BOL LOOPS(1)

BY

D. A. ROBINSON

1. **Introduction.** Loops $(G, \cdot)$ satisfying the property that

$$(1) \qquad (xy\cdot z)y = x(yz\cdot y),$$

for all $x, y, z \in G$, seem to have first made their appearance in the geometric considerations of G. Bol [2]. They have also been mentioned by R. H. Bruck [4, p. 116]; [5, p. 77]. Although we presently single out for attention those loops satisfying (1), it should be recognized that there exists a duality between those loops satisfying (1) and those loops for which

$$(1') \qquad y(z\cdot yx) = (y\cdot zy)x$$

for all $x, y, z \in G$. For an account of the geometric origins of loops $(G, \cdot)$ satisfying (1) and/or (1′), one may consult Bruck [5] and the references cited therein.

Loops $(G,\cdot)$ satisfying (1) (or (1′)) are more general than Moufang loops. In fact, it should be clear either from geometric or algebraic reasons that a loop $(G,\cdot)$ is Moufang if and only if $(G,\cdot)$ satisfies (1) and (1′). And it should be equally evident that Moufang loops are exactly those loops which satisfy (1) (or (1′)) and are di-associative.

It is our purpose in this paper to initiate a study of the algebraic properties of loops $(G,\cdot)$ for which (1) holds. With this in mind, we now formally state

DEFINITION 1.1. A loop $(G\cdot)$ is a *Bol loop* if and only if (1) holds for all $x, y, z \in G$.

In §2, basic properties of Bol loops are discussed. For example, it is shown that Bol loops are right alternative, satisfy the right inverse property, and are power-associative. Autotopisms are discussed and employed in a manner analogous to their role in the algebraic theory of Moufang loops. Notably, by using pseudo-automorphisms, a factorization of autotopisms is effected (see Theorem 2.6) which is similar to that achieved by Bruck [3, p. 300]; [4, p. 112] for inverse property loops and by R. Artzy [1] for crossed-inverse property loops. We conclude §2 by describing situations (see Theorems 2.7 and 2.8) in which a Bol loop is Moufang—thus amplifying some comments already made in the present section.

Received by the editors January 25, 1965.

(1) This paper constitutes a portion of the author's Ph. D. thesis written at the University of Wisconsin under the supervision of Professor J. Marshall Osborn. The research was done while the author was a National Science Foundation Science Faculty Fellow (Fellowship number 63006).

In §3, isotopy theory for Bol loops is considered. By means of geometric arguments (see Bruck [5, p. 76]), one can deduce the following: A loop $(G,\cdot)$ is a Bol loop if and only if every loop isotopic to $(G,\cdot)$ satisfies the right inverse property. We choose to present an algebraic proof (see Theorem 3.1) of this isotopy characterization of Bol loops by using techniques similar to those employed by J. M. Osborn [6]. If $(G,\cdot)$ is a Bol loop satisfying the automorphic inverse property, we are able to give a necessary and sufficient condition (see Theorem 3.2) for a loop isotopic to $(G,\cdot)$ to also satisfy the automorphic inverse property. Furthermore, we obtain a necessary and sufficient condition (see Theorem 3.3) for a loop isotopic to a Bol loop $(G,\cdot)$ to be, in fact, isomorphic to $(G,\cdot)$. We conclude §3 with an isotopic invariant for Bol loops.

The notation and terminology employed throughout this paper appears to be standard and, for the most part, coincides with that appearing in our comprehensive references [3](2), [4]. However, for the convenience of the reader, we note the following:

All mappings are tacitly assumed to be single-valued. If T is a mapping of a set G into itself or some other set and if $x\in G$, then xT shall denote the unique image of x under T. A mapping T of a set G is a *permutation* of G if and only if T is a one-to-one mapping of G onto G. If $(G,\cdot)$ is a binary system and if $x\in G$, then the mappings $R(x)$ and $L(x)$ are defined by $yR(x)=yx$ and $yL(x)=xy$ for all $y\in G$. The *left nucleus* N_λ, the *middle nucleus* N_μ, and the *right nucleus* N_ρ of a loop $(G,\cdot)$ are defined by

$$N_\lambda=\{\text{all } x\in G \mid x\cdot yz=xy\cdot z, \text{ all } y, z\in G\},$$

$$N_\mu=\{\text{all } y\in G \mid x\cdot yz=xy\cdot z, \text{ all } x, z\in G\},$$

$$N_\rho=\{\text{all } z\in G \mid x\cdot yz=xy\cdot z, \text{ all } x, y\in G\}.$$

2. **Preliminary results.** If $(G,\cdot)$ is a loop, then 1 shall denote the identity element of $(G,\cdot)$ and, for each $x\in G$, x^λ and x^ρ shall designate those unique elements in G such that $x^\lambda x=xx^\rho=1$.

THEOREM 2.1. *If $(G,\cdot)$ is a Bol loop, then*
(i) *$(G,\cdot)$ satisfies the right inverse property,*
(ii) *$y^\lambda=y^\rho$ for all $y\in G$,*
(iii) *$(G,\cdot)$ is right alternative.*

Proof. (i) In (1), let $z=y^\rho$. Then $(xy\cdot y^\rho)y=x(yy^\rho\cdot y)=xy$ for all $x,y\in G$. Hence, $xy\cdot y^\rho=x$ for all $x,y\in G$.

