DECOMPOSITIONS OF $E^3$ WITH A COMPACT
0-DIMENSIONAL SET OF NONDEGENERATE
ELEMENTS(1)

BY
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1. Introduction. The purpose of this paper is to study monotone upper semi-
continuous decompositions $G$ of $E^3$ such that the image, under the projection
map, of the union of all the nondegenerate elements of $G$ is contained in a compact
0-dimensional set. Such decompositions are of interest since a number of
examples that have been studied [4], [6], [7], [9], [15] satisfy the conditions
imposed above on $G$.

Suppose then that $G$ is a monotone upper semicontinuous decomposition of
$E^3$. Let $E^3/G$ denote the associated decomposition space and let $P$ denote the
projection map from $E^3$ onto $E^3/G$. Let $H_G$ denote the union of all the nondegenerate
elements of $G$. Suppose that $P[H_G]$ is contained in a compact 0-dimensional set.

In §§3 and 4, point-like decompositions are considered. In §4, we prove that
if $G$ is point-like and $E^3/G$ is a 3-manifold, then $E^3/G$ is homeomorphic to $E^3$.
This settles a special case of a question raised by Bing in [7]. Other special cases
of this question have been settled in [1], [2], [14], and [19]. Also in §4, we give
a condition which is both necessary and sufficient in order that $E^3/G$ be homeo-
 morphic to $E^3$ in case $G$ is a point-like decomposition satisfying the conditions
above. This condition is in terms of the existence of homeomorphisms from $E^3$
onto $E^3$ that shrink the nondegenerate elements of $G$ to small size.

In §§ 5 and 6, we study the following question: Is it true that if $G$ satisfies the
conditions imposed above and $E^3/G$ is homeomorphic to $E^3$, then each element
of $G$ is point-like? It is known [12] that if the set of nondegenerate elements is
countable, the question has an affirmative answer. Furthermore, there is an
due to Bing [10, p. 7] of a monotone decomposition $B$ of $E^3$ into non-
point-like sets such that $E^3/B$ is homeomorphic to $E^3$. In this case, $P[H_B]$ is
an arc. Consequently, the condition imposed above on $P[H_G]$ cannot be omitted
completely if an affirmative answer to the question is to be obtained. Although
we do not settle this question, we give some partial affirmative solutions. We

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give a solution in case each nondegenerate element of \( G \) is an arc, or, in fact, any tree-like continuum. Additional partial solutions of this question are given in [3] and [13].

In §7, we give a number of special results concerning point-like decompositions \( G \) of \( E^3 \) such that \( E^3/G \) is homeomorphic to \( E^3 \).

2. Notation and terminology. If \( X \) is a topological space and \( G \) is an upper semi-continuous decomposition of \( X \), then \( X/G \) denotes the associated decomposition space, \( P \) denotes the projection from \( X \) onto \( X/G \), and \( H_G \) denotes the union of all the nondegenerate elements of \( G \).

The statement that the upper semicontinuous decomposition \( G \) of \( E^3 \) is monotone means that each set of \( G \) is a compact continuum. A compact continuum \( K \) in \( E^3 \) is point-like if and only if \( E^3 - K \) is homeomorphic to \( E^3 - \{0\} \). By a point-like decomposition of \( E^3 \) is meant a monotone decomposition of \( E^3 \) into point-like sets.

If \( n \) is a positive integer, the statement that \( M \) is an \( n \)-manifold means that \( M \) is a separable metric space, each point of which has a neighborhood \( U \) in \( M \) such that \( U \) is an open \( n \)-cell. The statement that \( M \) is an \( n \)-manifold-with-boundary means that \( M \) is a separable metric space such that each point of \( M \) has a neighborhood \( U \) in \( M \) such that \( U \) is an \( n \)-cell. If \( M \) is an \( n \)-manifold-with-boundary, then the boundary of \( M \), denoted by \( \text{Bd} M \), is the set of all points \( p \) of \( M \) which do not have open \( n \)-cell neighborhoods in \( M \), and the interior of \( M \), \( \text{Int} M \), is \( M - \text{Bd} M \).

3. Preliminary results. The main result of this section, Theorem 1, will be used in §4 to construct homeomorphisms from \( E^3 \) onto \( E^3 \) having certain properties. Lemmas 1 and 2 are preliminary results for Theorem 1.

Suppose that \( C \) is a polyhedral 3-cell in \( E^3 \). If \( x \) and \( y \) are distinct points of \( \text{Bd} C \), the statement that \( \alpha \) is an unknotted chord of \( C \) from \( x \) to \( y \) means that \( \alpha \) is a polygonal arc with endpoints \( x \) and \( y \) such that (1) \( (\text{Int} \alpha) \subset \text{Int} C \) and (2) if \( \beta \) is any polygonal arc on \( \text{Bd} C \) from \( x \) to \( y \), then \( \alpha \cup \beta \) is the boundary of a polyhedral disc \( D \) such that \( (\text{Int} D) \subset \text{Int} C \). It may be shown [20] that if \( \alpha \) and \( \alpha' \) are two unknotted chords of the polyhedral 3-cell \( C \) such that \( \alpha \) and \( \alpha' \) have the same endpoints, there is a homeomorphism \( h \) from \( C \) onto \( C \) such that if \( p \in \text{Bd} C \), \( h(p) = p \) and \( h[\alpha] = \alpha' \).

**Lemma 1.** If \( C \) is a polyhedral 3-cell in \( E^3 \), \( x \) and \( y \) are distinct points of \( \text{Bd} C \), and \( B \) is a compact 0-dimensional subset of \( C \) containing neither \( x \) nor \( y \), then there is an unknotted chord \( \alpha \) of \( C \) such that \( \alpha \) has endpoints \( x \) and \( y \) and is disjoint from \( B \).

**Proof.** We may assume that \( C \) is the cube

\[
\{(x, y, z): (x, y, z) \in E^3, \, |x| \leq 1, \, |y| \leq 1, \, \text{and} \, |z| \leq 1\}
\]
that $B$ intersects neither the top nor bottom face of $C$, and that $x$ and $y$ belong to the top and bottom faces, respectively, of $C$. Let $D$ be the common part of $C$ and the $XZ$-plane. $D$ is a disc in $C$ such that $\text{Bd} D \subset \text{Bd} C$, and $x$ and $y$ lie on $\text{Bd} D$.

