A CHARACTERIZATION OF THE CUTPOINT-ORDER 
ON A TREE

BY

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1. Introduction. A tree is a continuum (compact, connected Hausdorff space) 
such that every two distinct points are separated by the omission of a third point. 
Let $X$ be a tree, and let $z$ be an arbitrary, fixed element of $X$. Let $Q(z)$ be the 
set of all pairs $(a, b)$ in $X \times X$ such that at least one of the following three con-
ditions is satisfied:

(i) $a = z$,
(ii) $a = b$,
(iii) $a$ separates $z$ and $b$ in $X$.

It turns out that $Q(z)$ is a continuous partial order on $X$, and with respect to this 
partial order $z$ is the unique minimal element. We shall refer to $Q(z)$, for any $z$ in $X$, 
as a cutpoint-order $[1]$ on the tree $X$. The purpose of this paper is to give a 
characterization of the cutpoint-order on a tree (Theorem 2). We also establish a 
new characterization of a tree from relation-theoretic and cohomological view-
points (Theorem 1).

2. Preliminaries. A relation $R$ on a space $X$ is a subset of the cartesian 
product $X \times X$. If $x \in X$, we write $xR = \{y \mid (x, y) \in R\}$, $Rx = \{y \mid (y, x) \in R\}$, 
$RA = \bigcup \{Rx \mid x \in A\}$ and $AR = \bigcup \{xR \mid x \in A\}$. Following Wallace $[11]$, we say 
that $R$ is left (right) monotone if each $Rx(xR)$ is connected. A relation $R$ is a 
quasi-order if it is reflexive and transitive; it is a partial-order if it is an anti-
symmetric quasi-order. $R$ is total if for every $(x, y)$ either $(x, y) \in R$ or $(y, x) \in R$; a 
total-order is a partial-order that is also total. A set $A \subseteq X$ is an $R$-chain if $R \cap (A \times A)$ 
is a total-order on $A$. An $R$-minimal element $a$ is an element such that $(x, a) \in R$ 
implies $(a, x) \in R$.

Definition 1. A space $X$ is unicoherent if and only if $X$ is connected and 
$X = A \cup B$ (with $A$ and $B$ closed and connected) implies that $A \cap B$ is connected. 
$X$ is hereditarily unicoherent if every subcontinuum of $X$ is unicoherent.

Perhaps the most useful of these characterizations is the following.

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Lemma 1 [2], [3]. A continuum $X$ is a tree if and only if it is locally connected and hereditarily unicoherent.

An excellent proof of this lemma may be found in Ward [15].

Definition 2. A space $X$ is said to be semi-locally-connected (abbreviated s.l.c.) at a point $x \in X$ if for each open set $U$ in $X$ containing $x$ there exists an open set $V$ such that $x \in V \subset U$ and such that $X - V$ has only a finite number of components. If $X$ is s.l.c. at each of its points, it is said to be s.l.c.

The Alexander-Kolmogoroff-Wallace cohomology groups will be used as explicated in [5], [6], [8] and [12]. In what follows, the coefficient group will be arbitrary but fixed, and will therefore not be mentioned. The following Mayer-Vietoris exact sequence will be used.

Lemma 2. If $X$ is a compact Hausdorff space and $X = A \cup B$, with $A$ and $B$ closed, then there is an exact sequence

$\cdots \rightarrow H^n(A \cap B) \xrightarrow{\Delta} H^{p+1}(X) \xrightarrow{J^*} H^{p+1}(A) \times H^{p+1}(B) \xrightarrow{I^*} \cdots$

such that $\Delta = 0$ if $p = 0$ and $A \cap B$ is connected.

Following Wallace [10], if $C \subseteq D$ and $h \in H^n(D)$, we denote by $h | C$ the image of $h$ under the natural homomorphism induced by the inclusion map of $C$ into $D$. Thus, if $h \in H^n(X)$, then $J^*(h) = (h | A, h | B)$.

