

THE HAUSDORFF-BESICOVICH DIMENSION OF THE LEVEL SETS OF PERRON'S MODULAR FUNCTION

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1. **Introduction.** The modular function M was introduced by O. Perron in [5]. $M(\xi)$ (for irrational ξ) is defined by the property that, for any $\varepsilon > 0$, the inequality

$$\left| \xi - \frac{p}{q} \right| < \frac{1 + \varepsilon}{M(\xi)q^2}$$

has infinitely many integer solutions (p, q) while the inequality

$$\left| \xi - \frac{p}{q} \right| < \frac{1 - \varepsilon}{M(\xi)q^2}$$

has only finitely many. Let

$$\xi = \left[x_1, x_2, \dots \right] = \frac{1}{x_1 +} \frac{1}{x_2 +} \dots$$

be the continued fraction expansion of ξ (we will only use the continued fraction expansion for numbers in the open interval $(0, 1)$) and set

$$M_k(\xi) = x_k + \left(\frac{1}{x_{k-1} +} \frac{1}{x_{k-2} +} \dots + \frac{1}{x_1} \right) + \left(\frac{1}{x_{k-1} +} \frac{1}{x_{k+2} +} \dots \right).$$

It is easily shown (see [5]) that

$$M(\xi) = \limsup_{k \rightarrow \infty} M_k(\xi).$$

For every positive number γ let

$$L(\gamma) = \left[\xi \mid M(\xi) = \gamma \right]$$

be the level set of M at γ . The theorem of this paper provides an estimate of the Hausdorff-Besicovich dimension of $L(\gamma)$ for sufficiently large γ . The Hausdorff-Besicovich dimension of a set S , which we will write $\dim(S)$, is defined as follows: let (I_i) be a covering of S by intervals, and let $|I_i|$ be the length of I_i ; then $\delta = \text{l.u.b. } |I_i|$ is called the norm of the covering;

$$\Gamma(\lambda, S) = \lim_{\delta \rightarrow \infty} \text{g.l.b. } \sum |I_i|^\lambda,$$

where the greatest lower bound is taken over all coverings of norm δ , is the

λ -dimensional Hausdorff measure of S and $\dim(S)$ is the number such that, for every positive ε ,

$$\Gamma(\dim(S) - \varepsilon, S) = \infty$$

and

$$\Gamma(\dim(S) + \varepsilon, S) = 0.$$

We will write E_N for the set of those numbers in $(0, 1)$ whose continued fraction expansions involve no integer bigger than N , s_N for the smallest number in E_N and l_N for the largest. It is easily seen that

$$s_N = [N, 1, N, 1, \dots] = \frac{(N^2 + 4N)^{1/2} - N}{2N}$$

and

$$l_N = [1, N, 1, N, \dots] = \frac{(N^2 + 4N)^{1/2} - N}{2}.$$

The following facts on continued fraction expansions can be found in [3]. Let $\xi = [x_1, x_2, \dots]$. Then the integers $P_k(\xi)$ and $Q_k(\xi)$ defined by

$$\begin{aligned} P_0(\xi) &= 0, P_1(\xi) = 1, \\ Q_0(\xi) &= 1, Q_1(\xi) = x_1, \\ P_{k+1}(\xi) &= x_{k+1} P_k(\xi) + P_{k-1}(\xi), \\ Q_{k+1}(\xi) &= x_{k+1} Q_k(\xi) + Q_{k-1}(\xi) \end{aligned}$$

satisfy

$$\begin{aligned} \frac{P_k(\xi)}{Q_k(\xi)} &= \frac{1}{x_1 +} \frac{1}{x_2 +} \dots + \frac{1}{x_k}, \\ P_{k-1}(\xi) Q_k(\xi) - P_k(\xi) Q_{k-1}(\xi) &= (-1)^k. \end{aligned}$$

If we set $\eta = [x_{k+1}, x_{k+2}, \dots]$ then

$$\xi = \frac{(x_k + \eta) P_{k-1}(\xi) + P_{k-2}(\xi)}{(x_k + \eta) Q_{k-1}(\xi) + Q_{k-2}(\xi)}.$$