Since $B$ is a compact 0-dimensional set, $B \cap D$ does not separate $x$ and $y$ in $D$. Hence there is polygonal arc $\alpha$ on $D$ such that $\alpha$ has endpoints $x$ and $y$, $\alpha$ is disjoint from $B$, and $(\text{Int} \, \alpha) \subset \text{Int} \, D$. It is easy to see that $\alpha$ is an unknotted chord of $C$.

The proof of Lemma 1 given above was shown to the author by R. H. Bing and is simpler than the author's original argument.

The following lemma is proved in [19].

**Lemma 2.** If $G$ is a point-like decomposition of $E^3$ such that $E^3/G$ is homeomorphic to $E^3$ and $U$ is a simply connected open set in $E^3/G$, then $P^{-1} [U]$ is simply connected.

We may now establish the main result of this section.

**Theorem 1.** Suppose that $G$ is a point-like decomposition $E^3$ such that $E^3/G$ is homeomorphic to $E^3$ and $\text{Cl} \, P[H_G]$ is a compact 0-dimensional set. Suppose that $M$ is a compact polyhedral 3-manifold-with-boundary in $E^3$ such that $\text{Bd} \, M$ is connected and is disjoint from $\text{Cl} \, H_G$. Then there is a homeomorphism $h$ from $M$ onto $P[M]$ such that $h \mid \text{Bd} \, M = P \mid \text{Bd} \, M$.

**Proof.** Let $K$ denote $P[\text{Bd} \, M]$; $K$ is homeomorphic to $\text{Bd} \, M$. Since $E^3/G$ is homeomorphic to $E^3$, $K$ has exactly one bounded complementary domain $U$ in $E^3/G$. Let $N$ denote $K \cup U$. It may be shown that $P[M] = N$.

Since $\text{Bd} \, M$ has a cartesian product neighborhood in $E^3$, $K$ has a cartesian product neighborhood in $E^3/G$. Hence $N$ is a compact 3-manifold-with-boundary, $\text{Bd} \, N = P[\text{Bd} \, M]$, and $\text{Int} \, N = P[\text{Int} \, M]$.

Since Theorem 1 is easily proved if $\text{Bd} \, M$ is a 2-sphere, we assume that $\text{Bd} \, M$ is not a 2-sphere. Let $T_0$ be a triangulation of $N$[11], [16] such that if $\sigma$ is any 3-simplex of $T_0$ and $S_2$ is the carrier of the 2-skeleton on $T_0$, then $S_2 - \text{Bd} \, \sigma$ is connected. Let $v_1, v_2, \ldots, v_k$ be the vertices of $T_0$ lying in $\text{Int} \, N$. Let $B_1, B_2, \ldots,$ and $B_k$ be mutually disjoint open balls such that if $i = 1, 2, \ldots,$ or $k$, $v_i \in B_i$ and $B_i \subset \text{Int} \, N$. If $i = 1, 2, \ldots,$ or $k$, then, since $\text{Cl} \, P[H_G]$ is 0-dimensional, $B_i - \text{Cl} \, P[H_G]$ exists, and let $v_i^*$ be a point of $B_i - \text{Cl} \, P[H_G]$. There is a homeomorphism $f_0$ from $N$ onto $N$ such that if $x \in N$ and $x \notin \bigcup_{i=1}^k B_i$, then $f_0(x) = x$, and if $i = 1, 2, \ldots,$ or $k$, $f_0[B_i] = B_i$ and $f_0(v_i) = v_i^*$. Let $T_1$ be $\{f_0[\sigma] : \sigma \in T_0\}$. Then $T_1$ is a triangulation of $N$, no vertex of $T_1$ belongs to $\text{Cl} \, P[H_G]$, and if $\sigma$ is any 3-simplex of $T_1$ and $S_2'$ is the carrier of the 2-skeleton of $T_1$, then $S_2' - \text{Bd} \, \sigma$ is connected.

If no 1-simplex of $T_1$ intersects $\text{Cl} \, P[H_G]$, let $T$ be $T_1$. Otherwise, let $s_1, s_2, \ldots,$ and $s_m$ be the 1-simplexes of $T_1$ which intersect $\text{Cl} \, P[H_G]$. There exist mutually
disjoint polyhedral (relative to \( T_1 \)) 3-cells \( R_1, R_2, \ldots, \) and \( R_m \) such that if \( i = 1, 2, \ldots, \) or \( m, \) then \((s_i \cap \text{Cl } \mathcal{P}[H_G]) \subset \text{Int } R_i,\) (2) \( R_i \subset \text{Int } N,\) and (3) \( s_i \cap R_i \) is an unknotted chord of \( R_i.\) Such 3-cells may be constructed in the following way: If \( i = 1, 2, \ldots, \) or \( m, \) let \( s'_i \) be a subarc of \( s_i, \) lying in \( \text{Int } s_i, \) and such that \((s'_i \cap \text{Cl } \mathcal{P}[H_G]) \subset \text{Int } s'_i.\) Since \( s'_i \) is polygonal (relative to \( T_1 \)), it may be thinned slightly to give \( R_i.\)

If \( i = 1, 2, \ldots, \) or \( m, \) let \( x_i \) and \( y_i \) be the two points of \( s_i \cap \text{Bd } R_i.\) By Lemma 1, there is an unknotted chord \( t_i \) of \( R_i \) from \( x_i \) to \( y_i, \) missing \( \text{Cl } \mathcal{P}[H_G].\) There is, then, a homeomorphism \( k_i \) from \( R_i \) onto \( R_i, \) such that if \( p \in \text{Bd } R_i, \) \( k_i(p) = p \) and \( k_i[s_i \cap R_i] = t_i.\) It is clear that there is a homeomorphism \( k \) from \( N \) onto \( N \) such that \((1) \) if \( x \notin \bigcup_{i=1}^m R_i, \) \( k(x) = x \) and \((2) \) if \( i = 1, 2, \ldots, \) or \( m, \) and \( x \in R_i, \) \( k(x) = k_i(x).\)

Let \( T \) be \( \{ k(\alpha) : \alpha \in T_i \}. \) Then \( T \) is a triangulation of \( N \) such that the carrier \( \Sigma_i \) of the 1-skeleton of \( T \) is disjoint from \( \text{Cl } \mathcal{P}[H_G].\) Let \( \Sigma_2 \) denote the carrier of the 2-skeleton of \( T.\)