Definition 3. If $X$ is a space, $A \subseteq X$, and $h$ is a nonzero member of $H^p(A)$, then a closed set $F (F \subseteq A)$ is a floor for $h$ if and only if $h | F \neq 0$ while $h | F_0 = 0$ for each closed proper subset $F_0$ of $F$.

Lemma 3 (Floor Theorem [12]). If $A$ a closed subset of a compact Hausdorff space $X$ and $h$ is a nonzero member of $H^p(A)$, then $h$ has a floor. Moreover, every floor is connected.

The closure of a set $A$ will be denoted by $A^*$, and the empty-set by $\emptyset$.

3. A characterization of trees. In 1953, A. D. Wallace [9] proved that a one-codimensional compact connected and locally connected topological semigroup with unit and zero is a tree (for the definition and properties of codimension, see Cohen [4]). L. W. Anderson and L. E. Ward, Jr. in 1961 [1] modified Wallace's result by eliminating the necessity of hypothesizing a unit. More precisely, they proved that if $X$ is a compact, connected, locally connected, one-codimensional topological semilattice, then $X$ is a tree. Wallace [10] improved this result by weakening the local connectedness of $X$ to semilocal connectedness of $X$. These elegant results on topological algebra motivated the following theorem, which bears a relation-theoretic analogy.

Theorem 1. If $X$ is an s.l.c. compact Hausdorff space of codimension one equipped with a relation $R$ such that
(i) $R = R^*$ and $RX = X$,
(ii) $H^1(Rx) = 0$ for each $x$ in $X$,
(iii) the collection $\{Rx \mid x \in X\}$ has the finite intersection property (abbreviated f.i.p.), and
(iv) $Ra \cap Rb$ is connected for each pair $(a, b)$ in $X \times X$,
then $X$ is a tree. Conversely, a tree satisfies all of the hypotheses described above.

**Lemma 4.** If $X$ is regular and is s.l.c. at $x \in X$, and if $x$ does not separate two points $a$ and $b$ in $X$, then there exists a closed and connected subset $N$ of $X$ such that $\{a, b\} \subseteq N \subseteq X - x$.

This lemma was first proved by G. T. Whyburn [16] for the particular case in which $X$ was assumed to be a metric continuum. The general case was implicit in a paper by Wallace [10].

**Proof of Theorem 1.** We shall show that $X$ is hereditarily unicoherent and then using this, together with Lemma 4, to show that $X$ is a tree. It follows from (i), (ii), and (iii) that

$$X = \bigcup \{Rx \mid x \in X\}$$

is connected, and thus $X$ is a continuum.

We first show $H^1(X) = 0$. If there were a nonzero $h \in H^1(X)$, then there would be a nonvoid maximal tower $\tau$ of closed subsets $A$ of $X$ such that $h \mid RA \neq 0$. Let $A_0 = \bigcap \{A \mid A \in \tau\}$. Then $h \mid RA_0 \neq 0$, for if $h \mid RA_0 = 0$, then by the reduction theorem [8] there would be an open $V \supset RA_0$ such that $h \mid V^* = 0$. Let $U = \{x \mid Rx \subseteq V\}$; then it follows from (i) that $U$ is an open set containing $A_0$ such that $RU \subseteq V$. Thus, there is an $A$ in $\tau$ with $A \subseteq U$ and $RA \subseteq RU \subseteq V^*$; therefore $h \mid RA = 0$, a contradiction.

**Case 1.** Card $A_0 = 1$; that is, $A_0 = \{x\}$. By (ii), $H^1(RA_0) = 0$, a contradiction.

**Case 2.** Card $A_0 > 1$. Write $A_0 = A_1 \cup A_2$, where both $A_1$ and $A_2$ are proper closed subsets of $A_0$. We consider the part

$$H^0(RA_1 \cap RA_2) \xrightarrow{\Delta} H^1(RA_0) \xrightarrow{J^*} H^1(RA_1) \times H^1(RA_2)$$

of the Mayer-Vietoris exact sequence (Lemma 2). Since by (iii) and (iv) the set

$$RA_1 \cap RA_2 = \bigcup \{Ra \cap Rb \mid (a, b) \in A_1 \times A_2\}$$

is connected, $\Delta = 0$ by Lemma 2, and

$$h \mid RA_0 \in \text{Ker } J^* = \text{Im } \Delta = 0,$$

a contradiction.

