INTEGRAL REPRESENTATION ALGEBRAS(1)

BY

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1. **Introduction.** Let RG denote the group ring of a finite group G over a commutative ring R. By an RG-module we shall mean a left finitely generated module which is R-torsion free. The *representation ring* $a(RG)$ is an abelian additive group defined by generators and relations: the generators are the symbols $[M]$, where M ranges over a full set of representatives of the isomorphism classes of RG-modules, with relations

$$[M] = [M'] + [M'']$$

whenever $M \cong M' \oplus M''$. Multiplication in $a(RG)$ is defined by forming tensor products of modules:

$$[M][N] = [M \otimes_R N],$$

where as usual G acts on the tensor product by the formula

$$g(m \otimes n) = gm \otimes gn, \qquad g \in G,\ m \in M,\ n \in N.$$

Let C be the complex field, and define the *integral representation algebra* $A(RG)$ by the formula

$$A(RG) = C \otimes_Z a(RG).$$

Such representation algebras have recently been studied by Conlon [1], Green [3], [4], and O'Reilly [10], for the special case in which R is a field. They have shown that under suitable hypotheses, the algebra $A(RG)$ is semisimple.

The present author investigated $a(RG)$ when R is a ring of integers (see [11], [12]). Of particular interest are the following choices for R:

Z (the ring of rational integers),

Z_p (the p-adic valuation ring in the rational field Q),

Z_p^* (the ring of p-adic integers in the p-adic completion of Q),

$Z' = \bigcap_{p \mid [G:1]} Z_p$, a semilocal ring of integers in Q.

To give the reader the proper perspective, we quote two earlier results.

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THEOREM 1. *If K is a field of characteristic p, and if G has a cyclic p-Sylow subgroup, then $A(KG)$ is a finite dimensional semisimple C-algebra (O'Reilly [10]).*

THEOREM 2. *Suppose that for some prime p, the group G contains an element of order p^2. Then both $a(Z_pG)$ and $a(Z_p^*G)$ contain at least one nonzero nilpotent element. The same is therefore true for $A(Z_pG)$ and $A(Z_p^*G)$ (Reiner [12]).*

The aim of the present paper is to present a partial converse to the latter theorem. We shall prove here:

THEOREM 3. *Let p be a fixed prime, and suppose that the p-Sylow subgroups of G are cyclic of order p. Then $A(Z_pG)$ and $A(Z_p^*G)$ contain no nonzero nilpotent elements.*

THEOREM 4. *If $[G:1]$ is squarefree, then $A(Z'G)$ contains no nonzero nilpotent element.*

In the course of the proof we shall establish the following fact, which is of independent interest, and is an immediate consequence of Theorem 5 below.

PROPOSITION. *Let p be an odd prime, and let G have a normal p-Sylow subgroup which is cyclic of order p. Let M and N be Z_pG-modules. Then $M \cong N$ if and only if $M/pM \cong N/pN$ as $(Z/pZ)G$-modules.*

2. **Preliminary remarks.** We collect here some definitions, remarks, and previously established results which will be needed in the paper.

(a) If R is a field, or if $R = Z_p^*$, the Krull-Schmidt theorem holds for RG-modules. (See [2, Theorem 14.5 and Theorem 76.26].)

(b) For R an arbitrary ring, every element of $a(RG)$ is expressible in the form $[M]-[N]$, where M and N are RG-modules, but is not uniquely so expressible. Furthermore, $[M]=[N]$ in $a(RG)$ if and only if there exists an RG-module X such that $M \oplus X \cong N \oplus X$. If the Krull-Schmidt Theorem holds for RG-modules, this last isomorphism implies that $M \cong N$. Furthermore, in this case $a(RG)$ has a Z-basis consisting of the symbols $[L]$, where L ranges over a full set of representatives of the isomorphism classes of indecomposable RG-modules.

(c) Let $A \to B$ be a monomorphism of abelian additive groups. Then also

$$C \otimes_Z A \to C \otimes_Z B$$

is a monomorphism. (See MacLane [7, p. 152, Theorem 6.2].)