Let \( \Delta_1, \Delta_2, \ldots, \) and \( \Delta_r \) be the 2-simplexes of \( T \) whose interiors lie in \( \text{Int } N.\) If \( i = 1, 2, \ldots, \) or \( r, \) there is an annulus \( A_i \) on \( \Delta_i \) such that \((1) \) \( A_i \) is disjoint from \( \text{Cl } \mathcal{P}[H_G] \) and \((2) \) \( \text{Bd } A_i \) is one boundary component of \( A_i.\) If \( i = 1, 2, \ldots, \) or \( r, \) let \( J_i \) be a centerline of \( A_i, \) and let \( D_i \) be the subdisc of \( \Delta_i \) having \( J_i \) as its boundary; \( D_i \cap A_i \) is an annulus \( B_i.\) Let \( V_1, V_2, \ldots, \) and \( V_r \) be mutually disjoint simply connected open sets such that if \( i = 1, 2, \ldots, \) or \( r, \) then \((1) \) \( \text{Cl } V_i \subset \text{Int } N,\) \((2) \) \( V_i \cap \Sigma_2 = \text{Int } D_i,\) and \((3) \) \( \text{Cl } V_i \cap \Sigma_2 = D_i. \) Such open sets may be constructed by slight thickenings of \( \text{Int } D_1, \) \( \text{Int } D_2, \ldots, \) and \( \text{Int } D_r.\)

If \( i = 1, 2, \ldots, \) or \( r, \) let \( U_i \) be \( \text{P}^{-1}[V_i].\) By Lemma 2, each of \( U_1, U_2, \ldots, \) and \( U_r \) is simply connected. It is easily seen, with the aid of Dehn's lemma \([18]\) that if \( i = 1, 2, \ldots, \) or \( r, \) there is a polyhedral disc \( E_i \) such that \((1) \) \( \text{Int } E_i \subset U_i \) and \((2) \) \( \text{Bd } E_i = \text{P}^{-1}[J_i].\) Note that \( E_i \subset M.\) If \( i = 1, 2, \ldots, \) or \( r, \) let \( Y_i \) be \( E_i \cup \text{P}^{-1}[A_i - B_i]; \) \( Y_i \) is a disc with boundary \( \text{P}^{-1}[A_i], \) and \( Y_i \subset M.\)

If \( i \) and \( j \) are distinct positive integers, neither greater than \( r, \) then \( \text{Int } Y_i \) and \( \text{Int } Y_j \) are disjoint, for \((1) \) \( \text{Int } A_i \) and \( \text{Int } A_j \) are disjoint and \((2) \) \( U_i \) and \( U_j \) are disjoint.

If \( i = 1, 2, \ldots, \) or \( r, \) let \( L_i \) denote the closure of \( A_i - B_i.\) It is clear that \( \text{P}^{-1}[[(\text{Bd } N) \cup (\bigcup_{i=1}^r L_i)]] \) is a homeomorphism \( h_0 \) from \( (\text{Bd } N) \cup (\bigcup_{i=1}^r L_i) \) onto \( (\text{Bd } M) \cup (\bigcup_{i=1}^r \text{P}^{-1}[L_i])\) such that \( h_0 | \text{Bd } N = \text{P}^{-1} | \text{Bd } N.\) It is easily seen that there is a homeomorphism \( g \) from \( \Sigma_2 \) onto \( (\text{Bd } M) \cup (\bigcup_{i=1}^r Y_i)\) such that \((1) \) \( g | \text{Bd } N = \text{P}^{-1} | \text{Bd } N \) and \((2) \) if \( i = 1, 2, \ldots, \) or \( r, \) \( g[A_i] = Y_i.\) It is clear that \( g[\Sigma_2] \subset M.\)

Suppose that \( \sigma \) is a 3-simplex of \( T \) and let \( S \) be \( \text{Bd } \sigma. \) We shall show now that \( g[\Sigma_2] \) is disjoint from \( \text{Int } g[S], \) the interior, in \( E^3, \) of the 2-sphere \( g[S].\)

Suppose that there is a point \( p \) of \( \Sigma_2 \) such that \( g(p) \in \text{Int } g[S].\) Clearly \( p \in \Sigma_2 - S. \) Now \( T \) has the property that if \( \sigma \) is any 3-simplex of \( T, \) then \( \Sigma_2 - \text{Bd } \sigma \) is connected. Therefore \( \Sigma_2 - S \) is connected. Consequently \( g[\Sigma_2] \subset (g[S] \cup \text{Int } g[S]), \) and thus \( \text{Bd } M \subset (g[S] \cup \text{Int } g[S]). \) Since \( \text{Bd } M \) is not a 2-sphere, there is a point
of \(g[S]\) not in \(M\). This is a contradiction, for \(g[\Sigma_2] \subset M\). Hence, no point of \(g[\Sigma_2]\) lies in \(\text{Int} g[S]\).

Let \(K_1, K_2, \ldots, K_n\) be the distinct 2-spheres in \(N\) which are boundaries of 3-simplexes of \(T\). Let \(W\) be \(\{q[K_i] \cup \text{Int} g[K_i] : i = 1, 2, \ldots, n\}\). We shall show now that \(W\) is a triangulation of \(M\).

First, \(W\) is a 3-complex. For suppose that \(i\) and \(j\) are distinct positive integers, neither greater than \(n\). Then \(\text{Int} g[K_i]\) and \(\text{Int} g[K_j]\) are disjoint. For, it was shown that \(g[K_i]\) and \(\text{Int} g[K_j]\) are disjoint, and so are \(g[K_j]\) and \(\text{Int} g[K_i]\). If \(\text{Int} g[K_j]\) and \(\text{Int} g[K_i]\) intersect, then one contains the other, say \(\text{Int} g[K_i] \subset \text{Int} g[K_j]\). Since \(g[K_i] \neq g[K_j]\), then \(g[K_i]\) intersects \(\text{Int} g[K_j]\). This is a contradiction, and hence \(\text{Int} g[K_i]\) and \(\text{Int} g[K_j]\) are disjoint. Since \(g[K_i] \cap g[K_j] = g[K_i \cap K_j]\), it is clear that \(W\) is a 3-complex.

Second, \(\bigcup \{w : w \in W\} = M\). Since \(g[\Sigma_2] \subset M\), it is clear that \(\bigcup \{w : w \in W\} \subset M\).

Hence we need only to show that \(M = \bigcup \{w : w \in W\}\).