If $\xi' = [x'_1, x'_2, \dots]$ where $x_i = x'_i$ for $i < k$ and if we set $\eta' = [x'_{k+1}, x'_{k+2}, \dots]$ then

$$\begin{aligned} (1) \quad |\xi - \xi'| &= \frac{|x_k + \eta - x'_k - \eta'|}{((x_k + \eta) Q_{k-1}(\xi) + Q_{k-2}(\xi)) ((x'_k + \eta') Q_{k-1}(\xi) + Q_{k-2}(\xi))} \\ &= \frac{|x_k + \eta - x'_k - \eta'|}{(Q_k(\xi) + \eta Q_{k-1}(\xi)) (Q_k(\xi') + \eta' Q_{k-1}(\xi'))}. \end{aligned}$$

$Q_k(\xi)$ is an increasing function of k and satisfies

$$(2) \quad \left(\prod_{k=j+1}^{j+l} x_k \right) Q_j(\xi) \leq Q_{j+l}(\xi) \left(\prod_{k=j+1}^{j+l} (x_k + 1) \right) Q_j(\xi).$$

2. **Preliminary lemmas.** We define intervals $A_N, B_N,$ and C_N by;

$$\begin{aligned} A_4 &= \left[5 + l_4 - \frac{1}{6 + l_4}, 5 + 2l_4 \right), \\ B_4 &= [5 + 2l_4, 5 + 2l_5), \\ A_N &= [N + 2l_N, N + 1 + 2l_4), \quad N \geq 5, \\ B_N &= [N + 1 + 2l_4, N + 1 + 2l_{N+1}), \quad N \geq 5, \\ C_N &= A_N \cup B_N, \quad N \geq 4. \end{aligned}$$

Our first lemma is a direct corollary of a theorem of Marshall Hall.

LEMMA 1. *If γ is in $C_N, N \geq 4$ we can write*

$$\gamma = F(\gamma) + \alpha + \beta$$

where α and β are in E_4 and $F(\gamma)$ is $N + 1$ for γ in A_N and is $N + 2$ for γ in B_N .

Proof. By Theorem 3.1 of [1] every number γ in $[M + 2s_4, M + 2l_4]$ can be written in the form $M + \alpha + \beta$ with α and β in E_4 . Straightforward computation shows that $[N + 1 + 2s_4, N + 1 + 2l_4]$ contains A_N and that $[N + 2 + 2s_4, N + 2 + 2l_4]$ contains B_N for all $N \geq 4$.

Suppose now that $\gamma = F(\gamma) + \alpha + \beta$ is in C_N with $\alpha = [a_1, a_2, \dots]$ and $\beta = [b_1, b_2, \dots]$ in E_4 . We choose a sequence (p_i) of integers satisfying

$$p_1 \geq 3$$

and

$$\lim_{k \rightarrow \infty} \frac{k^2}{S_k} = 0$$

where

$$\begin{aligned} S_k &= 0 \quad \text{if } k = 0 \\ &= \sum_{i=1}^k p_i \quad \text{if } k > 0. \end{aligned}$$

For any $\xi = [x_1, x_2, \dots]$ in E_N we define $\phi(\xi) = [y_1, y_2, \dots]$ by setting

$$\begin{aligned} y_j &= x_{j-n(n-1)} \quad \text{if } S_{n-1} + n(n-1) < j \leq S_n + n(n-1), \\ &= a_{n+1-l} \quad \text{if } j = S_n + n(n-1) + l, & l = 1, \dots, n, \\ &= F(\gamma) \quad \text{if } j = S_n + n^2 + 1, \\ &= b_l \quad \text{if } j = S_n + n^2 + 1 + l, & l = 1, \dots, n-1, \end{aligned}$$

for $n = 1, 2, \dots$. If we set

$$\pi(j) = j + n(n - 1) \quad \text{if } S_{n-1} < j \leq S_n$$

then $x_j = y_{\pi(j)}$, j and $\pi(j)$ have the same parity and

$$1 \leq \frac{\pi(j)}{j} = 1 + \frac{n(n - 1)}{j} \leq 1 + \frac{n(n - 1)}{S_{n-1}} \rightarrow 1.$$

LEMMA 2. ϕ is an order preserving map of E_N into $L(\gamma)$.