(d) For M a Z_pG-module, let $M^* = Z_p^* \otimes_{Z_p} M$. The map $[M] \to [M^*]$ gives a ring homomorphism $a(Z_pG) \to a(Z_p^*G)$, which we claim is a monomorphism. For let M and N be Z_pG-modules such that $[M^*]=[N^*]$. Then $M^* \cong N^*$, which implies that $M \cong N$ (see Maranda [8], or [2, Theorem 76.9]). This proves that the above homomorphism is monic, so by the preceding remark, $A(Z_pG) \to A(Z_p^*G)$ is also monic.

Next, there is a ring homomorphism

$$a(Z'G) \to \prod_{p \mid [G:1]} a(Z_pG),$$

gotten by mapping $[M]$ onto the element whose pth component is $[Z_p \otimes_{Z'} M]$. This map is monic, since if M and N are $Z'G$-modules such that

$$Z_p \otimes_{Z'} M \cong Z_p \otimes_{Z'} N, \qquad p \mid [G:1],$$

then by Maranda [9] (see [2, Theorem 81.2]) it follows that $M \cong N$. The map

$$A(Z'G) \to \prod_{p \mid [G:1]} A(Z_pG)$$

is also monic, by (c) above.

(e) Suppose that R is either a field of characteristic p, or $R = Z_p^*$, and let H be a subgroup of G. An RG-module M is called *(G,H)-projective* if M is a direct summand of an induced module L^G for some RH-module L. If H is a p-Sylow subgroup of G, then every RG-module is (G,H)-projective (see [2, §63]). The $(G,\{1\})$-projective modules are just the ordinary projective RG-modules.

For D a subgroup of G, let $a_D(RG)$ be the ideal of $a(RG)$ generated by the set of all (G,D)-projective RG-modules. Denote by $a'_D(RG)$ the ideal generated by the (G,D')-projectives, where D' ranges over the proper subgroups of D. Define

$$w_D(RG) = a_D(RG)/a'_D(RG),$$

and set

$$W_D(RG) = C \otimes_Z w_D(RG), \qquad A_D = C \otimes a_D, \qquad A'_D = C \otimes a'_D.$$

Then we have:

TRANSFER THEOREM. *The algebra $A(RG)$ is semisimple if $W_D(R \cdot N_GD)$ is semisimple for each p-subgroup D of G. Here, N_GD is the normalizer of D in G (Green [4]).*

Using this, O'Reilly [10] was able to prove Theorem 1 by showing:

THEOREM. *If k is a field of characteristic p, and if G has a cyclic p-Sylow subgroup, then $W_D(k \cdot N_GD)$ is semisimple for each p-subgroup D of G.*

(f) Now let H be a normal subgroup of G, and suppose that the Krull-Schmidt Theorem holds for RG-modules. Let L be an RH-module, and let $x \in G$. We may form a new RH-module L^x, called a *conjugate* of L, by letting L^x have the same elements as L, but where each $h \in H$ acts on L^x as does xhx^- on L.

For an RG-module M, denote by $\text{res}_H M$ the RH-module gotten from M by restriction of operators from G to H. From the Mackey Subgroup Theorem (see [2, Theorem 44.2]), it follows at once that $\operatorname{res}_H(L^G)$ is a direct sum of conjugates of L.

We note further that if L_1 and L_2 are RH-modules, then

$$L_1^G \otimes L_2^G \cong \sum{}^{\oplus} (L_1 \otimes L_2^y)^G,$$

the sum extending over certain elements $y \in G$ (see [2, Theorem 44.3]).

(g) Starting with a (left) RG-module M, we may form another (left) RG-module M^*, called the *contragredient* of M (see [2, §43]). As R-module, M^* is just $\operatorname{Hom}_R(M,R)$. Each $x \in G$ acts on M^* in the same way that x^{-1} acts on the right RG-module $\operatorname{Hom}_R(M,R)$. Then $(M_1 \oplus M_2)^* \cong M_1^* \oplus M_2^*$. Also, if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of RG-modules, then so is

$$0 \to M_3^* \to M_2^* \to M_1^* \to 0.$$

Finally, we have $(RG)^* \cong RG$, and therefore contragredients of projective modules are again projective.