(ii) In (1), let $z=y^\lambda$. Then $(xy\cdot y^\lambda)y=x(yy^\lambda\cdot y)$ for all $x,y\in G$. Now using the right inverse property and the fact that $y=(y^\lambda)^\rho$, we obtain $xy=x(yy^\lambda\cdot y)$ for all $x,y\in G$. Therefore $yy^\lambda=1$ and, hence, $y^\lambda=y^\rho$ for all $y\in G$.

(2) Note that in [3] the term "associator" is used in place of the more current term "nucleus."

(iii) In (1), let $z = 1$ and get $xy \cdot y = x \cdot yy$ for all $x, y \in G$.

In view of (ii) of the preceding theorem, if $(G,\cdot)$ is a Bol loop, define x^{-1} by $x^{-1} = x^{\lambda} = x^{\rho}$ for all $x \in G$. This brings us to the following definition.

DEFINITION 2.1. If x is an element of a Bol loop $(G,\cdot)$ and n is a nonnegative integer, define x^n recursively by $x^0 = 1$ and $x^n = x^{n-1} \cdot x$ for $n > 0$. For any negative integer n, now define x^n by $x^n = (x^{-1})^{|n|}$.

LEMMA 2.1. *If $(G,\cdot)$ is a Bol loop, then*

$$xy^n = xy^{n-1} \cdot y = xy \cdot y^{n-1} \tag{2}$$

for all $x, y \in G$ and all integers n.

Proof. Clearly (2) holds for $n = 0$ and for $n = 1$. Now assume that, for $k > 1$,

$$xy^k = xy^{k-1} \cdot y = xy \cdot y^{k-1} \tag{3}$$

for all $x, y \in G$. (In particular, $y^k = y^{k-1}y = yy^{k-1}$ for all $y \in G$.) Then $xy^{k+1} = x \cdot y^k y = x(yy^{k-1} \cdot y) = (xy \cdot y^{k-1})y = xy^k \cdot y$ for all $x, y \in G$. Then, replacing x by xy in (3), we get

$$xy \cdot y^k = (xy \cdot y^{k-1})y = x(yy^{k-1} \cdot y) = x(y^{k-1}y \cdot y) = x \cdot y^k y = xy^{k+1}$$

for all $x, y \in G$. Thus, (2) holds for all integers $n \geqq 0$.

Now, for all integers $n > 0$ and all $x, y \in G$, expression (2) applied to x and y^{-1} gives

$$x(y^{-1})^{n+1} = x(y^{-1})^n \cdot y^{-1} = xy^{-n} \cdot y^{-1}$$

and (2) applied to xy and y^{-1} gives

$$xy \cdot (y^{-1})^{n+1} = (xy \cdot y^{-1})(y^{-1})^n = xy^{-n}.$$

Hence, $xy^{-n} = xy^{-n-1} \cdot y = xy \cdot y^{-n-1}$ and the proof of Lemma 2.1 is complete.

THEOREM 2.2. *If $(G,\cdot)$ is a Bol loop, then*

$$xy^m \cdot y^n = xy^{m+n} \tag{4}$$

for all $x, y \in G$ and all integers m and n. In particular, Bol loops are power-associative.

Proof. The desired result clearly holds for $n = 0$ and, by Lemma 2.1, it also holds for $n = 1$.

For any integer $n > 1$, assume that (4) holds for all integers m and all $x, y \in G$. Then, by Lemma 2.1, $xy^{m+n+1} = xy^{m+n} \cdot y = (xy^m \cdot y^n)y = xy^m \cdot y^{n+1}$ for all $x, y \in G$ and all integers m. So (4) holds for all $x, y \in G$, all integers m, and all nonnegative integers n. (In particular, for use below, $(y^n)^{-1} = y^{-n}$ for all nonnegative integers n and all $y \in G$.) Replacing m by $m - n$, we have $xy^{m-n} \cdot y^n = xy^m$

and, hence, $xy^{m-n} = xy^m \cdot (y^n)^{-1} = xy^m \cdot y^{-n}$ for all integers $n \geqq 0$, all integers m, and all $x, y \in G$.

In particular, $y^m y^n = y^{m+n}$ for all $y \in G$ and all integers m and n. Consequently, the Bol loop $(G, \cdot)$ is power-associative.

We now define autotopisms for loops in the customary manner. (For instance, see Bruck [3, p. 285].)

DEFINITION 2.2. Let U, V, and W be permutations of the set G. Then the ordered triple (U, V, W) is an *autotopism* of the loop $(G, \cdot)$ if and only if

$$xU \cdot yV = (xy)W$$

for all $x, y \in G$.

Recall that the set of all autotopisms of a loop $(G, \cdot)$ forms a group with the "*componentwise multiplication*"

$$(U_1, V_1, W_1)(U_2, V_2, W_2) = (U_1U_2, V_1V_2, W_1W_2).$$

Note that the identity element of this group is (I, I, I) where $xI = x$, all $x \in G$, and

$$(U, V, W)^{-1} = (U^{-1}, V^{-1}, W^{-1}).$$

We have occasional need for the following easily proved result.