Proof. If $y_k = 4$

$$\begin{aligned} M_k(\phi(\xi)) &\leq \max\left(4 + l_4 + \frac{1}{1 + (1/F(\gamma) + l_4)}, 4 + l_4 + l_N\right) + O(1) \\ &= 5 + l_4 - \frac{1}{F(\gamma) + 1 + l_4} + O(1), \end{aligned}$$

and if $y_k = N$

$$M_k(\phi(\xi)) \leq N + 2l_N + O(1).$$

It follows from this that

$$\begin{aligned} \limsup_{k \rightarrow \infty} M_k(\phi(\xi)) &= \lim_{n \rightarrow \infty} M_{S_n + n^2 + 1}(\phi(\xi)) \\ &= F(\gamma) + \alpha + \beta = \gamma, \end{aligned}$$

i.e. that $\phi(\xi)$ is in $L(\gamma)$.

Now suppose that $\xi = [x_1, x_2, \dots] < \xi' = [x'_1, x'_2, \dots]$ and let k be the first integer for which $x_k \neq x'_k$. Then either k is odd and $x'_k < x_k$ or k is even and $x'_k > x_k$. If $\phi(\xi) = [y_1, y_2, \dots]$ and $\phi(\xi') = [y'_1, y'_2, \dots]$ then $y_j = y'_j$ for $j < \pi(k)$ and either $\pi(k)$ is odd and $y'_{\pi(k)} = x'_k < x_k = y_{\pi(k)}$ or $\pi(k)$ is even and $y'_{\pi(k)} = x'_k > x_k = y_{\pi(k)}$. Hence $\phi(\xi) < \phi(\xi')$.

Set, for $k \geq 2$,

$$\rho(k) = \pi(k) - k + \pi(k - 1) - (k - 1).$$

Note that ρ is an increasing function of k and that

$$\lim_{k \rightarrow \infty} \frac{\rho(k)}{k} = 2 \lim_{k \rightarrow \infty} \left(\frac{\pi(k)}{k} - 1 \right) = 0.$$

LEMMA 3. For $k \geq 2$

$$Q_k(\xi) \leq Q_{\pi(k)}(\phi(\xi)) \leq (N + 3)^{\rho(k)} Q_k(\xi).$$

Proof. Since $p_1 \geq 3$ the lemma holds for $k = 2$ or 3 . Suppose it holds for all $j \leq k$. Then if $\xi = [x_1, x_2, \dots]$ is in E_N and $\phi(\xi) = [y_1, y_2, \dots]$ we have

$$\begin{aligned} Q_{\pi(k+1)}(\phi(\xi)) &= y_{\pi(k+1)}Q_{\pi(k+1)-1}(\phi(\xi)) + Q_{\pi(k+1)-2}(\phi(\xi)) \\ &\geq x_{k+1}Q_{\pi(k)}(\phi(\xi)) + Q_{\pi(k-1)}(\phi(\xi)) \\ &\geq x_{k+1}Q_k(\xi) + Q_{k-1}(\xi) = Q_{k+1}(\xi) \end{aligned}$$

so the first inequality holds for $k + 1$.

Now suppose that $\pi(k \pm 1) = \pi(k) \pm 1$ and hence that $\rho(k + 1) = \rho(k)$.

Then

$$\begin{aligned} Q_{\pi(k+1)}(\phi(\xi)) &= y_{\pi(k+1)}Q_{\pi(k)}(\phi(\xi)) + Q_{\pi(k-1)}(\phi(\xi)) \\ &\leq (N + 3)^{\rho(k)}(x_{k+1}Q_k(\xi) + Q_{k-1}(\xi)) \\ &= (N + 3)^{\rho(k)}Q_{k+1}(\xi). \end{aligned}$$

