which have the properties (i)–(v) of the forms $g_N(\delta)$ defined in Lemma 5, but with the Δ_{jp}^r replacing the $D^r f(\omega_j)$. Given the existence of such forms $h_N(\delta)$ we can set $\delta = 1$ and conclude Theorem II for the Δ_{jp}^r. As a first step we shall show that the linear forms $g_N(\delta)$ in the $D^r f(\omega_j)$ may be rewritten as linear forms in the Δ_{jp}^r with rational coefficients. Now $g_N(\delta)$ is an integral multiple of $(1/2\pi i)\int_C f(z) z^h (\phi(z))^{-m} dz$ where

$$N = (kt + \delta)m - h.$$

Recall that

$$\phi(z) = \left(\prod_{j=0}^{t-1} (z - \omega_j)\right)^k \cdot z^\delta.$$

We note that

$$\frac{1}{2\pi i}\int_C \frac{f(z) z^h dz}{(\phi(z))^m} = \frac{1}{2\pi i}\int_C \frac{f(z) \cdot z^h \cdot \dfrac{(z-\omega_j)^{km} dz}{\phi(z-\omega_j) + \omega_j}}{(z-\omega_j)^{km}}$$

from which it is not hard to see that the process of evaluation of the residue at ω_j and also the use of the differential equation (4), at $z = \omega_j \neq 0$ to express the residue in terms of $\{D^r f(\omega_j);\ 0 \leqq r \leqq k-1\}$ will be such as to yield conjugate coefficients at conjugate points, i.e., if

$$g_N(\delta) = \sum_{r=0}^{k-1} \sum_{j=0}^{t-1} \varepsilon_{rj} D^r f(\omega_j) + \varepsilon \tag{47}$$

then $\sigma(\varepsilon_{rj}) = \varepsilon_{rj_1}$. Now ε is rational since the residue at $z = 0$ is rational and if ε_j' is the contribution to ε from the residue at ω_j then $\sigma\varepsilon_j' = \varepsilon_{j_1}'$. Hence

$$g_N(\delta) = \sum_j{}' \left(\sum_{r=0}^{k-1} \sum_{p=0}^{\lambda_j - 1} \frac{\lambda_j}{l_j} \operatorname{trace} L_j/_Q(\varepsilon_{rj}\omega_j^p)\Delta_{jp}^r\right) + \varepsilon, \tag{48}$$

where the sum over j indexes a set of representatives of the classes of conjugate

elements of $\omega_0, \cdots, \omega_{t-1}$. If M is a positive integer such that $M \cdot \omega_j$ is an algebraic integer for each j, set $h_N(\delta) = M^{-1} g_N(\delta)$. Then $h_N(\delta)$ is a linear form in the Δ^r_{jp} and 1 with integral coefficients.

We wish to show the analogue of the statements (i)–(v) in Lemma 5 for the $h_N(\delta)$. Parts (iv) and (v) hold but for different constants. Parts (ii) and (iii) hold for the $h_N(\delta)$ as for the $g_N(\delta)$. To prove the analogue of (i) we need only show that Δ^r_{jp} for $0 \leqq j \leqq t-1$, $0 \leqq p \leqq \lambda_j - 1$, and $0 \leqq r \leqq k-1$ can be expressed as a linear combination with rational coefficients of the $g_{N+\alpha}(\delta)$, hence the $h_{N+\alpha}(\delta)$, for $0 \leqq \alpha \leqq (k+2)\ (kt+\delta)$ in the case $N = 1$, since Lemma 1 will then yield the cases $N = 2, 3, \cdots$. Now

$$D^r f(\omega_j) = \frac{r!}{2\pi i} \int_C \frac{f(z)dz}{(z-\omega_j)^{r+1}} = \frac{1}{2\pi i} \int_C \frac{f(z)\left[\dfrac{\phi(z)}{(z-\omega_j)}\right]^{r+1} dz}{(\phi(z))^{r+1}},$$

so

$$D^r f(\omega_j) = \sum_{N=1}^{(k+2)(kt+\delta)+1} Y^r_{jN} g_N(\delta)$$

where $\sigma Y^r_{jN} = Y^r_{j_1 N}$. Recalling the definition of the matrix A, from (46), we see that

$$(\Delta^r_{j1}, \cdots, \Delta^r_{j,\lambda_j-1}) \cdot A = (D^r f(\sigma_0 \omega_j), \cdots, D^r f(\sigma_{\lambda_j - 1}\omega_j))$$

where $\sigma_0 \omega_j = \omega_j$, $\sigma_1 \omega_j, \cdots, \sigma_{\lambda_j - 1}\omega_j$ are a complete set of conjugate numbers. Hence writing

$$\Delta^r_{jp} = \sum_j{}' X^r_{jp} D^r f(\omega_j)$$

where the sum is over a complete set of conjugates, we see that

$$\sigma X^r_{jp} = X^r_{j_1 p}\,.$$

Consequently

$$\Delta^r_{jp} = \sum_{N=1}^{(k+2)(kt+\delta)+1} \lambda_j / l_j \text{ trace } L_j/_Q(X^r_{jp} Y^r_{jN}) g_N(\delta).$$

This proves Theorem III.

Proof of Theorem VI. We need only set $\delta = 0$ in the proof of Theorem III and change the references to Theorems I and II into references to Theorems III and IV respectively.

Theorem VI implies Theorems IV and V obviously. It is less obvious that Theorem III implies Theorems I and II, since the ω_i in Theorems I and II may be in $Q(-n)^{1/2}$ but not in Q. We set

$$\theta_1(E) = \left(\prod_{j=0}^{t_1-1} (E-\omega_j)(E-\bar{\omega}_j)\right) \prod_{j=t_1}^{t_1+t_2-1} (E-\omega_j).$$

Our $k \cdot (t_1 + 2t_2)$ linearly independent sequences consist of the

$$d!/(d-\alpha)!\,\frac{\omega_j^d + \overline{\omega}_j^d}{2} \text{ and } d!/(d-a)!\,\frac{\omega_j^d - \overline{\omega}_j^d}{2(-n)^{1/2}}$$

for $0 \leqq \alpha \leqq k-1$ and $0 \leqq j \leqq t_1 - 1$ along with the $d!/(d-\alpha)!\omega_j^d$ for $0 \leqq \alpha \leqq k-1$ and $t_1 \leqq j \leqq t_1 + t_2 - 1$. Applying Theorem III under these circumstances yields Theorems I and II.

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