Suppose that \(M = \bigcup \{w : w \in W\}\) exists. Since \(\bigcup \{w : w \in W\}\) is closed, there is a point \(p\) of \(\text{Int} M\) such that \(p \notin \bigcup \{w : w \in W\}\). Let \(q\) be a point such that for some 3-simplex \(w\) of \(W\), \(q \in \text{Int} w\). Then \(q \in \text{Int} M\) and there is an arc \(pq\) in \(\text{Int} M\); we may assume that \(pq\) is disjoint from the carrier of the 1-skeleton of \(W\). Let \(b\) be the first point of \(\bigcup \{w : w \in W\}\) on \(pq\) in the order from \(p\) to \(q\); \(pb - \{b\}\) is disjoint from \(\bigcup \{w : w \in W\}\). There is a 2-simplex \(\tau\) of \(W\) such that \(b \in \text{Int} \tau\), and there is a 3-simplex \(\sigma\) of \(W\) such that \(\tau \subset \text{Bd} \sigma\). Now \(\tau \notin \text{Bd} M\). For if \(\tau \subset \text{Bd} M\), then clearly the arc \(pb\) intersects \(E^3 - M\), which is impossible. Thus \(\tau \notin \text{Bd} M\). It follows that \(g^{-1}[\tau] \notin \text{Bd} N\).

There is exactly one 3-simplex \(t_1\) of \(T\) such that \(\text{Bd} \sigma = g[\text{Bd} t_1]\). There is a 3-simplex \(t_2\) of \(T\) such that \(\text{Int} t_1\) and \(\text{Int} t_2\) are disjoint and \(\text{Bd} t_1 \cap \text{Bd} t_2 = g^{-1}[\tau]\). Let \(\sigma'\) be \(g[\text{Bd} t_2] \cup \text{Int} g[\text{Bd} t_2]\). Then \(\sigma' \in W\) and \(\sigma \cap \sigma' = \tau\). Since \(\text{Int} \sigma\) and \(\text{Int} \sigma'\) are disjoint and \(\text{Bd} \sigma \cap \text{Bd} \sigma' = \tau\), it follows that \((pb - \{b\}) \subset \text{Int} \sigma'\). This is a contradiction, and it follows that \(M = \bigcup \{w : w \in W\}\).

Consequently, \(W\) is a triangulation of \(M\). It is clear that there is an extension \(f\) of \(g\) such that \((1) f\) is a homeomorphism from \(N\) onto \(M\) and \((2) f|\text{Bd} N = P^{-1}|\text{Bd} N\) and \(f|\text{Bd} M = P|\text{Bd} M\). This concludes the proof of Theorem 1.

4. Results on point-like decompositions. In this section, we establish the main results of the paper relative to point-like decompositions \(G\) of \(E^3\) such that \(E^3/G\) either is homeomorphic to \(E^3\) or is a 3-manifold.

**Theorem 2.** Suppose that \(G\) is a point-like decomposition of \(E^3\) such that \((1) E^3/G\) is homeomorphic to \(E^3\) and \((2) \text{Cl} P[H_G] \) is a compact 0-dimensional set. Suppose that \(U\) is an open set in \(E^3\) containing \(\text{Cl} H_G\) and \(e\) is a positive
number. Then there is a homeomorphism $h$ from $E^3$ onto $E^3$ such that (1) if $x \notin U$, $h(x) = x$ and (2) if $g$ is any nondegenerate element of $G$, $(diam h[g]) < \varepsilon$.

**Proof.** Since $Cl P[H_G]$ is a compact 0-dimensional subset of $E^3/G$ and $E^3/G$ is homeomorphic to $E^3$, there exists a sequence $N_1, N_2, N_3, \ldots$ such that (1) for each positive integer $i$, $N_i$ is a compact polyhedral 3-manifold-with-boundary such that (a) each component of $N_i$ has connected boundary and diameter less than $1/i$, and (b) $N_{i+1} \subseteq \text{Int} N_i$, and (2) $Cl P[H_G] = \bigcup_{i=1}^{\infty} N_i$. For each positive integer $i$, let $M_i$ denote $P^{-1}[N_i]$; $M_i$ is a compact 3-manifold-with-boundary. Since, for each positive integer $i$, $N_i$ is polyhedral, then $M_i$ is tame in $E^3$.

Since $Cl H_G \subset U$, $P[U]$ is open in $E^3/G$ and there is, accordingly, a positive integer $k$ such that $N_k \subset P[U]$. It follows that $M_k \subset U$. Let $M_{k1}, M_{k2}, \ldots$, and $M_{km}$ be the components of $M_k$. If $j = 1, 2, \ldots, m$, there is, by Theorem 1, a homeomorphism $f_j$ from $M_{kj}$ onto $P[M_{kj}]$ such that $f_j \mid Bd M_{kj} = P \mid Bd M_{kj}$. Let $f$ be the homeomorphism from $M_k$ onto $P[M_k]$ such that if $j = 1, 2, \ldots, m$, $f(M_{kj}) = f_j$. Now $f^{-1}$ is uniformly continuous and there exists a positive integer $r$ greater than $k$ such that if $L$ is any component of $P[M_r]$, then $(diam f^{-1}[L]) < \varepsilon$. Let $M_{r1}, M_{r2}, \ldots$, and $M_{rn}$ be the components of $M_r$. With the aid of Theorem 1, it follows that if $i = 1, 2, \ldots, n$, there is a homeomorphism $k_i$ from $M_{ri}$ onto $P^{-1}[P[M_{ri}]]$ such that $k_i \mid Bd M_{ri} = f^{-1}P \mid Bd M_{ri}$. Let $k$ be the homeomorphism from $M_r$ onto $P^{-1}[P[M_r]]$ such that if $i = 1, 2, \ldots, n$, $k(M_{ri}) = k_i$.

Define a homeomorphism $h$ as follows: (1) If $x \notin M_k$, then $h(x) = x$. (2) If $x \in (M_k - M_r)$, then $h(x) = f^{-1}(P(x))$. (3) If $x \in M_r$, then $h(x) = k(x)$. Then $h$ is a homeomorphism from $E^3$ onto $E^3$. If $x \notin U$, then since $x \notin M_k$, $h(x) = x$.

If $g$ is any nondegenerate element of $G$, then for some positive integer $i$ not greater than $n$, $g \subset M_{ri}$. Hence $h[g] = h[M_{ri}]$; but $h[M_{ri}] = f^{-1}[P[M_{ri}]]$, and $(diam f^{-1}[PM_{ri}]) < \varepsilon$. Therefore $(diam h[g]) < \varepsilon$ and the proof of Theorem 2 is complete.