LEMMA 2.2. *Let (U, V, W) be an autotopism of the loop $(G, \cdot)$ and let mappings R and L be defined by $xR = x^\rho$ and $xL = x^\lambda$ for all $x \in G$.*

(i) *If $(G, \cdot)$ satisfies the right inverse property, then (W, RVR, U) is an autotopism of $(G, \cdot)$.*

(ii) *If $(G, \cdot)$ satisfies the left inverse property, then (LUL, W, V) is an autotopism of $(G, \cdot)$.*

THEOREM 2.3. *A loop $(G, \cdot)$ is a Bol loop if and only if, for each $x \in G$, $(R(x)^{-1}, L(x)R(x), R(x))$ is an autotopism of $(G, \cdot)$.*

Proof. The loop $(G, \cdot)$ is a Bol loop if and only if $a(xb \cdot x) = (ax \cdot b)x$ for all $a, b, x \in G$ or, equivalently, if and only if $aR(x)^{-1} \cdot (xb \cdot x) = ab \cdot x$ for all $a, b, x \in G$.

DEFINITION 2.3. A permutation T of a set G is a *semiautomorphism* of the loop $(G, \cdot)$ if and only if

$$1T = 1$$

and

$$(xy \cdot x)T = (xT \cdot yT) \cdot xT$$

for all $x, y \in G$.

DEFINITION 2.4. A loop $(G, \cdot)$ satisfies the *automorphic inverse property* if and only if $x \to x^\rho$ is an automorphism of $(G, \cdot)$ and satisfies the *semiautomorphic inverse property* if and only if $x \to x^\rho$ is a semiautomorphism of $(G, \cdot)$.

THEOREM 2.4. *If $(G,\cdot)$ is a Bol loop, then $(G,\cdot)$ satisfies the semiautomorphic inverse property.*

Proof. Clearly $1^{-1}=1$. Now, for all $x,y\in G$, $(xy\cdot x)(x^{-1}y^{-1}\cdot x^{-1}) = [((xy\cdot x)x^{-1})y^{-1}]x^{-1} = (xy\cdot y^{-1})x^{-1} = xx^{-1} = 1$. So $(xy\cdot x)^{-1} = x^{-1}y^{-1}\cdot x^{-1}$ for all $x,y\in G$.

A connection between autotopisms and semiautomorphisms of Bol loops is afforded by the following result.

THEOREM 2.5. *If (U,T,U) is an autotopism of a Bol loop $(G,\cdot)$, then T is a semiautomorphism of $(G,\cdot)$.*

Proof. $aU\cdot bT=(ab)U$ for all a, $b\in G$. Setting $b=1$, we get $1T=1$. Let $u=1U$. Then, for $a=1$, we get $U=TL(u)$. So $xTL(u)\cdot yT=(xy)TL(u)$ for all $x,y\in G$. That is,

$$(u\cdot xT)\cdot yT = u\cdot (xy)T \tag{5}$$

for all $x,y\in G$. Replacing x by yx in (5), we get

$$[u\cdot (yx)T]\cdot yT = u\cdot (yx\cdot y)T \tag{6}$$

for all $x,y\in G$. Using (5) (with x and y interchanged) in (6), we obtain

$$[(u\cdot yT)\cdot xT]\cdot yT = u\cdot (yx\cdot y)T$$

for all $x,y\in G$. Then by (1),

$$u[(yT\cdot xT)\cdot yT] = u\cdot (yx\cdot y)T$$

and

$$(yT\cdot xT)\cdot yT = (yx\cdot y)T$$

for all $x,y\in G$.

COROLLARY 2.5.1. *If $(G,\cdot)$ is a Bol loop and if $x^2\in N_\rho$, where N_ρ is the right nucleus of $(G,\cdot)$, then $L(x)R(x)^{-1}$ is a semiautomorphism of $(G,\cdot)$.*

Proof. Since $x^2\in N_\rho$, $(I, R(x)^{-2}, R(x)^{-2})$ is an autotopism of $(G,\cdot)$. So

$$(R(x)^{-1}, L(x)R(x), R(x))(I, R(x)^{-2}, R(x)^{-2}) = (R(x)^{-1}, L(x)R(x)^{-1}, R(x)^{-1})$$

is also an autotopism of $(G,\cdot)$.

As an immediate consequence of the preceding corollary, we have

COROLLARY 2.5.2. *If $(G,\cdot)$ is a Bol loop of exponent 2, then $L(x)R(x)^{-1}$ is a semiautomorphism of $(G,\cdot)$ for each $x\in G$.*

In view of the preceding corollaries, the following examples are of interest.

EXAMPLE 2.1. Let $(R,+,\cdot)$ be the ring of integers modulo 2 and let $G = R \times R \times R$. For (i,j,k) and (p,q,r) in G, define

$$(i,j,k) \circ (p,q,r) = (i+p, j+q, k+r+jpq).$$

One can verify by a direct computation that $(G,\circ)$ is a Bol loop of order 8 with the property that $x^2 \in N_\rho$ for all $x \in G$. Note that $(G,\circ)$ is not Moufang.

EXAMPLE 2.2. Select G as in the preceding example but now, for (i,j,k) and (p,q,r) in G, define

$$(i,j,k) \circ (p,q,r) = (i+p, j+q, k+r+jp(q+1)).$$

Then $(G,\circ)$ is a Bol loop of order 8 and exponent 2. Again $(G,\circ)$ is not Moufang.

One can construct Bol loops $(G,\cdot)$ with the property that not all the mappings $L(x)R(x)^{-1}$, $x \in G$, are semiautomorphisms. (For such a construction, see Robinson [7, Chapter V].) Contrast this with the fact (see Bruck [4, p. 117]) that all inner mappings of a Moufang loop are semiautomorphisms.