If $\pi(k + 1) = \pi(k) + 2n + 1$, i.e. if $k = S_n$, then $\pi(k - 1) = \pi(k) - 1$ and $\rho(k + 1) = \rho(k) + 2n$. We have, using (2),

$$\begin{aligned} Q_{\pi(k+1)}(\phi(\xi)) &= y_{\pi(k+1)}Q_{\pi(k+1)-1}(\phi(\xi)) + Q_{\pi(k+1)-2}(\phi(\xi)) \\ &\leq (N + 3)^{2n}(x_{k+1}Q_{\pi(k)}(\phi(\xi)) + Q_{\pi(k-1)}(\phi(\xi))) \\ &\leq (N + 3)^{2n+\rho(k)}(x_{k+1}Q_k(\xi) + Q_{k-1}(\xi)) \\ &= (N + 3)^{\rho(k+1)}Q_{k+1}(\xi). \end{aligned}$$

The case $k - 1 = S_n$, $\pi(k + 1) = \pi(k) + 1$, $\pi(k - 1) = \pi(k) - 2n - 1$, $\rho(k + 1) = \rho(k) + 2n$ is handled similarly.

LEMMA 4. *If $\xi = [x_1, x_2, \dots]$ and $\xi' = [x'_1, x'_2, \dots]$ are in E_M and $x_i = x'_i$ for $i < k$ but $x_k \neq x'_k$ then*

$$|\xi - \xi'| \geq \frac{1 - (l_M - s_M)}{4Q_k(\xi)Q_k(\xi')} \geq \frac{1 - (l_M - s_M)}{4(M + 1)^{2k}}.$$

Proof. Setting $\eta = [x_{k+1}, x_{k+2}, \dots]$ and $\eta' = [x'_{k+1}, x'_{k+2}, \dots]$ and using formulas (1) and (2) we have

$$\begin{aligned} |\xi - \xi'| &= \frac{|x_k + \eta - x'_k - \eta'|}{(Q_k(\xi) + \eta Q_{k-1}(\xi))(Q_k(\xi') + \eta' Q_{k-1}(\xi'))} \\ &\geq \frac{1 - (l_M - s_M)}{4Q_k(\xi)Q_k(\xi')} \geq \frac{1 - (l_M - s_M)}{4(M + 1)^{2k}}. \end{aligned}$$

LEMMA 5. *If $\xi = [x_1, x_2, \dots]$ and $\xi' = [x'_1, x'_2, \dots]$ are in E_N and $x_i = x'_i$ for $i < k$ but $x_k \neq x'_k$ then there is an A depending only on N such that*

$$|\phi(\xi) - \phi(\xi')| \geq A(N + 3)^{-2\rho(k)}|\xi - \xi'|.$$

Proof. By Lemmas 3 and 4

$$\begin{aligned}
 |\phi(\xi) - \phi(\xi')| &\geq \frac{1 - (l_{N+2} - s_{N+2})}{4Q_{\pi(k)}(\phi(\xi))Q_{\pi(k)}(\phi(\xi'))} \\
 &\geq \frac{1 - (l_{N+2} - s_{N+2})}{4(N+3)^{2\rho(k)}} \frac{1}{Q_k(\xi)Q_k(\xi')}.
 \end{aligned}$$

But, if $\eta = [x_k, x_{k+1}, \dots]$ and $\eta' = [x'_k, x'_{k+1}, \dots]$, then

$$\begin{aligned}
 |\xi - \xi'| &= \frac{|x_k + \eta - (x'_k + \eta')|}{(Q_k(\xi) + \eta Q_{k-1}(\xi))(Q_k(\xi') + \eta' Q_{k-1}(\xi'))} \\
 &\leq \frac{N}{Q_k(\xi) Q_k(\xi')}
 \end{aligned}$$

so the lemma holds with

$$A = (1 - (l_{N+2} - s_{N+2}))/4N .$$

3. The dimension of $L(\gamma)$. In §7 of [4] a probability measure ω^* on $[0, 1]$ was defined for every pair of positive integers M_1 and M_2 with $M_1 > M_2 + 1$. We take ω_N^* to be one that corresponds to $M_1 = N + 1$ and $M_2 = 1$ so that $\omega_N^*(E_N) = 1$. Let

$$h_N = -2 \int_0^1 \log t \omega_N^*(dt).$$

It is clear from the estimates in [4] that $0 < h_N < \infty$. For every $\varepsilon > 0$ and integer l set

$$E'_N(\varepsilon, l) = \left[\xi \mid \xi \in E_N \text{ and } \left| \frac{P_k(\xi)}{Q_k(\xi)} - \xi \right| \leq \exp[-k(h_N - \varepsilon)] \text{ for all } k \geq l \right].$$