(h) SCHANUEL'S LEMMA (SEE SWAN [13]). *Suppose we are given two exact sequences of RG-modules*:

$$0 \to U_i \to P_i \to V_i \to 0, \quad i = 1,2,$$

in which P_1 and P_2 are projective. If $V_1 \cong V_2$, then

$$U_1 \oplus P_2 \cong U_2 \oplus P_1.$$

3. **Main theorem.** Throughout this section we fix a prime p, and set $R = Z_p^*$, $\bar{R} = R/pR$. Let G contain a p-Sylow subgroup H which is cyclic of order p. In view of 2(d) above, Theorem 3 will be established as soon as we show that $A(RG)$ contains no nonzero nilpotent element.

Up to isomorphism, there are exactly three indecomposable RH-modules, namely R_H, I_H, and RH (see Heller and Reiner [6]). Here, R_H is the module R on which H acts trivially; I_H is the augmentation ideal of the group ring RH, and there is an exact sequence of RH-modules:

$$0 \to I_H \to RH \to R_H \to 0. \tag{1}$$

By 2(e) each indecomposable RG-module is a direct summand of one of the induced modules $(I_H)^G$, RG, $(R_H)^G$. (We have used the obvious isomorphism: $(RH)^G \cong RG$.) Thus the number of isomorphism classes of indecomposable RG-modules is finite, so $A(RG)$ is a finite dimensional commutative C-algebra. We must show that $A(RG)$ is semisimple, or equivalently, that $A(RG)$ contains no nonzero nilpotent element. To prove this, by 2(e) it is enough to show that $W_D(R \cdot N_G D))$. is semisimple for each p-subgroup D of G. But W_D is unchanged when D is replaced by one of its conjugates, and therefore we need only show that

$$W_{\{1\}}(RG) \text{ and } W_H(R \cdot N_G H)$$

are both semisimple.

The algebra $W_{\{1\}}(RG)$ is generated by the projective RG-modules. As is well known (see [2, §77]), there is a one-to-one isomorphism-preserving correspondence between the indecomposable direct summands of RG and those of $\bar{R}G$. In other words, we have

$$W_{\{1\}}(RG) \cong W_{\{1\}}(\bar{R}G),$$

the isomorphism being given by $[M] \to [M/pM]$. But $\bar{R}$ is a field of characteristic p, so by O'Reilly's Theorem of 2(e) it follows that $W_{\{1\}}(\bar{R}G)$ is semisimple. (This can also be proved easily by use of Brauer characters; see, for example, Conlon [1].)

It remains for us to show that $W_H(R \cdot N_G H)$ is semisimple. Changing notation, we may hereafter assume that G has a normal p-Sylow subgroup H which is cyclic of order p, and we must prove that $W_H(RG)$ is semisimple. For p odd, this is an immediate consequence of O'Reilly's Theorem together with the following result:

THEOREM. 5. *Let G have a normal p-Sylow subgroup H which is cyclic of order p, where p is an odd prime. Then the algebra homomorphism*

$$W_H(RG) \to W_H(\bar{R}G)$$

is monic.

(Before starting the proof, we may remark that the theorem fails to be true when $p = 2$. Nevertheless, most of the details of the proof are valid for $p = 2$, and will be used for that case in the following section.)

Proof. As was pointed out earlier in this section, the nonisomorphic indecomposable direct summands of $(R_H)^G$, $(I_H)^G$, and RG, give a full set of indecomposable RG-modules. The direct summands of RG are RG-projective, and generate the ideal $a_H(RG)$ of $a_H(RG)$. Thus $w_H(RG)$ is generated as Z-module by the indecomposable direct summands of $(R_H)^G$ and $(I_H)^G$. Let us write

$$(R_H)^G = N_1 \oplus \cdots \oplus N_k, \; N_i \text{ indecomposable,} \tag{2}$$

where the summands are numbered so that the first m of them are a full set of nonisomorphic modules from the set of summands. It will turn out that

$$(I_H)^G = L_1 \oplus \cdots \oplus L_k, \; L_i \text{ indecomposable,} \tag{3}$$

with the first m summands a full set of nonisomorphic modules from the set $\{L_1, \cdots, L_k\}$. Thus $\{[N_1], \cdots, [N_m], [L_1], \cdots, [L_m]\}$ form a Z-basis for $w_H(RG)$.