DEFINITION 2.5. A permutation A of a set G is a *pseudo-automorphism* of the loop $(G,\cdot)$ if and only if there exists an element $c \in G$ such that $(A, AR(c), AR(c))$ is an autotopism of $(G,\cdot)$. Then c is referred to as a *companion* of A. (See Bruck [4, p. 113].)

We now prove that every autotopism of a Bol loop can be expressed as a product of an autotopism of the type appearing in Definition 2.5 and the inverse of an autotopism of the type presented in Theorem 2.3. Specifically,

THEOREM 2.6. *Let (U,V,W) be an autotopism of the Bol loop $(G,\cdot)$, let $u = 1U$, and let $v = 1V$. Then $A = UR(u)^{-1}$ is a pseudo-automorphism of $(G,\cdot)$ with companion $c = uv\cdot u$ such that*

$$(U,V,W) = (A, AR(c), AR(c))(R(u)^{-1}, L(u)R(u), R(u))^{-1}.$$

Proof. By Theorem 2.3, $(R(u)^{-1}, L(u)R(u), R(u))$ is an autotopism of $(G,\cdot)$. Hence,

$$(A,B,C) = (U,V,W)(R(u)^{-1}, L(u)R(u), R(u)), \tag{7}$$

where $A = UR(u)^{-1}$, $B = VL(u)R(u)$, and $C = WR(u)$, is also an autotopism of $(G,\cdot)$. That is,

$$aA \cdot bB = (ab)C$$

for all $a, b \in G$. Since $1A = 1$, setting $a = 1$, we get $B = C$. Then, for $b = 1$, we get $B = AR(1B)$. But $1B = 1VL(u)R(u) = uv\cdot u$. Hence, A is a pseudo-automorphism with companion $c = uv \cdot u$. Substitution of this information into (7) completes the proof.

We conclude the present section with two theorems which describe situations in which a Bol loop is Moufang.

THEOREM 2.7. *Let $(G,\cdot)$ be a Bol loop. Then the following statements are equivalent.*

(i) *$(G,\cdot)$ is Moufang.*

(ii) *$(G,\cdot)$ is di-associative.*

(iii) *$xy\cdot x = x\cdot yx$ for all $x, y \in G$.*

(iv) *$(G,\cdot)$ is left alternative.*

(v) *$(G,\cdot)$ satisfies the left inverse property.*

(vi) *$(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in G$.*

Proof. From well-known properties of Moufang loops (see Bruck [4, Chapter VII]), it follows that (i) implies each of the remaining statements.

(v) implies (i): By Theorem 2.3, $(R(x)^{-1}, L(x)R(x), R(x))$ is an autotopism of $(G,\cdot)$ for all $x \in G$. Now by (v) and Lemma 2.2, $(JR(x)^{-1}J, R(x), L(x)R(x))$ is also an autotopism of $(G,\cdot)$ for all $x \in G$ where $J: x \to x^{-1}$. But in an inverse property loop, $JR(x)^{-1}J = L(x)$ for all $x \in G$. So $(L(x), R(x), L(x)R(x))$ is an autotopism of $(G,\cdot)$ for all $x \in G$ and this is equivalent (see Bruck [4]) to $(G,\cdot)$ being Moufang.

(iii) implies (i): Using (iii) in expression (1), we obtain $(xy\cdot z)y = x(y\cdot zy)$ for all $x, y, z \in G$ and $(G,\cdot)$ is Moufang.

To complete the proof, it suffices to show that each of (ii), (iv), (vi) implies (iii).

(ii) implies (iii): This is obvious.

(iv) implies (iii): $(x\cdot xy)x = (xx\cdot y)x = x(xy\cdot x)$. So $(x\cdot xy)x = x(xy\cdot x)$ for all $x, y \in G$ and (iii) holds.

(vi) implies (iii): By Theorem 2.4, $(xy\cdot x)^{-1} = x^{-1}y^{-1}\cdot x^{-1}$ for all $x, y \in G$. But, by (vi), $(xy\cdot x)^{-1} = x^{-1}(xy)^{-1} = x^{-1}\cdot y^{-1}x^{-1}$ for all $x, y \in G$. So $x^{-1}y^{-1}\cdot x^{-1} = x^{-1}\cdot y^{-1}x^{-1}$ for all $x, y \in G$ and (iii) holds.

THEOREM 2.8. *A Bol loop $(G,\cdot)$ is Moufang if $(G,\cdot)$ satisfies any one of the following:*

(i) *inverse property,*

(ii) *crossed-inverse property (see Artzy [1]),*

(iii) *weak inverse property (see Osborn [6]).*

Proof. Assume that $(G,\cdot)$ satisfies (iii). Then $(xy)^{-1}x = y^{-1}$ for all $x, y \in G$. So, by the right inverse property, $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in G$. By Theorem 2.7, the Bol loop $(G,\cdot)$ is Moufang. Since each of (i) and (ii) implies (iii), the proof is complete.

In view of Theorem 2.8, it should be pointed out that a Bol loop may satisfy the automorphic inverse property without being Moufang. This is certainly illustrated by Example 2.2. For an important class of Bol loops satisfying the automorphic inverse property see Robinson [7, Chapter V].

3. **Isotopy theory of Bol loops.** For a general discussion of isotopy the reader may consult Bruck [3, Chapter I].