The following theorem may be compared with Theorem 2 of [1].

**Theorem 3.** Suppose that $G$ is a point-like decomposition of $E^3$ such that $P[H_G]$ is a compact 0-dimensional set. Then $E^3/G$ is homeomorphic to $E^3$ if and only if for each open set $U$ in $E^3$ containing $H_G$ and each positive number $\varepsilon$, there is a homeomorphism $h$ from $E^3$ onto $E^3$ such that (1) if $x \notin U$, $h(x) = x$ and (2) if $g$ is any nondegenerate element of $G$, $(diam h[g]) < \varepsilon$.

**Proof.** If, for each open set $U$ in $E^3$ containing $H_G$ and each positive number $\varepsilon$, there exists a homeomorphism $h$ from $E^3$ onto $E^3$ having the properties stated above, then it follows from the proof of Theorem 1 of [4] that $E^3/G$ is homeomorphic to $E^3$. Hence the condition stated is sufficient. By Theorem 2, it is necessary.

The following theorem may be compared with Theorem 5 of [1].
Theorem 4. Suppose that $G$ is a point-like decomposition of $E^3$ such that $P[H_G]$ is a compact 0-dimensional set. If $E^3/G$ is a 3-manifold, then $E^3/G$ is homeomorphic to $E^3$.

Proof. If $E^3/G$ is a 3-manifold, then a local version of Theorem 1 holds. Hence the proof of Theorem 2 is valid and thus the conclusion of that theorem holds. Then by Theorem 3, $E^3/G$ is homeomorphic to $E^3$.

Corollary 1. If $G$ is a point-like decomposition of $S^3$ such that $P[H_G]$ is a compact 0-dimensional set and $S^3/G$ is a 3-manifold, then $S^3/G$ is a 3-sphere.

5. Monotone decompositions of $E^3$. In this section and the next, we study monotone decompositions $G$ of $E^3$ such that (1) $P[H_G]$ is contained in a compact 0-dimensional set and (2) $E^3/G$ is homeomorphic to $E^3$. We want to determine conditions under which it can be concluded that each element of $G$ is point-like.

A compact continuum $K$ in $E^3$ is cellular if and only if there exists a sequence $C_1, C_2, C_3, \ldots$ of 3-cells in $E^3$ such that (1) if $n$ is any positive integer, $C_{n+1} \subset \text{Int } C_n$, and (2) $K = \bigcap_{i=1}^\infty C_i$. It is well known that for compact continua in $E^3$, “point-like” and “cellular” are equivalent; see [21].

Theorem 5. Suppose that $G$ is a monotone decomposition of $E^3$ such that $P[H_G]$ is contained in a compact 0-dimensional set and $E^3/G$ is homeomorphic to $E^3$. Suppose that if $U$ is any open set in $E^3$ containing $C_1 \cap H_G$ and $\varepsilon$ is any positive number, there exists a homeomorphism $f$ from $E^3$ onto $E^3$ such that (1) if $x \notin U$, $f(x) = x$, and (2) if $g$ is any nondegenerate element of $G$, then $(\text{diam } f[g]) < \varepsilon$. Then each element of $G$ is point-like.

Proof. Since $\text{Cl } P[H_G]$ is compact and 0-dimensional, there exists a sequence $N_1, N_2, N_3, \ldots$ of compact 3-manifolds-with-boundary in $E^3/G$ such that (1) if $i$ is any positive integer, $N_{i+1} \subset \text{Int } N_i$, and each component of $N_i$ has diameter less than $1/i$, and (2) $\bigcap_{i=1}^\infty N_i = P[H_G]$. For each positive integer $i$, let $M_i$ be $P^{-1}[N_i]$. Then for each positive integer $i$, $M_i$ is a compact 3-manifold-with-boundary and $M_{i+1} \subset \text{Int } M_i$. Further, $\bigcap_{i=1}^\infty M_i = \text{Cl } H_G$.

Suppose that $g$ is a nondegenerate element of $G$. For each positive integer $i$, let $K_i$ be the component of $M_i$ containing $g$. Then for each positive integer $i$, $K_i$ is a compact 3-manifold-with-boundary, and $K_{i+1} \subset \text{Int } K_i$. Further, $\bigcap_{i=1}^\infty K_i = g$.

Suppose now that $j$ is a positive integer. We shall show that there is a 3-cell $C$ such that $g \subset \text{Int } C$ and $C \subset \text{Int } K_j$. Let $\{R_1, R_2, \ldots, R_m\}$ be a finite set of 3-cells, each contained in $\text{Int } K_j$ and such that $\{\text{Int } R_1, \text{Int } R_2, \ldots, \text{Int } R_m\}$ covers $K_{j+1}$. There exists a positive number $\varepsilon$ such that if $A$ is any subset of $K_{j+1}$ of diameter less than $\varepsilon$, then $A$ is contained in some one of $\text{Int } R_1, \text{Int } R_2, \ldots$, and $\text{Int } R_m$. By hypothesis, there is a homeomorphism $f$ from $E^3$ onto $E^3$ such that (1) if $x \notin \text{Int } M_{j+1}$, then $f(x) = x$, and (2) $(\text{diam } f[g]) < \varepsilon$. It follows that $f[K_{j+1}] \subset K_{j+1}$, and hence there is a positive integer $k$ not greater than $m$ such that $f[K_k] \subset \text{Int } R_k$.

Let $C = f^{-1}[R_k]$; clearly $C$ is a 3-cell such that $g \subset \text{Int } C$ and $C \subset \text{Int } K_j$.
It now follows that $g$ is cellular. There exists a 3-cell $C_1$ such that $g \subset \text{Int } C_1$ and $C_1 \subset \text{Int } K_1$. Let $n_1$ be 1. There is a positive integer $n_2$ such that $K_{n_2} \subset \text{Int } C_1$. There is a 3-cell $C_2$ such that $g \subset \text{Int } C_2$ and $C_2 \subset \text{Int } K_{n_2}$. A continuation of this process yields an increasing sequence $n_1, n_2, n_3, \ldots$ of positive integers and a sequence $C_1, C_2, C_3, \ldots$ of 3-cells such that for each positive integer $i$, $g \subset \text{Int } C_i$ and $C_i \subset \text{Int } K_n$. It is clear that for each positive integer $i$, $C_{i+1} \subset \text{Int } C_i$, and that $\bigcap_{i=1}^{\infty} C_i = g$. Hence $g$ is cellular. Hence $g$ is point-like and Theorem 5 is established.