Let $\bar{N}_i = N_i/pN_i$, $\bar{L}_i = L_i/pL_i$, viewed as $\bar{R}G$-modules. In order to prove that $W_H(RG) \to W_H(\bar{R}G)$ is monic, it suffices by 2(c) to show that $w_H(RG) \to w_H(\bar{R}G)$ is monic; and for this, we need only show that $\{[\bar{N}_1], \cdots, [\bar{N}_m], [\bar{L}_1], \cdots, [\bar{L}_m]\}$ are linearly independent (over Z) in $w_H(\bar{R}G)$.

By Schur's Theorem [2, Theorem 7.5], there exists a subgroup F of G such that $G = HF$, $F \cong G/H$. It is easily seen that

$$(R_H)^G \cong RF \qquad \text{as } RG\text{-modules},$$

where H acts trivially on the module RF. Indeed, $(R_H)^G = RG \otimes_{RH} R_H$, and the isomorphism is given by

$$\sum_{x \in F;\, y \in H} \alpha_{x,y}(xy \otimes 1) \to \sum_{x,y} \alpha_{x,y} x, \qquad \alpha_{x,y} \in R.$$

Hence the $\{N_i\}$ occurring in (2) are gotten by decomposing RF into a direct sum of indecomposable left ideals. However, $p \nmid [F:1]$, so if K is the quotient field of R, we have

$$KF = KN_1 \oplus \cdots \oplus KN_k,$$

where the $\{KN_i\}$ are minimal left ideals of KF. Furthermore, it follows from [2, Theorem 76.17 and Theorem 76.23], that each $\bar{N}_i$ is indecomposable, and that for $1 \leqq i, j \leqq k$,

$$N_i \cong N_j \Leftrightarrow KN_i \cong KN_j \Leftrightarrow \bar{N}_i \cong \bar{N}_j.$$

Thus $\{\bar{N}_1, \cdots, \bar{N}_m\}$ are distinct indecomposable $\bar{R}G$-modules, on each of which H acts trivially.

Turning next to the consideration of $(I_H)^G$, we observe first that forming induced modules preserves exactness, and so from (1) we obtain an exact sequence of RG-modules

$$0 \to (I_H)^G \to RG \to (R_H)^G \to 0. \tag{4}$$

Each N_i is a quotient module of $(R_H)^G$, hence also of RG, and so there exist exact sequences

$$0 \to M_i \to RG \to N_i \to 0, \qquad 1 \leqq i \leqq k. \tag{5}$$

If $N_i \cong N_j$, then by 2(h) we have $M_i \cong M_j$. Conversely, if $M_i \cong M_j$, then taking contragredients (see 2(g)) and using 2(h) again, we obtain $N_i^* \cong N_j^*$, and $N_i \cong N_j$.

If $RG^{(k)}$ denotes a direct sum of k copies of RG, then from (5) we obtain an exact sequence

$$0 \to M_1 \oplus \cdots \oplus M_k \to RG^{(k)} \to N_1 \oplus \cdots \oplus N_k \to 0.$$

Comparing this with (4) and using 2(h), we find that

(6) $$M_1 \oplus \cdots \oplus M_k \cong (I_H)^G \oplus RG^{(k-1)}.$$

For each i, $1 \leqq i \leqq k$, let us write

$$M_i = L_i \oplus P_i,$$

where P_i is projective, and L_i has no projective direct summand. It follows from the Krull-Schmidt Theorem for RG-modules that M_i determines L_i and P_i uniquely, up to isomorphism. By (6), each L_i is a direct summand of $(I_H)^G$. On the other hand, $(I_H)^G$ has no projective direct summand, since $\mathrm{res}_H(I_H)^G$ is a direct sum of conjugates of I_H, hence of copies of I_H, whereas for X a projective RG-module, $\mathrm{res}_H X$ is free. Consequently

$$L_1 \oplus \cdots \oplus L_k \cong (I_H)^G,$$

and $\{L_1, \cdots, L_m\}$ are a full set of nonisomorphic modules from the set $\{L_1, \cdots, L_k\}$. To show that each L_i is indecomposable, we shall establish the stronger result that $\{\bar{L}_1, \cdots, \bar{L}_m\}$ are a set of distinct indecomposable $\bar{R}G$-modules.