DEFINITION 3.1. Groupoids $(G,\cdot)$ and $(H,\circ)$ are *isotopic* (and $(H,\circ)$ is an *isotope* of $(G,\cdot)$) if and only if there exist one-to-one mappings U, V, and W of G onto H such that

$$xU\circ yV=(x\cdot y)W \tag{8}$$

for all $x, y\in G$.

The following two lemmas and their proofs are well known (see Bruck [3, Chapter I]) and are included here for reference purposes.

LEMMA 3.1. *Let $(G,\cdot)$ be a quasigroup and let $f,g\in G$. For all $x,y\in G$, let*

$$x\circ y=xR(g)^{-1}\cdot yL(f)^{-1}. \tag{9}$$

Then $(G,\circ)$ is a loop and $(G,\cdot)$ and $(G,\circ)$ are isotopic.

A loop $(G,\circ)$ obtained from a quasigroup $(G,\cdot)$ in the manner described in the preceding lemma is called a *principal isotope* of $(G,\cdot)$.

LEMMA 3.2. *If the quasigroup $(G,\cdot)$ and the loop $(H,\circ)$ are isotopic, then $(H,\circ)$ is isomorphic to a principal isotope of $(G,\cdot)$.*

Now let $(G,\cdot)$ be a loop which satisfies the right inverse property and let $f, g\in G$. For all $x,y\in G$, define $x\circ y$ by (9). For $x\in G$, recall that x^{λ} and x^{ρ} are those unique elements in G such that $x^{\lambda}\cdot x=x\cdot x^{\rho}=1$. Since $f\cdot g$ is the identity element for $(G,\circ)$, for each $x\in G$, let $x^{\rho\circ}$denote that unique element in G such that $x\circ x^{\rho\circ}=f\cdot g$. Define mappings L, R, and R_0 by

$$xL=x^{\lambda},\ xR=x^{\rho},\ xR_0=x^{\rho\circ} \tag{10}$$

for all $x\in G$. Then, for all $x\in G$,

$$xR(g)^{-1}\cdot xR_0L(f)^{-1}=f\cdot g$$

and, by the right inverse property for $(G,\cdot)$,

$$xR(g)^{-1}=(f\cdot g)\cdot(xR_0L(f)^{-1}R).$$

So we have

$$R_0=R(g)^{-1}L(f\cdot g)^{-1}LL(f). \tag{11}$$

Note that $(x\circ y)\circ yR_0=x$, for all $x,y\in G$, if and only if

$$(xR(g)^{-1}\cdot yL(f)^{-1})R(g)^{-1}\cdot yR_0L(f)^{-1}=x$$

for all $x,y\in G$. The latter holds, replacing x by ug and y by fv, if and only if

$$(u \cdot v) R(g)^{-1} \cdot (f \cdot v) R_0 L(f)^{-1} = u \cdot g$$

for all $u, v \in G$ or, using the right inverse property for $(G, \cdot)$, if and only if

$$(u \cdot v) R(g)^{-1} = (u \cdot g) \cdot (f \cdot v) R_0 L(f)^{-1} R$$

for all $u, v \in G$. Now, using (11), this is equivalent to

$$\alpha(f, g) = (R(g), L(f) R(g)^{-1} L(f \cdot g)^{-1}, R(g)^{-1})$$

being an autotopism of $(G, \cdot)$. We have, therefore, proved the following:

LEMMA 3.3. *Let $(G, \cdot)$ be a loop satisfying the right inverse property, let f, $g \in G$, and let $x \circ y$ be defined by* (9) *for all $x, y \in G$. Then the principal isotope $(G, \circ)$ also satisfies the right inverse property if and only if $\alpha(f, g)$ is an autotopism of $(G, \cdot)$.*

The following theorem constitutes an isotopy characterization of Bol loops.

THEOREM 3.1. *If $(G, \cdot)$ is a loop, then the following statements are equivalent.*
(i) *$(G, \cdot)$ is a Bol loop.*
(ii) *Each loop isotopic to $(G, \cdot)$ satisfies the right inverse property.*
(iii) *Each loop isotopic to $(G, \cdot)$ is right alternative.*

Proof. By virtue of Lemma 3.2, we need only be concerned with principal isotopes of $(G, \cdot)$.

It is immediate from Definition 1.1 that a loop $(G, \cdot)$ is a Bol loop if and only if $L(xy) R(y) = L(y) R(y) L(x)$ for all $x, y \in G$ or, equivalently, if and only if

$$R(y)^{-1} L(y)^{-1} = L(x) R(y)^{-1} L(xy)^{-1} \tag{12}$$

for all $x, y \in G$.

Assume that (i) holds. Then $(G, \cdot)$ satisfies the right inverse property and $(R(g)^{-1}, L(g) R(g), R(g))$ is an autotopism of $(G, \cdot)$ for each $g \in G$. Taking the inverse of this autotopism and using (12), we see that $\alpha(f, g)$ is an autotopism of $(G, \cdot)$ for all $f, g \in G$. So, by Lemma 3.3, statement (ii) holds.