The following theorem is closely related to Theorem 1. In the proof of Theorem 6 we construct triangulations in the order opposite to that used in the proof of Theorem 1.

**Theorem 6.** Suppose that $G$ is a monotone decomposition of $E^3$ such that (1) $E^3/G \approx E^3$ and (2) $\text{Cl } P[H_G]$ is a compact 0-dimensional set. Suppose that $M$ is a compact polyhedral 3-manifold-with-boundary in $E^3$ such that $\text{Bd } M$ is a connected and is disjoint from $H_G$. Suppose that $M$ has a triangulation $T$ such that the carrier $\Sigma_1$ of the 1-skeleton of $T$ is disjoint from $\text{Cl } H_G$. Then there is a homeomorphism $h$ from $M$ onto $P[M]$ such that $h \mid \text{Bd } M = P \mid \text{Bd } M$.

**Proof.** Let $\Delta_1, \Delta_2, \ldots$, and $\Delta_n$ be the 2-simplexes of $T$ that intersect $\text{Int } M$. Let $i = 1, 2, \ldots, n$, let $F_i$ be a disc in $\text{Int } \Delta_i$ and such that $\text{Int } F_i$ contains $\Delta_i \cap \text{Cl } H_G$. If $i = 1, 2, \ldots, n$, there is an annulus $A_i$ on $\Delta_i$ such that $\text{Bd } A_i = (\text{Bd } A_i) \cap \text{Int } F_i$.

Let $N$ be $P[M]$; it may be shown that $N$ is a compact 3-manifold-with-boundary, $\text{Bd } N = P[\text{Bd } M]$, and $\text{Int } N = P[\text{Int } M]$. If $i = 1, 2, \ldots, n$, then $P[D_i]$ is a singular disc lying in $\text{Int } N$, and having no singularities on $P[B_i]$. Further, there is a neighborhood $U$ of $\Sigma_1$ such that (1) $P[U \cup (\bigcup_{i=1}^{n} A_i)]$ is a homeomorphism and (2) if $x \notin U \cup (\bigcup_{i=1}^{n} A_i)$, then $P(x) \notin P[U \cup (\bigcup_{i=1}^{n} A_i)]$.

With the aid of Dehn’s lemma [18], it may be shown that there exists discs $K_1, K_2, \ldots, K_n$ such that (1) if $i = 1, 2, \ldots, n$, (a) $K_i \subset \text{Int } N$, (b) $\text{Bd } K_i = P[\text{Bd } D_i]$, and (c) $K_i \cap P[U \cup (\bigcup_{j=1}^{i} A_j)] \subset P[B_i]$, and (2) if $i$ and $j$ are distinct positive integers, neither greater than $n$, then $K_i$ and $\text{Bd } K_j$ are disjoint.

The discs $K_1, K_2, \ldots, K_n$ are not necessarily mutually disjoint. By an argument similar to that given in the proof of Theorem 9 of [7], it may be shown that there exist mutually disjoint discs $K'_1, K'_2, \ldots, K'_n$ such that if $i = 1, 2, \ldots, n$, $\text{Bd } K'_i = \text{Bd } K_i$, $K'_i \subset \text{Int } N$, and $K'_i$ is disjoint from $P[\bigcup_{j=1}^{i}(A_j - B_j)]$. Then if $i = 1, 2, \ldots, n$, $K'_i \cup P[A_i - B_i]$ is a disc $\Delta'_i$.

By an argument similar to that used in the proof of Theorem 1, it may be
shown that there is a homeomorphism \( h \) from \( M \) onto \( P[M] \) such that 
\[ h \mid \text{Bd } M = P \mid \text{Bd } M. \]
This completes the proof of Theorem 6.

**Corollary 2.** Suppose that \( G \) is a monotone decomposition of \( E^3 \) such that (1) \( E^3/G \) is homeomorphic to \( E^3 \) and (2) \( \text{Cl } P[H_G] \) is a compact 0-dimensional set. Suppose that there exists a sequence \( M_1, M_2, M_3, \ldots \) of compact 3-manifolds-with-boundary in \( E^3 \) such that (1) if \( n \) is any positive integer, \( M_n \subseteq M_{n+1} \) and \( M_n \) has a triangulation such that the carrier of its 1-skeleton is disjoint from \( \text{Cl } H_G \), and (2) \( \bigcap_{i=1}^{\infty} M_i = \text{Cl } H_G. \) Then each element of \( G \) is point-like.

**Proof.** By an argument similar to that given to prove Theorem 2, but using Theorem 6 in place of Theorem 1, we may show that under the hypothesis of the corollary, the following holds: If \( U \) is any open set in \( E^3 \) containing \( H_G \) and \( \varepsilon \) is any positive number, there exists a homeomorphism \( h \) from \( E^3 \) onto \( E^3 \) such that (1) if \( x \notin U \), \( h(x) = x \), and (2) if \( g \) is any nondegenerate element of \( G \), then 
\[ (\text{diam } h[g]) < \varepsilon. \]
It then follows by Theorem 5 that each element of \( G \) is point-like.

6. Decompositions of \( E^3 \) into continua of type \( T \). The statement that a compact metric continuum \( M \) is of type \( T \) means that if \( K \) is any subcontinuum of \( M \) that can be embedded in a 2-sphere and \( f \) is an embedding of \( K \) in some 2-sphere \( S \), then \( f[K] \) does not separate \( S \). Since for planar continua, separating the plane is a topological invariant, the criterion used is meaningful. Continua of type \( T \) include tree-like continua [8], snake-like continua [8], dendrons, and arcs.

**Theorem 7.** If \( G \) is a monotone decomposition of \( E^3 \) such that (1) \( E^3/G \) is homeomorphic to \( E^3 \), (2) \( P[H_G] \) is contained in a compact 0-dimensional set, and (3) each element of \( G \) is a continuum of type \( T \), then each element of \( G \) is point-like.

**Proof.** First it will be shown that if \( M \) is a compact polyhedral 3-manifold-with-boundary in \( E^3 \) such that \( \text{Bd } M \) is connected and is disjoint from \( \text{Cl } H_G \), then \( M \) has a triangulation \( L \) such that the carrier of the 1-skeleton of \( L \) is disjoint from \( \text{Cl } H_G \).