From (5) we obtain exact sequences

$$0 \to \bar{N}_i^* \to \bar{R}G \to \bar{L}_i^* \oplus \bar{P}_i^* \to 0, \qquad 1 \leqq i \leqq m.$$

If $\bar{L}_i \cong \bar{L}_j$ for some i, j, where $1 \leqq i, j \leqq m$, the above implies (using 2(h)) that $\bar{N}_i^* \oplus \bar{P}_j^* \cong \bar{N}_j^* \oplus \bar{P}_i^*$. But $\bar{N}_i^*$ is indecomposable, and is not projective because H acts trivially on $\bar{N}_i^*$. Hence $\bar{N}_i^* \cong \bar{N}_j^*$, so $N_i \cong N_j$ and $i = j$.

Next, suppose $\bar{L}_i$ decomposable; then so is $\bar{L}_i^*$, and we may write $\bar{L}_i^* = U_1 \oplus U_2$, say. Each U_j is a homomorphic image of $\bar{R}G$, so there exist $\bar{R}G$-modules X_1, X_2 with

$$0 \to X_j \to \bar{R}G \to U_j \to 0, \qquad j = 1, 2,$$

exact. Thus

$$0 \to X_1 \oplus X_2 \to \bar{R}G^{(2)} \oplus \bar{P}_i^* \to U_1 \oplus U_2 \oplus \bar{P}_i^* \to 0$$

is exact. By 2(h) it follows that

$$\bar{N}_i^* \oplus \bar{R}G \oplus \bar{P}_i^* \cong X_1 \oplus X_2.$$

But $\bar{N}_i^*$ is indecomposable, and $\bar{P}_i^*$ is projective, so either X_1 or X_2 must be projective; say X_1 is projective. Then $\mathrm{res}_H X_1$ is free. On the other hand, X_1 is a direct summand of $\bar{L}_i^*$, and $\mathrm{res}_H \bar{L}_i^*$ is a direct sum of copies of $\bar{I}_H$ (since $\bar{I}_H^* \cong \bar{I}_H$). This gives a contradiction, and so indeed each $\bar{L}_i$ is indecomposable.

We may remark that in terms of the loop space functor Ω introduced by Heller [5], we have $L_i = \Omega(N_i)$.

Let us show at once that $\bar{N}_i \cong \bar{L}_j$ is impossible, and it is precisely for this purpose that the hypothesis $p > 2$ is needed. We know that $\mathrm{res}_H \bar{N}_i$ is a direct sum of copies of $\bar{R}_H$, whereas $\mathrm{res}_H \bar{L}_j$ is a direct sum of copies of $\bar{I}_H$. But $\bar{I}_H$ is indecomposable, and for $p > 2$, $\bar{I}_H$ is not isomorphic to $\bar{R}_H$. Thus $\bar{N}_i \not\cong \bar{L}_j$ for any i,j, and we have shown that $\{\bar{N}_1, \cdots, \bar{N}_m, \bar{L}_1, \cdots, \bar{L}_m\}$ are a set of nonisomorphic indecomposable $\bar{R}G$-modules. Obviously none of them lies in $a_H'(\bar{R}G)$, and so they are Z-linearly independent when considered as elements of $w_H(\bar{R}G)$. This completes the proof of Theorem 5.

Since we have already shown that $W_{\{1\}}(RG) \cong W_{\{1\}}(\bar{R}G)$, it follows that the map $A(RG) \to A(\bar{R}G)$ is monic when restricted to $A_H'(RG)$. Combining this with Theorem 5, we may conclude that the algebra homomorphism

$$A(RG) \to A(\bar{R}G)$$

is also monic, provided the hypotheses of Theorem 5 are satisfied. But this establishes the validity of the proposition stated at the end of §1.