Now assume that (ii) holds. In particular, $(G, \cdot)$ satisfies the right inverse property and, by Lemma 3.3, $\alpha(f, g)$ is an autotopism of $(G, \cdot)$ for all $f, g \in G$. Recall that $\alpha(f, g) = (R(g), Y, R(g)^{-1})$ where $Y = L(f) R(g)^{-1} L(fg)^{-1}$. So $ag \cdot bY = (ab) R(g)^{-1}$ for all $a, b \in G$. For $a = 1$, we get $Y = R(g)^{-1} L(g)^{-1}$. Comparing the two expressions for Y, we get $R(g)^{-1} L(g)^{-1} = L(f) R(g)^{-1} L(fg)^{-1}$ for all $f, g \in G$. So (12) holds for all $x, y \in G$ and $(G, \cdot)$ is a Bol loop.

Recall that isotopy is an equivalence relation on the set of all loops. Hence, from the equivalence of (i) and (ii) of Theorem 3.1, it follows that *each loop isotopic to a Bol loop is a Bol loop*. It is evident, therefore, that (i) implies (iii).

Now assume that (iii) holds. That is,

$$[x R(g)^{-1} \cdot y L(f)^{-1}] R(g)^{-1} \cdot y L(f)^{-1} = x R(g)^{-1} \cdot [y R(g)^{-1} \cdot y L(f)^{-1}] L(f)^{-1} \tag{13}$$

for all x, y, f, $g \in G$. For $x = g$ and $y = f$ in (13), we get $g^{\lambda} = fR(g)^{-1}L(f)^{-1}$ for all f, $g \in G$. Consequently,

$$R(g)^{-1} = R(g^{\lambda}) \tag{14}$$

for all $g \in G$. Letting $x = g$, replacing y by $yL(f)$ in (13), and using (14), we obtain

$$f(yg^{\lambda} \cdot y) = (fy \cdot g^{\lambda})y \tag{15}$$

for all $f, g, y \in G$. Hence, $(G, \cdot)$ is a Bol loop and the proof of Theorem 3.1 is complete.

From Theorem 2.8 and from the fact that every loop isotopic to a Moufang loop is Moufang (see Bruck [3, p. 304]), we get the following:

COROLLARY 3.1.1. *If the Bol loop $(G, \cdot)$ is isotopic to the loop $(H, \circ)$ and if $(H, \circ)$ satisfies any of the following properties:*
(i) *inverse property,*
(ii) *crossed-inverse property,*
(iii) *weak inverse property,*
then $(G, \cdot)$ and $(H, \circ)$ are isotopic Moufang loops.

In the case of Bol loops, Lemma 3.2 can be somewhat improved. (For a proof of the following lemma, see Bruck [4, p. 129]. Although Bruck's result is stated for Moufang loops, his proof holds more generally for Bol loops.)

LEMMA 3.4. *Let $(G, \cdot)$ be a Bol loop. Each loop isotopic to $(G, \cdot)$ is isomorphic to a principal isotope $(G, \circ)$ where $x \circ y = xR(f) \cdot yL(f)^{-1}$ for all $x, y \in G$ and some $f \in G$.*

THEOREM 3.2. *Let $(G, \cdot)$ be a Bol loop with the automorphic inverse property, let $f \in G$, and let*

$$x \circ y = xR(f) \cdot yL(f)^{-1}$$

for all $x, y \in G$. Then $(G, \circ)$ satisfies the automorphic inverse property if and only if $f \in N_{\lambda}$ where N_{λ} is the left nucleus of $(G, \cdot)$.

Proof. Define the mappings L, R, and R_0 as in (10). Since $(G, \cdot)$ is a Bol loop, we may let $J = L = R$. Then using (11) with $g = f^{-1}$, we get

$$R_0 = R(f)JL(f). \tag{16}$$

The loop $(G, \circ)$ has the automorphic inverse property if and only if

$$(xR(f) \cdot yL(f)^{-1})R_0 = xR_0R(f) \cdot yR_0L(f)^{-1} \tag{17}$$

for all $x, y \in G$. Using (16), setting $x = uR(f)^{-1}$, and setting $y = vL(f)$, expression (17) becomes

$$(u \cdot v)R(f)JL(f) = uJL(f)R(f) \cdot vL(f)R(f)J.$$

The latter holds for all $u, v \in G$ if and only if

$$\alpha = (JL(f)R(f),\ L(f)R(f)J,\ R(f)JL(f))$$

is an autotopism of $(G,\cdot)$. Since $(G,\cdot)$ satisfies the automorphic inverse property, (J,J,J) is an autotopism of $(G,\cdot)$. So α is an autotopism of $(G,\cdot)$ if and only if

$$\beta = \alpha(J,J,J)(R(f^{-1})^{-1},\ L(f^{-1})R(f^{-1}),\ R(f^{-1}))$$

is an autotopism of $(G,\cdot)$. Since $(G,\cdot)$ is a Bol loop,

$$xL(f)R(f)L(f^{-1})R(f^{-1}) = [f^{-1}(fx\cdot f)]f^{-1} = [(f^{-1}f\cdot x)f]f^{-1} = x$$

for all $x \in G$. That is, $L(f)R(f)L(f^{-1})R(f^{-1}) = I$. Also, since J is an automorphism of $(G,\cdot)$, we have $R(f)J = JR(f^{-1})$ and $L(f)J = JL(f^{-1})$. Thus,

$$\beta = (L(f^{-1}),\ I,\ R(f)L(f^{-1})R(f^{-1})).$$

Hence, $(G,\circ)$ satisfies the automorphic inverse property if and only if β is an autotopism of $(G,\cdot)$.

