A FACTORIZATION ALGORITHM FOR $q \times q$ MATRIX-VALUED FUNCTIONS ON THE REAL LINE R

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1. Introduction. Let $\boldsymbol{H}_\delta(\Delta^+)$, $0 < \delta \leqq \infty$, be the set of all $q \times q$ matrix-valued functions $\boldsymbol{F}^+ = [f_{ij}^+]$, $1 \leqq i, j \leqq q$, on the upper half-plane Δ^+ such that each entry f_{ij}^+ is in the Hardy class $H_\delta(\Delta^+)$. Let $\boldsymbol{L}_\delta^{0+}(R)$ be the set of all $q \times q$ matrix-valued functions $\boldsymbol{F}$ on the real line R which are the nontangential limits of the functions $\boldsymbol{F}^+$ in $\boldsymbol{H}_\delta(\Delta^+)$. Similarly let $\boldsymbol{H}_\delta(\Delta^-)$ and $\boldsymbol{L}_\delta^{0-}(R)$ denote the appropriate classes of $q \times q$ matrix-valued functions on the lower half-plane Δ^- and on R respectively.

An important problem in multivariate prediction theory with continuous time is, given a nonnegative hermitian $q \times q$ matrix-valued function $\boldsymbol{F}$ on R such that $\boldsymbol{F} \in \boldsymbol{L}_1(R)$ and $\{\log\det \boldsymbol{F}(\lambda)\}/(1+\lambda^2) \in L_1(R)$, to find a $q \times q$ matrix-valued function $\boldsymbol{\Phi}$ on R such that

$$\boldsymbol{F}(\lambda) = \boldsymbol{\Phi}(\lambda)\boldsymbol{\Phi}^*(\lambda) \text{ a.e. on } R,$$

where $\boldsymbol{\Phi}(\lambda) = \int_0^\infty \boldsymbol{C}(t)e^{i\lambda t}\,dt$, $\boldsymbol{C}(\cdot) \in \boldsymbol{L}_2(R)$, and if $\boldsymbol{\Phi}^+$ is the holomorphic extension of $\boldsymbol{\Phi}$ to Δ^+, then

$$\boldsymbol{\Phi}^+(i) > \boldsymbol{0} \text{ and } \det\boldsymbol{\Phi}^+(i) = \exp \frac{1}{\pi}\int_{-\infty}^{\infty} \log\det \boldsymbol{F}(\lambda)\frac{d\lambda}{1+\lambda^2} > 0.$$

An iterative procedure which yields an infinite series for $\boldsymbol{\Phi}$ in terms of $\boldsymbol{F}$ has been given by Wiener and Masani in [9] for the case that $\boldsymbol{F}$ is defined on the unit circle C. To carry out the algorithm they assumed

$$(1) \qquad \boldsymbol{F}(e^{i\theta}) = \boldsymbol{I} + \boldsymbol{M}(e^{i\theta}) \;\&\; \underset{0 \leqq \theta \leqq 2\pi}{\text{ess.l.u.b.}} |\boldsymbol{M}(e^{i\theta})|_B \leqq \mu < 1 \quad (|\ |_B = \text{Banach norm}).$$

When $\boldsymbol{F}$ is nonnegative hermitian-valued on C they showed that the algorithm would hold under a weaker condition, namely that there exist constants c_1, c_2, $0 < c_1 \leqq c_2 < \infty$, such that

$$(2) \qquad c_1\boldsymbol{I} \leqq \boldsymbol{F} \leqq c_2\boldsymbol{I}.$$

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In [7] Masani was able to improve the results he and Wiener gave in [9] by assuming in lieu of condition (2) that

(i) $\boldsymbol{F}$ is nonnegative hermitian-valued on C such that $\boldsymbol{F} \in \boldsymbol{L}_1(C)$.

(3) (ii) $\boldsymbol{F}^{-1}$ exists a.e. on C and $\boldsymbol{F}^{-1} \in \boldsymbol{L}_1(C)$.

(iii) If $\nu(e^{i\theta}), \mu(e^{i\theta})$ denote the smallest and largest eigenvalues of $\boldsymbol{F}(e^{i\theta})$, then $\mu/\nu \in L_1(C)$.

In his work Masani repeatedly made use of the fact that $\boldsymbol{F}^{-1} \in \boldsymbol{L}_1(C)$. However for many situations this condition is not necessarily satisfied. Moreover to accomplish his algorithm he had to factor the scalar-valued function $(\nu + \mu)$ which is not known.

In [10] an attempt by Wong and Thomas was made to obtain results similar to those in [9] for the case that $\boldsymbol{F}$ is nonnegative hermitian-valued and on the real line R. Their proof is ambiguous and incorrect. The correct result can be obtained as consequence of our work.