Let \( L_0 \) be any triangulation of \( M \). Let \( v_1, v_2, \ldots, v_n \) denote the vertices of \( L_0 \). Since \( \text{Cl } P[H_G] \) is compact and 0-dimensional, there are points \( v'_1, v'_2, \ldots, \) and \( v'_n \) of \( \text{Int } M \) not in \( \text{Cl } H_G \). There is a homeomorphism \( h \) from \( M \) onto \( M \) such that \( h \mid \text{Bd } M = \text{the identity} \) and if \( i = 1, 2, \ldots, \) or \( n, h(v_i) = v'_i \). Let \( L_1 \) denote \( \{h[\sigma]: \sigma \in L_0 \} \). \( L_1 \) is a triangulation of \( M \) and no vertex of \( L_1 \) belongs to \( \text{Cl } H_G \).

If \( s \) is a 1-simplex of \( L_1 \), let \( C_s \) be a 3-cell containing \( s \cap \text{Cl } H_G \), obtained by a slight thickening of a subarc \( s' \) of \( s \) and such that \( C_s \cap s = s' \). It is to be true that if \( s \) and \( t \) are distinct 1-simplexes of \( L_1 \), \( C_s \) and \( C_t \) are disjoint. Now by hypothesis, if \( g_0 \) is any subcontinuum of an element of \( G \), \( g_0 \) does not separate any
2-sphere containing \( g_0 \). In addition, each component of \((\text{Bd } C_s) \cap \text{Cl } H_G\) is a subcontinuum of some element of \( G \). Hence no component of \((\text{Bd } C_s) \cap \text{Cl } H_G\) separates \( \text{Bd } C_s \). By unicoherence, \((\text{Bd } C_s) - \text{Cl } C_s\) is connected. There is, therefore, an arc \( s' \) on \( \text{Bd } C_s \), disjoint from \( \text{Cl } H_G \) and having as endpoints the endpoints of \( s' \). It is now easy to construct a triangulation \( L \) of \( M \) such that the carrier of the 1-skeleton of \( L \) is disjoint from \( \text{Cl } H_G \).

By using Theorem 6 in place of Theorem 1, the argument used to establish Theorem 2 shows that the hypothesis of Theorem 5 is satisfied. Hence by Theorem 5, each element of \( G \) is point-like.

**Corollary 3.** If \( G \) is a monotone decomposition of \( E^3 \) such that (1) \( E^3/G \) is homeomorphic to \( E^3 \), (2) \( P[H_G] \) is contained in a compact 0-dimensional set, and (3) each element of \( G \) is a tree-like continuum, then each element of \( G \) is point-like.

**Corollary 4.** If \( G \) is a monotone decomposition of \( E^3 \) into arcs and one-point sets such that \( E^3/G \) is homeomorphic to \( E^3 \) and \( \text{Cl } P[H_G] \) is a compact 0-dimensional set, then each element of \( G \) is point-like \((2)\).

7. **Results on point-like decompositions.** In this section, we present some results on point-like decompositions of \( E^3 \). We begin with two results on 3-cells-with-handles.

The statement that \( K \) is a 3-cell-with-handles means that \( K \) is an orientable 3-manifold-with-boundary such that there exist 3-cells \( C_0, C_1, \ldots, C_n \) such that (1) any two of \( C_0, C_1, \ldots, C_n \) are disjoint, (2) if \( i = 1, 2, \ldots, n \), \( C_0 \cap C_i = (\text{Bd } C_0) \cap (\text{Bd } C_i) \) and \( C_0 \cap C_i \) is the union of two disjoint discs, and (3) \( K = \bigcup_{i=0}^{n} C_i \).

**Lemma 3.** If \( K \) is a polyhedral 3-cell-with-handles and \( L \) is a polyhedral 3-cell such that \( K \cap L = (\text{Bd } K) \cap (\text{Bd } L) \) and \( K \cap L = D_1 \cup D_2 \cup \cdots \cup D_n \) where \( D_1, D_2, \ldots, D_n \) are mutually disjoint polyhedral discs, then \( K \cup L \) is a polyhedral 3-cell-with-handles.

**Lemma 4.** If \( B \) is a compact 0-dimensional subset of \( E^3 \) and \( U \) is any open set containing \( B \), there exists a polyhedral 3-manifold-with-boundary \( M \) such that \( B \subset \text{Int } M \), \( M \subset U \), and each component of \( M \) is a 3-cell-with-handles.

**Proof.** There exists a polyhedral 3-manifold-with-boundary \( N \) such that \( B \subset N \) and \( N \subset U \). It follows with the aid of Lemma 1 that there is a triangulation \( T \) of \( N \) such that the carrier of the 1-skeleton of \( T \) is disjoint from \( B \).

Let \( N_0 \) be a component of \( N \), and let \( S_0^* \) be the carrier of the 1-skeleton of the triangulation \( T_0 \) of \( N_0 \) induced by \( T \). There exists a polyhedral tubular neighborhood \( S_0^* \) of \( S_0 \) such that (1) \( S_0^* \) is disjoint from \( B \), (2) if \( \sigma \) and \( \sigma' \) are distinct

\( (2) \) Joseph M. Martin has recently established this result without requiring that \( \text{Cl } P[H_G] \) be compact and 0-dimensional.
3-simplexes of \( T_0 \) and \( \sigma - (\text{Int} \ S^*_0) \) and \( \sigma' - (\text{Int} \ S^*_0) \) intersect, then their common part is a disc lying in the interior of some 2-simplex of \( T_0 \), and (3) \( N_0 - \text{Int} \ S^*_0 \) is connected. With the aid of Lemma 3, it follows that \( N_0 - \text{Int} \ S^*_0 \) is a 3-cell-with-handles. Lemma 4 now follows easily.

Suppose that \( G \) is an upper semicontinuous decomposition of \( E^3 \). The statement that \( H_G \) is definable by 3-cell-with-handles means that there exists a sequence \( M_1, M_2, M_3, \ldots \) such that (1) for each positive integer \( n \), \( M_n \) is a polyhedral 3-manifold-with-boundary such that each component of \( M_n \) is a 3-cell-with-handles and \( M_{n+1} \subset \text{Int} \ M_n \), (2) \( \text{Cl} \ H_G = \bigcap_{i=1}^\infty M_i \), and (3) \( g \) is a nondegenerate element of \( G \) if and only if \( g \) is a nondegenerate component of \( \bigcap_{i=1}^\infty M_i \). It is clear that if \( H_G \) is definable by 3-cells-with-handles, then \( \text{Cl} \ P[H_G] \) is a compact 0-dimensional set.