Now assume that β is an autotopism of $(G,\cdot)$. Then

$$xL(f^{-1})\cdot y = (xy)R(f)L(f^{-1})R(f^{-1})$$

for all $x, y \in G$. For $y = 1$, we get $L(f^{-1}) = R(f)L(f^{-1})R(f^{-1})$. Therefore, $\beta = (L(f^{-1}), I, L(f^{-1}))$ and $f^{-1} \in N_\lambda$. Hence, $f \in N_\lambda$.

On the other hand, suppose that $f \in N_\lambda$. Then $\gamma = (L(f), I, L(f))$ is an autotopism of $(G,\cdot)$. But $f \in N_\lambda$ implies that $L(f)^{-1} = L(f^{-1}) = R(f)L(f^{-1})R(f^{-1})$. Hence, $\beta = \gamma^{-1}$ and β is an autotopism of $(G,\cdot)$.

COROLLARY 3.2.1. *Let $(G,\cdot)$ be a Bol loop with the automorphic inverse property. Then all loops isotopic to $(G,\cdot)$ satisfy the automorphic inverse property if and only if $(G,\cdot)$ is a commutative group.*

Proof. Suppose that every loop isotopic to $(G,\cdot)$ satisfies the automorphic inverse property. Then $f \in N_\lambda$ for all $f \in G$. Hence, $(G,\cdot)$ is a group. Furthermore, $y^{-1}x^{-1} = (xy)^{-1} = x^{-1}y^{-1}$ for all, $x, y \in G$. So $(G,\cdot)$ is commutative. The converse is evident if one recalls (see Bruck [3, p. 255]) that every loop isotopic to a group is an isomorphic group.

The next corollary is immediate.

COROLLARY 3.2.2. *Let $(G,\cdot)$ be a Bol loop with the automorphic inverse property. Then every loop isotopic to $(G,\cdot)$ is isomorphic to $(G,\cdot)$ if and only if $(G,\cdot)$ is a commutative group.*

THEOREM 3.3. *Let $(G,\cdot)$ be a Bol loop, let $f \in G$, and let $x \circ y = xR(f)\cdot yL(f)^{-1}$ for all $x, y \in G$. Then $(G,\cdot)$ and $(G,\circ)$ are isomorphic if and only if there exists a pseudo-automorphism of $(G,\cdot)$ with companion f.*

Proof. The loops $(G,\cdot)$ and $(G,\circ)$ are isomorphic if and only if there exists a permutation T of G such that $xT\circ yT=(x\cdot y)T$ for all $x,y\in G$ or, equivalently, if and only if $xTR(f)\cdot yTL(f)^{-1}=(x\cdot y)T$ for all $x,y\in G$. The latter holds if and only if $\alpha=(TR(f),\ TL(f)^{-1},\ T)$ is an autotopism of $(G,\cdot)$. And α is an autotopism of $(G,\cdot)$ if and only if $\beta=\alpha(R(f)^{-1},\ L(f)R(f),\ R(f))$ is an autotopism of $(G,\cdot)$. But $\beta=(T,\ TR(f),\ TR(f))$.

COROLLARY 3.3.1. *Let $(G,\cdot)$ be a Bol loop, let $f\in G$, and let $x\circ y=xR(f)\cdot yL(f)^{-1}$ for all $x,y\in G$. If $f\in N_\rho$, where N_ρ is the right nucleus of $(G,\cdot)$, then $(G,\cdot)$ and $(G,\circ)$ are isomorphic.*

Proof. $f\in N_\rho$ implies that $I:x\to x$ is a pseudo-automorphism of $(G,\cdot)$ with companion f.

COROLLARY 3.3.2. *Let $(G,\cdot)$ be a Bol loop. Then every loop isotopic to $(G,\cdot)$ is isomorphic to $(G,\cdot)$ if and only if each element in G is a companion for a pseudo-automorphism of $(G,\cdot)$.*

Proof. This is an immediate consequence of Lemma 3.4 and Theorem 3.3.

We deem it appropriate at this stage to raise the following question: If $(G,\cdot)$ is a Bol loop which is isomorphic to all of its loop isotopes, is $(G,\cdot)$ necessarily Moufang?

We now wish to present an isotopic invariant for Bol loops. For this purpose, we introduce the "core" of a Bol loop $(G,\cdot)$. It should be noted that our construction reduces to that of Bruck [4, p. 120] when $(G,\cdot)$ is Moufang.

DEFINITION 3.2. Let $(G,\cdot)$ be a Bol loop. For all $x,y\in G$, define $x+y$ by $x+y=xy^{-1}\cdot x$. The groupoid $(G,+)$ is called the *core* of $(G,\cdot)$.

LEMMA 3.5. *Let $(G,\cdot)$ be a Bol loop, let $f\in G$, and let $x\circ y=xR(f)\cdot yL(f)^{-1}$ for all $x,y\in G$. Let T be a permutation of G and let $(G,+)$ and $(G,\oplus)$ be the cores of $(G,\cdot)$ and $(G,\circ)$ respectively. Then*

$$xT\oplus yT=(x+y)T \tag{18}$$

for all $x,y\in G$ if and only if

$$[(fx\cdot y^{-1})x]T^{-1}=[(fx)T^{-1}\cdot(fy)T^{-1}J]\cdot(fx)T^{-1} \tag{19}$$

for all $x,y\in G$ where $J:x\to x^{-1}$.