If

$$E'_N = \lim_{\varepsilon \rightarrow 0} \lim_{l \rightarrow \infty} E'_N(\varepsilon, l)$$

then it follows from Theorems 2.2 and 7.1 of [4] that

$$\dim(E'_N) \geq \frac{2}{h_N} \left[\log N - \int_0^1 \log((N-1)t + 1) \frac{d\omega}{d\lambda}(t) dt \right] > 0$$

where

$$\frac{d\omega}{d\lambda}(t) = \frac{N-1}{\log \left(\frac{N(2N+1)}{(N+1)^2} \right)} \left(\frac{1}{N+1+(N-1)t} - \frac{1}{N^2+1+(N-1)t} \right).$$

The estimate in Theorem 8.1 of [4] gives

$$\dim(E'_N) \geq 1 - \frac{2}{\tau(N-1)} + O\left(\frac{1}{N^2}\right)$$

where $\tau = \pi^2/(6 \log 2)$ is Khintchine's constant.

THEOREM. *If $\gamma \geq (7/2) + (7/4) 2^{1/2}$ then $\dim(L(\gamma)) > 0$. For γ in C_N*

$$\dim(L(\gamma)) \geq 1 - \frac{2}{\tau(N-1)} + O\left(\frac{1}{N^2}\right).$$

For γ in C_N , $N \geq 6$

$$\dim(L(\gamma)) \leq 1 - \frac{1}{8(N+2) \log(N+2)}.$$

Proof. Set $\gamma = F(\gamma) + \alpha + \beta$ as in the previous section. We will first show that $\dim(\phi(E'_N)) \geq \dim(E'_N)$ which will establish the first and second assertions. For small enough ε and large enough l we can find $\lambda < 1$ such that

$$k \left[\lambda(h_N - \varepsilon) - 2\frac{\rho(k)}{k} \log(N+3) \right] + \log A - \lambda \log(2e^{h_N}) > 0$$

for all $k \geq l$. By Lemma 4 with $M = N + 2$ we can choose δ so small that $|\phi(\xi) - \phi(\eta)| \leq \delta$ implies that $\phi(\xi)$ and $\phi(\eta)$ have the same first $\pi(l-1)$ continued fraction coefficients and hence that ξ and η have the same first $l-1$ coefficients. Let (I_i) be a covering of $\phi(E'_N(\varepsilon, l))$ of norm at most δ . We can assume that each I_i is the smallest interval containing $I_i \cap \phi(E'_N(\varepsilon, l))$. Let $J_i = \phi^{-1}(I_i)$. Since ϕ preserves order, there are sequences (ξ_j) and (η_j) contained in J_i such that

$$\begin{aligned} \frac{|I_i|}{|J_i|^{1+\lambda}} &= \lim_{j \rightarrow \infty} \frac{|\phi(\xi_j) - \phi(\eta_j)|}{|\xi_j - \eta_j|^{1+\lambda}} \\ &\geq \limsup_{j \rightarrow \infty} \frac{A}{(N+3)^{2\rho(k_j)}} \frac{1}{|\xi_j - \eta_j|^\lambda} \end{aligned}$$

where $k_j \geq l$ is the first place where the continued fraction coefficients of ξ_j and η_j differ. But

$$\begin{aligned} |\xi_j - \eta_j| &\leq \left| \xi_j - \frac{P_{k_j-1}(\xi_j)}{Q_{k_j-1}(\xi_j)} \right| + \left| \frac{P_{k_j-1}(\eta_j)}{Q_{k_j-1}(\eta_j)} - \eta_j \right| \\ &\leq 2 \exp[-(k_j-1)(h_N - \varepsilon)] \leq 2e^{h_N} \exp[-k_j(h_N - \varepsilon)] \end{aligned}$$

so that

$$\frac{|I_i|}{|J_i|^{1+\lambda}} \geq \limsup_{j \rightarrow \infty} \frac{A}{(2 \exp(h_N))^2} \exp \left\{ k_j \left[\lambda(h_N - \varepsilon) - \frac{2\rho(k_j)}{k_j} \log(N + 3) \right] \right\} \geq 1.$$