**Theorem 8.** Suppose that \( G \) is a point-like decomposition of \( E^3 \) such that (1) \( E^3/G \) is homeomorphic to \( E^3 \) and (2) \( \text{Cl} \ P[H_G] \) is a compact 0-dimensional set. Then \( H_G \) is definable by 3-cells-with-handles.

**Proof.** Since \( E^3/G \) is homeomorphic to \( E^3 \), there is, by Lemma 4, a sequence \( M_1, M_2, M_3, \ldots \) such that (1) for each positive integer \( i \), \( M_i \) is a compact polyhedral 3-manifold-with-boundary, \( M_{i+1} \subset \text{Int} \ M_i \), and each component of \( M_i \) is a 3-cell-with-handles, and (2) \( \text{Cl} \ P[H_G] = \bigcap_{i=1}^\infty M_i \). It is clear that \( \text{Cl} \ H_G = \bigcap_{i=1}^\infty P^{-1}[M_i] \).

Suppose that \( j \) is a positive integer and \( L \) is a component of \( M_j \). Since \( L \) is a 3-cell-with-handles, there exist polyhedral 3-cells \( C_0, C_1, \ldots, C_r \) such that (1) any two of \( C_1, C_2, \ldots, C_r \) are disjoint, (2) if \( i = 1, 2, \ldots, r \),

\[
C_0 \cap C_i = (\text{Bd} \ C_0) \cap (\text{Bd} \ C_i)
\]

and \( C_0 \cap C_1 \) is the union of two disjoint polyhedral discs \( D'_i \) and \( D_i \), and (3) \( L = \bigcup_{i=0}^r C_i \). An argument similar to that given in proving Theorem 1 may be used to show that there exists 3-cells \( C'_0, C'_1, \ldots, C'_r \) such that \( \bigcup_{i=0}^r C'_i = P^{-1}[L] \) and \( C'_0, C'_1, \ldots, C'_r \) satisfy conditions relative to \( P^{-1}[L] \) analogous to those satisfied by \( C_0, C_1, \ldots, C_r \) relative to \( L \). It follows that \( P^{-1}[L] \) is a 3-cell-with-handles. It is easily seen that \( H_G \) is definable by 3-cells-with-handles.

The following result, which was announced in [3], provides a converse to Theorem 8. It may be proved by line of argument similar to that used to establish Theorem 6 and Corollary 2.

**Theorem 9.** Suppose that \( G \) is a monotone decomposition of \( E^3 \) such that (1) \( E^3/G \) is homeomorphic to \( E^3 \) and (2) \( P[H_G] \) is a compact 0-dimensional set. If \( H_G \) is definable by 3-cells-with-handles, then each element of \( G \) is point-like.

Our last result concerns the construction of a point-like decomposition of \( E^3 \) for which the associated decomposition space has certain given properties.
THEOREM 10. If $K$ is any compact 0-dimensional subset of $E^3$, there exist a point-like decomposition $G$ of $E^3$ and a homeomorphism $h$ from $E^3/G$ onto $E^3$ such that $h[P[H_G]] = K$.

Proof. Let $M_1, M_2, M_3, \ldots$ be a sequence of polyhedral 3-manifolds-with-boundary such that (1) if $i$ is any positive integer, $M_{i+1} \subset \text{Int } M_i$, and (2) $\bigcap_{i=1}^{\infty} M_i = K$. Let $P_1$ and $P_2$ be disjoint planes in $E^3$.

There is a homeomorphism $h_1$ from $E^3$ onto $E^3$ such that if $M_1'$ is any component of $M_1$, $h_1[\text{Int } M_1']$ intersects both $P_1$ and $P_2$. There is a homeomorphism $h_2$ from $E^3$ onto $E^3$ such that if $x \in E^3 - \text{Int } M_1$, $h_2(x) = x$ and if $M_2'$ is any component of $M_2$, then $h_2h_1[\text{Int } M_1']$ intersects both $P_1$ and $P_2$. Suppose that $i$ is a positive integer and $h_1, h_2, \ldots, h_i$ have been defined. There is a homeomorphism $h_{i+1}$ from $E^3$ onto $E^3$ such that if $x \in E^3 - \text{Int } M_i$, $h_{i+1}(x) = x$, and if $M_{i+1}'$ is any component of $M_{i+1}$, then $h_{i+1}h_i \cdots h_1[\text{Int } M_{i+1}']$ intersects both $P_1$ and $P_2$. Therefore there exists a sequence $h_1, h_2, h_3, \ldots$ of homeomorphisms from $E^3$ onto $E^3$ having properties indicated above. For each positive integer $i$, let $N_i$ be $h_ih_{i-1} \cdots h_1[M_i]$.

Let $G$ be the decomposition of $E^3$ whose nondegenerate elements are the components of $\bigcap_{i=1}^{\infty} N_i$. It may be proved, using [17, Chapter V, Theorem 20], that $G$ is upper semicontinuous. Note that $H_G = \bigcap_{i=1}^{\infty} N_i$.

Now we shall show that if $U$ is any open set in $E^3$ containing $H_G$ and $\varepsilon$ is any positive number, there is a homeomorphism $f$ from $E^3$ onto $E^3$ such that if $x \notin U$, $f(x) = x$ and if $g$ is any nondegenerate element of $G$, $(\text{diam } f[g]) < \varepsilon$. Suppose $U$ is open in $E^3$ and $\varepsilon$ is any positive number. There is a positive integer $j$ such that $N_j \subset U$. Since (1) for each positive integer $n$, $M_n$ is compact, (2) $h_jh_{j-1} \cdots h_1$ is a homeomorphism, and (3) the maximum of the diameters of the components of $M_n$ approaches 0 as $n$ increases without bound, there is a positive integer $k$ greater than $j$ such that if $M'_k$ is any component of $M_k$, $(\text{diam } h_kh_{k-1} \cdots h_1[M'_k]) < \varepsilon$. Let $f$ be $(h_kh_{k-1} \cdots h_{j+1})^{-1}$. Then $f[N_j] = h_jh_{j-1} \cdots h_1[M_k]$ and if $x \in E^3 - N_j$, then $f(x) = x$. Hence $f$ is a homeomorphism from $E^3$ onto $E^3$ having the specified properties.

It may now be seen, from a consideration of the proof of Theorem 1 of [4], that there is a homeomorphism $h$ from $E^3/G$ onto $E^3$ such that $h[P[H_G]] = K$. With the aid of Theorem 5, it may be proved that each element of $G$ is point-like.

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