Since $X$ is a continuum and $H^1(X) = 0$, $X$ is unicoherent ([2] and [3]). $X$ being of codimension one and $H^1(X) = 0$ imply that $H^1(K) = 0$ for every closed subset $K$ of $X$ [4], and thus every subcontinuum of $X$ is unicoherent.

We now prove that every two points of $X$ are separated in $X$ by a third point.
Suppose there were two points \( a \) and \( b \) such that no point separates \( a \) and \( b \) in \( X \). Then by Lemma 4, for any \( p \) different from both \( a \) and \( b \), there would exist a continuum \( P \) that is irreducible from \( a \) to \( b \) and does not contain \( p \). If \( q \) were an element of \( P \) distinct from \( a \) and \( b \), there would also exist a continuum \( Q \), irreducible from \( a \) to \( b \), that does not contain \( q \). But then \( P \cup Q \) would be a sub-continuum of \( X \) that is not unicoherent, since (by our selection of \( P \) and \( Q \)) \( P \cap Q \) would obviously not be connected. This contradiction proves that \( X \) is a tree.

Our proof depends heavily on Wallace [10].

The proof for the converse of this theorem is included in the proof of the next theorem.

4. A characterization of the cutpoint-order on a tree. The main purpose of the next theorem is to characterize the cutpoint-order on a tree from relation-theoretic and cohomological stand-points. A relation \( R \) on a space \( X \) is said to be closed if it is closed in the product \( X \times X \).

**Theorem 2.** If \( X \) is a compact Hausdorff space and \( P \) is a relation on \( X \), then the conditions

(i) \( X \) is of codimension one and s.l.c.,

(ii) \( P \) is a closed partial order,

(iii) \( P \) is both left and right monotone and \( H^1(Px) = 0 \) for every \( x \) in \( X \), and

(iv) \( \{Px \mid x \in X\} \) has the f.i.p.

are necessary and sufficient that \( X \) be a tree and that \( P \) be a cutpoint-order.

**Proof.** We first prove the sufficiency. Conditions (ii), (iv) and the first half of (iii) imply that

\[
Pa \cap Pb = \bigcup \{Px \mid x \in Pa \cap Pb\}
\]

is connected, and thus Theorem 1 implies that \( X \) is a tree.

Since \( X \) is compact and \( \{Px \mid x \in X\} \) has the f.i.p., \( \bigcap \{Px \mid x \in X\} \) is is a single point, the unique \( P \)-minimal element of \( X \). Let us denote by \( \{0\} \) the set

\[
\bigcap \{Px \mid x \in X\}.
\]

We prove that \( P = Q(0) \). If \((a, b) \in Q(0)\) and \( a = 0 \) or \( a = b \), then clearly \((a, b)\) is also in \( P \). If \( a \) separates 0 and \( b \) in \( X \), then since \( Pb \) is a continuum containing 0 and \( b \), it must contain \( a \), and we again conclude that \((a, b)\) is in \( P \). Thus \( Q(0) \subseteq P \). Conversely, if \((a, b)\) is in \( P \), then since \( a \) is in \( aP \cap Pb \), and since both \( aP \) and \( Pb \) are continua, \( aP \cup Pb \) is a subcontinuum of the tree \( X \), and therefore by Lemma 1 it is unicoherent. Thus \( aP \cap Pb \) is also a continuum. Now, by virtue of Hausdorff's Maximality Principle, \( aP \cap Pb \) has a maximal \( P \)-chain \( C \); such a \( P \)-chain will be shown to be closed, connected, and unique. The closedness of \( C \) is proved in [7]. Suppose \( C \) were not connected, then there would exist two non-void disjoint closed sets \( A \) and \( B \) such that \( C = A \cup B \) and \( b \in B \). The set \( A \) contains a maximal element \( m \). Define \( A' \) and \( B' \) by the equations
Then $B' \subseteq mP$, and since $A \subseteq A'$, it follows that $B' \subseteq B$. Now