Now (J_i) is a covering of $E'_N(\varepsilon, l)$ of norm at most $\delta^{1/(1+\lambda)}$ so for all small enough δ and

$$\mu < \frac{\dim(E'_N(\varepsilon, l))}{1 + \lambda}$$

we have

$$\sum |I_i|^\mu \geq \sum |J_i|^{\mu(1+\lambda)} \geq 1$$

and hence

$$\dim(\phi(E'_N(\varepsilon, l))) \geq \frac{\dim(E'_N(\varepsilon, l))}{1 + \lambda}.$$

As we let $l \rightarrow \infty$ we can take $\lambda \rightarrow 0$ so

$$\dim(\phi(\lim_{l \rightarrow \infty} E'_N(\varepsilon, l))) \geq \dim\left(\lim_{l \rightarrow \infty} E'_N(\varepsilon, l)\right).$$

The first part of the proof is now completed by letting $\varepsilon \rightarrow 0$.

Next we will show that $\dim L(\gamma) \leq \dim(E_{N+2})$ and this plus Jarnik's theorem (see [2]) that

$$\dim(E_N) \leq 1 - \frac{1}{8N \log N}, \quad N > 8$$

will complete the proof.

Clearly if $\xi = [x_1, x_2, \dots]$ has infinitely many $x_i \geq N + 3$ then $M(\xi) > N + 3$ so $\xi \notin L(\gamma)$. Thus $L(\gamma) \subset \bigcup_{\vec{k} \in K} G_{\vec{k}}$ where K is the set of all finite vectors $\vec{k} = (k_1, k_2, \dots, k_{2n})$, each k_i being a positive integer, and $G_{\vec{k}}$ is the set of all $\xi = [x_1, x_2, \dots]$ with

$$\begin{aligned} x_i &= k_i, & 1 \leq i \leq 2n, \\ x_i &\leq N + 2, & 2n < i. \end{aligned}$$

Since K is countable it will be sufficient to show that $\dim(G_{\vec{k}}) \leq \dim(E_{N+2})$ for each \vec{k} . We define an order preserving map Ψ of E_{N+2} onto $G_{\vec{k}}$ by setting, for $\xi = [x_1, x_2, \dots]$ in E_{N+2} , $\Psi(\xi) = [k_1, k_2, \dots, k_{2n}, x_1, x_2, \dots]$. If also $\xi' = [x'_1, x'_2, \dots]$ is in E_{N+2} and $x_i = x'_i$ for $i < l$ but $x_l \neq x'_l$ then, letting $N' = \max(k_1, \dots, k_{2n}, N + 2)$,

$$\begin{aligned} |\xi - \xi'| &\geq \frac{1 - (l_{N+2} - s_{N+2})}{4Q_l(\xi)Q_l(\xi')} \geq A \frac{N'}{Q_l(\xi)Q_l(\xi')} \\ &\geq A \frac{N'}{Q_{2n+l}(\Psi(\xi))Q_{2n+l}(\Psi(\xi'))} \\ &\geq A |\Psi(\xi) - \Psi(\xi')|. \end{aligned}$$

We have used here Lemma 4, the fact that $Q_i(\eta) \leq Q_{2n+1}(\Psi(\eta))$, and (1) in that order. If (I_i) is a covering of E_{N+2} of norm δ , each I_i being the smallest interval containing $I_i \cap E_{N+2}$, then $(\Psi(I_i))$ is a covering of G_k of norm at most δ/A . For any λ less than $\dim(G_k)$ and for small enough δ we have

$$\sum |I_i|^\lambda \geq A^\lambda \sum |\Psi(I_i)|^\lambda > 1$$

so $\dim(E_{N+2}) \geq \dim(G_k)$.

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