Proof. Define R_0 as in (10). Then, using (11), we get $R_0=R(f)JL(f)$. Then

$$\begin{aligned} x\oplus y &= (x\circ yR_0)\circ x\\ &= [xR(f)\cdot yR_0L(f)^{-1}]R(f)\cdot xL(f)^{-1}\\ &= [xR(f)\cdot yR(f)J]R(f)\cdot xL(f)^{-1}. \end{aligned}$$

Then (18) holds for all $x, y \in G$ if and only if

$$[xTR(f) \cdot yTR(f)J]R(f) \cdot xTL(f)^{-1} = (xy^{-1} \cdot x)T$$

for all $x, y \in G$. Replacing x by $xL(f)T^{-1}$ and y by $yR(f)^{-1}T^{-1}$, the latter holds if and only if

$$(xL(f)R(f) \cdot y^{-1})R(f) \cdot x = \{[xL(f)T^{-1} \cdot yR(f)^{-1}T^{-1}J] \cdot xL(f)T^{-1}\}T$$

for all $x, y \in G$. And, using Bol loop properties, this can be rewritten as

$$\{[(fx)(fy^{-1} \cdot f)]x\}T^{-1} = \{(fx)T^{-1} \cdot [f(f^{-1}y \cdot f^{-1})]T^{-1}J\} \cdot (fx)T^{-1}$$

for all $x, y \in G$. Now letting $z = f^{-1}y \cdot f^{-1}$, we get

$$[(fx \cdot z^{-1})x]T^{-1} = [(fx)T^{-1} \cdot (fz)T^{-1}J] \cdot (fx)T^{-1}$$

which is, aside from an obvious notational change, expression (19).

THEOREM 3.4. *The core is an isotopic invariant for Bol loops. More precisely, isotopic Bol loops have isomorphic cores.*

Proof. In view of Lemma 3.4, we consider only those isotopes $(G, \circ)$ where $x \circ y = xR(f) \cdot yL(f)^{-1}$ for all $x, y \in G$. Let $(G, +)$ and $(G, \oplus)$ be the cores of $(G, \cdot)$ and $(G, \circ)$ respectively.

Since $(G, \cdot)$ is a Bol loop, $(fx \cdot y^{-1})x = f(xy^{-1} \cdot x)$ for all $x, y \in G$. Then $[(fx \cdot y^{-1})x]L(f)^{-1} = xy^{-1} \cdot x$ for all $x, y \in G$. And

$$[(fx \cdot y^{-1})x]L(f)^{-1} = [(fx)L(f)^{-1} \cdot (fy)L(f)^{-1}J] \cdot (fx)L(f)^{-1}$$

for all $x, y \in G$. Hence, (19) holds for all $x, y \in G$ with $T = L(f)$. So, by Lemma 3.5, $(G, +)$ and $(G, \oplus)$ are isomorphic.

As one final application of Lemma 3.5, we present the following characterization of Moufang loops.

THEOREM 3.5. *A Bol loop $(G, \cdot)$ is Moufang if and only if, for each principal isotope $(G, \circ)$ where $x \circ y = xR(f) \cdot yL(f)^{-1}$,*

$$x + y = x \oplus y$$

for all $x, y \in G$ where $(G, +)$ and $(G, \oplus)$ are the cores of $(G, \cdot)$ and $(G, \circ)$ respectively.

Proof. If $(G, \cdot)$ is Moufang, then $x + y = x \oplus y$, all $x, y \in G$, for a l principal isotopes $(G, \circ)$ of $(G, \cdot)$. (See Bruck [4, p. 121].)

Now suppose that $x + y = x \oplus y$ for all $x, y \in G$ and all principal isotopes $(G, \circ)$ of the form indicated above. Then, by Lemma 3.5 with $T = I$,

$$(fx \cdot z^{-1})x = [(fx)(fz)^{-1}](fx)$$

for all x, z, $f \in G$. In particular, for $x = f^{-1}$, we get $z^{-1}f^{-1} = (fz)^{-1}$ for all $z, f \in G$. So, by Theorem 2.7, the loop $(G, \cdot)$ is Moufang.

ADDENDA. Recently G. Glauberman (*On loops of odd order*, J. Algebra **1** (1964), 374–396) has studied automorphic inverse property loops satisfying our condition (1′). He, furthermore, assumes that each element of such a loop has finite odd order and refers to such loops as B-loops. Those results so kindly attributed to the author by Dr. Glauberman in his paper appear as Theorem 2.2 of the present paper.

REFERENCES

1. R. Artzy, *Crossed-inverse and related loops*, Trans. Amer Math. Soc. **91** (1959), 480–492.
2. G. Bol, *Gewebe und Gruppen*, Math. Ann. **114** (1937), 414–431.
3. R. H. Bruck, *Contributions to the theory of loops*, Trans. Amer. Math. Soc. **60** (1946), 245–354.
4. ———, *A survey of binary systems*, Springer-Verlag, Berlin, 1958.
5. ———, *What is a loop?*, Studies in Modern Mathematics Vol. 2, Math. Assoc. America, Prentice-Hall, Englewood Cliffs, N. J., 1963, pp. 59–99
6. J. M. Osborn, *Loops with the weak inverse property*, Pacific J. Math. **10** (1960), 295–404.
7. D. A. Robinson, *Bol loops*, Ph. D. Thesis, University of Wisconsin, Madison, Wis., 1964.

GEORGIA INSTITUTE OF TECHNOLOGY,
ATLANTA, GEORGIA