$$A' \cap B' = Pm \cap (mP \cap B) = (Pm \cap mP) \cap B = \emptyset,$$

therefore

$$C = A' \cup B'$$

is a separation. If $b_0$ designates the minimal element in $B'$, then by the maximality of $C$

$$mP \cap Pb_0 = \{m, b_0\};$$

this contradicts the connectedness of $mP \cap Pb_0$. Therefore, any maximal $P$-chain in $aP \cap Pb$ is connected. We now show that $C$ is unique. For, if $C$ and $C'$ were two distinct maximal $P$-chains in $aP \cap Pb$, then both $C$ and $C'$ would contain $a$ and $b$; therefore, $C \cup C'$ would be connected, and hence $C \cap C'$ would be connected. But for $x \in C - C'$,

$$C \cap C' = (px \cup xP) \cap C \cap C'$$

$$= (px \cap C \cap C') \cup (xP \cap C \cap C')$$

is obviously a separation; this is a contradiction. Throughout the rest of proof, the unique $P$-chain containing $a$ and $b$ will be denoted by $C_p(a, b)$.

Since $(0, b) \in Q(0) \subseteq P$ and $X$ is a tree, there exists a unique connected $Q$-chain $[14] C_q(0, b) \subseteq Pb$ that contains both $0$ and $b$. $Pb$ must also have a connected $P$-chain containing both $0$ and $b$, and this $P$-chain must be unique. We denote by $C_p(0, b)$ the unique connected $P$-chain in $Pb$ containing $0$ and $b$. Since a $Q$-chain is also a $P$-chain,

$$C_p(0, b) = C_q(0, b).$$

Similarly, there is a unique connected $P$-chain $C_p(0, a)$ in $Pa$ containing both $0$ and $a$. Clearly,

$$C_p(0, a) \cup C_p(a, b) = C_p(0, b) = C_q(0, b).$$

As a consequence, $a \in C_q(0, b)$, and hence $(a, b) \in Q$, which was to be proved.

We next prove the necessity. Let $X$ be tree, and let $P$ be the cutpoint-order on $X$ with respect to a point $z$ in $X$. We shall prove that $X$ and $P$ satisfy the conditions (i), (ii), (iii), and (iv) stated in the theorem.

**Proof of (i).** By Ward [15], a tree is a compact connected commutative idempotent semigroup with zero; therefore, it is acyclic [10]. Hence, in particular, $H^1(X) = 0$. We now show that $H^1(A) = 0$ for every $A = A^* \subseteq X$ and thus $X$ is of codimension one, unless $X$ is degenerate. Suppose on the contrary that
$H^1(A) \neq 0$ for some closed subset $A$ of $X$. If $h$ is a nonzero member of $H^1(A)$, then by the Floor Theorem (Lemma 3) there exists a floor $F \subset A$ for $h$, which is connected. The set $F$ being a subcontinuum of a tree is itself a tree and hence is acyclic. Therefore $H^1(F) = 0$, which contradicts the fact that $F$ is a floor, and thus $H^1(A) = 0$. The semilocality connectedness of $X$ follows from the fact that $X$ is compact and locally connected (Lemma 1).

**Proof of (ii).** This is proved in Ward [15].

**Proof of (iii).** The cutpoint-order $P$ is order dense [14], and since $P = P^*$ by (ii), every maximal $P$-chain in $Px$ is connected [13]; thus $Px$ is connected. Similarly, each $xP$ is connected. Indeed, $Px$ itself is a tree and therefore, as has been proved in (i), $H^1(Px) = 0$.

**Proof of (iv).** This is obvious, since $z \in Px$ for every $x \in X$.

**Bibliography**


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