4. **The case** $p = 2$. In this section we shall prove Theorem 3 for the case $p = 2$. We use the notation of the preceding section, and we are assuming now that G has a cyclic 2-Sylow subgroup H of order 2. As we have seen, we need only show that $W_H(RG)$ contains no nonzero nilpotent element, and it suffices to prove this for the case where H is normal in G. Furthermore, in order to prove that $W_H(RG)$ has no nilpotent elements except 0, it is enough to show that if $x \in W_H(RG)$ satisfies $x^2 = 0$, then necessarily $x = 0$.

As in §3, we let $\{N_1, \cdots, N_m\}$ be the nonisomorphic indecomposable summands of $(R_H)^G$, and $\{L_1, \cdots, L_m\}$ those of $(I_H)^G$. Now $R_H \not\cong I_H$, even for $p = 2$, so by considering restrictions to H it is clear that $N_i \not\cong L_j$ for any i,j. Hence $W_H(RG)$ has C-basis $\{[N_1], \cdots, [N_m], [L_1], \cdots, [L_m]\}$. Furthermore, we know from §3 that $\{\bar{N}_1, \cdots, \bar{N}_m\}$ are the nonisomorphic indecomposable summands of $(\bar{R}_H)^G$, while $\{\bar{L}_1, \cdots, \bar{L}_m\}$ are those of $(\bar{I}_H)^G$. However, since $p = 2$ we have $\bar{R}_H \cong \bar{I}_H$, and so the $\bar{L}$'s are a rearrangement of the $\bar{N}$'s. Thus the maps $W_H(RG) \to W_H(\bar{R}G)$, $A(RG) \to A(\bar{R}G)$, are no longer monomorphisms.

In this case we have $[G:F] = 2$, so F is normal in G, and $G/F \cong H$. If h is the generator of H, we may form the RH-module Y having the same elements as R, but where

$$h\alpha = -\alpha, \qquad \alpha \in Y.$$

Then use the homomorphism of G onto H to turn Y into an RG-module, that is, let F act trivially on Y. The RG-module thus obtained will also be denoted by Y. Obviously

$$\bar{Y} \cong \bar{R}_G, \qquad Y \otimes Y \cong R_G,$$

where R_G is the trivial RG-module.

Consider now the RG-modules $Y \otimes N_1, \cdots, Y \otimes N_m$. Each is indecomposable, since

$$\overline{Y \otimes N_i} \cong \bar{Y} \otimes \bar{N}_i \cong \bar{N}_i.$$

Furthermore, it cannot happen that $Y \otimes N_i \cong N_j$, since h acts on $Y \otimes N_i$ as multiplication by -1, whereas h acts trivially on N_j. Thus, the modules $\{Y \otimes N_i : 1 \leqq i \leqq m\}$ coincide with the modules $\{L_i : 1 \leqq i \leqq m\}$ in some order.

Let us set $Q_i = Y \otimes N_i$, $1 \leqq i \leqq m$. The above discussion shows that $\{[N_1], \cdots, [N_m], [Q_1], \cdots, [Q_m]\}$ is a Z-basis for $w_H(RG)$, hence also a C-basis for $W_H(RG)$. Furthermore, the kernel of the algebra homomorphism $W_H(RG) \to W_H(\bar{R}G)$ has C-basis $\{[N_i] - [Q_i] : 1 \leqq i \leqq m\}$.

We shall now investigate $N_i \otimes N_j$. Since N_i and N_j are direct summands of $(R_H)^G$, their tensor product is a direct summand of $(R_H)^G \otimes (R_H)^G$. By 2(f) we see that this latter module is a direct sum of modules of the form $(R_H \otimes R_H^y)^G$, for some elements $y \in G$. However, $R_H^y \cong R_H$ and $R_H \otimes R_H \cong R_H$. Therefore $N_i \otimes N_j$ is a direct sum of copies of $N_1, \cdots, N_m$; suppose that N_s occurs with multiplicity α_{ijs} as a direct summand of $N_i \otimes N_j$. We have then

$$[N_i][N_j] = \sum_{s=1}^{m} \alpha_{ijs}[N_s] \qquad \text{in } W_H(RG).$$

Furthermore we obtain

$$Q_i \otimes Q_j = (Y \otimes N_i) \otimes (Y \otimes N_j) \cong (Y \otimes Y) \otimes (N_i \otimes N_j) \cong N_i \otimes N_j,$$

so

$$[Q_i][Q_j] = \sum_{s=1}^{m} \alpha_{ijs}[N_s] \qquad \text{in } W_H(RG).$$

Finally we note that

$$[Q_i][N_j] = [Y \otimes (N_i \otimes N_j)] = \sum_{s} \alpha_{ijs}[Y \otimes N_s] = \sum_{s} \alpha_{ijs}[Q_s] \qquad \text{in } W_H(RG).$$

Suppose now that $x \in W_H(RG)$ and $x^2 = 0$; we are trying to prove that x must be 0. Since the image of x in $W_H(\bar{R}G)$ is also nilpotent, and since $W_H(\bar{R}G)$ contains no nonzero nilpotent element, it follows that x lies in the kernel of the map $W_H(RG) \to W_H(\bar{R}G)$. Thus we may write

$$x = \sum_{i=1}^{m} c_i([N_i] - [Q_i]), \qquad c_i \in C.$$

Then

$$x^2 = \sum_{i,j=1}^{m} \{c_i c_j [N_i][N_j] - 2c_i c_j [N_i][Q_j] + c_i c_j [Q_i][Q_j]\}$$

$$= \sum_{i,j,s=1}^{m} 2c_i c_j \alpha_{ijs}([N_s] - [Q_s]).$$

But $x^2 = 0$, and so

$$2 \cdot \sum_{i,j=1}^{m} c_i c_j \alpha_{ijs} = 0, \qquad 1 \leqq s \leqq m.$$

Therefore $\sum_{i,j} c_i c_j \alpha_{ijs} = 0$, which shows that $\sum_{i=1}^{m} c_i[N_i]$ has square 0. Thus $\sum_i c_i[\bar{N}_i] = 0$ in $W_H(\bar{R}G)$, and consequently each $c_i = 0$. This proves that $x = 0$, and completes the demonstration of Theorem 3 for the case $p = 2$.

5. **Concluding remarks.** Let us show that Theorem 4 is an easy consequence of Theorem 3. Suppose that $[G:1]$ is squarefree; then by using Theorem 3 for each prime p dividing $[G:1]$, we see that each $A(Z_p^* G)$ contains no nonzero nilpotent element. Hence also the product

$$\prod_{p \mid [G:1]} A(Z_p^* G)$$

contains no nonzero nilpotent element. But by 2(d) the algebra $A(Z'G)$ may be embedded in the above product, and hence also $A(Z'G)$ contains no nonzero nilpotent element.

It would be of interest to consider the corresponding question for $A(ZG)$. The difficulty seems to arise from the fact that the map $A(ZG) \to A(Z'G)$ need not be monic.

Conjecture 1. The kernel of the map $A(ZG) \to A(Z'G)$ is a torsion Z-module.

As remarked in Theorem 2, if G contains an element of order p^2, then $A(Z_p^* G)$ contains nonzero nilpotent elements. On the other hand, we have shown that if the p-Sylow subgroup of G is cyclic of order p, then $A(Z_p^* G)$ is semisimple. We are left with a large class of groups which fall into neither category, for example an elementary abelian (p, p) group.

Conjecture 2. If the p-Sylow subgroup of G is not cyclic of order p, then $A(Z_p^* G)$ contains nonzero nilpotent elements.

We may remark that Theorem 5 is best possible, in the following sense. Let $R = Z_p^*$, and let H be a p-Sylow subgroup of G. If H is not normal in G, or if H is not cyclic of order p, then the maps

$$W_H(RG) \to W_H(\bar{R}G), \quad A(RG) \to A(\bar{R}G),$$

are not monic. Indeed, even when G is cyclic of order p^2, the map $A(RG) \to A(\bar{R}G)$ is not monic.

Finally, the proof of Theorem 5 suggests that the proposition at the end of §1 may be a special case of a more general result. This will be investigated more fully in a future work (to appear in Michigan Math. J.)

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