§2 is devoted to preliminary results which are used in the rest of this paper. In §3 we consider the factorization problem for any $q \times q$ matrix-valued function $\boldsymbol{F}$ on R, not necessarily hermitian, such that

$$(4) \qquad \boldsymbol{F}(\lambda) = \frac{1}{\pi}\,\frac{1}{1+\lambda^2}\,\{\boldsymbol{I} + \boldsymbol{M}(\lambda)\}, \text{ where } \operatorname*{ess.\,l.u.b.}_{-\infty<\lambda<\infty} |\boldsymbol{M}(\lambda)|_B \leqq \mu < 1.$$

We show that $\boldsymbol{F}$ can be factored in the form

$$\boldsymbol{F}(\lambda) = \boldsymbol{\Phi}_1(\lambda)\boldsymbol{\Phi}_2(\lambda) \text{ a.e. on } R,$$

and obtain an algorithm similar to that given in [9] by Wiener and Masani for the factors $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$. In §4 we consider the case that $\boldsymbol{F}$ is nonnegative hermitian-valued and on the real line R. Our condition pertains to the boundedness of the eigenvalues of $\boldsymbol{F}(\lambda)$ and is weaker than condition (4). In §5 we look at a special case when $\boldsymbol{F}$ is a rational function on R.

2. **Preliminary results.** As in [9] bold face letters $\boldsymbol{A}$, $\boldsymbol{B}$, etc. will denote $q \times q$ matrices with complex entries a_{ij}, b_{ij}, etc. and bold face letters $\boldsymbol{F}, \boldsymbol{\Phi}$, etc. will denote functions whose values are such matrices. tr, det, $*$, will be reserved for the trace, determinant and adjoint of matrices. $|\boldsymbol{A}|_B$, $|\boldsymbol{A}|_E$ will denote the Banach and Euclidean norms of $\boldsymbol{A}$ [9, II]. The $(k,+)$th and $(k,-)$th Laguerre functions l_k^+ and l_k^- are defined on the extended complex plane by $l_k^+(w) = i\pi^{-1/2}\{(w-i)^k/(w+i)^{k+1}\}$ and $l_k^-(w) = -i\pi^{-1/2}\{(w+i)^k/(w-i)^{k+1}\}$ respectively.

We shall be concerned with the sets $\boldsymbol{L}_\delta(R)$ of $q \times q$ matrix-valued functions $\boldsymbol{F}$ on R each entry of which is in $L_\delta(R)$, $0 < \delta \leqq \infty$. If $\boldsymbol{F}$ is in $\boldsymbol{L}_\delta(R)$, $1 \leqq \delta < \infty$, then for each $k \geqq 0$, the matrix $\boldsymbol{A}_k^{\pm} = \int_{-\infty}^{\infty} l_k^{\mp}(\lambda)\,\boldsymbol{F}(\lambda)d\lambda$ is called the $(k, \pm)$th Laguerre coefficient of $\boldsymbol{F}$. For $1 \leqq \delta < \infty$, $\boldsymbol{L}_\delta^{0+}(R)$ will denote the subset of functions in $\boldsymbol{L}_\delta(R)$ whose $(k,-)$th Laguerre coefficients $\boldsymbol{A}_k^-$, $k \geqq 0$, vanish and $\boldsymbol{L}_\delta^+(R)$ will consist of functions in $\boldsymbol{L}_\delta^{0+}$ whose $(0,+)$th Laguerre coefficient are zero. Similarly

$L_\delta^{0-}(R)$ and $L_\delta^-(R)$ may be defined. This definition of $L_\delta^{0\pm}(R)$ is equivalent to the one mentioned in §1.

If $F \in L_2(R)$ and has the Laguerre coefficients $(A_k, \pm)$, $k \geqq 0$, then F_{0+} will denote the function in $L_2^{0+}(R)$ whose $(k,+)$th Laguerre coefficients are A_k^+ for $k \geqq 0$, and whose remaining Laguerre coefficients are zero. F_+ will denote the function $F_{0+} - A_0^+ l_0^+$. Similarly F_{0-} and F_- may be defined in $L_2^{0-}(R)$. In $L_2(R)$ we introduce the Gramian, inner product and norm

$$(\Phi, \Psi) = \int_{-\infty}^{\infty} \Phi(\lambda)\Psi^*(\lambda)d\lambda,$$

$$((\Phi, \Psi)) = \operatorname{tr}(\Phi, \Psi), \qquad \|\Phi\| = (\operatorname{tr}(\Phi,\Phi))^{1/2}.$$

We now state a lemma the proof of which is in [8, §4].

2.1 Lemma. *Let* $F \in L_\delta^{0+}(R)$. *Then* (a) $\det F \in L_{\delta/q}^{0+}(R)$.

(b) *Either* $\det F = 0$ *a.e. on* R *or* $\{\log|\det F(\lambda)|\}/(1+\lambda^2) \in L_1(R)$.

2.2 Definition. (a) Φ is said to be of *full-rank* iff $|\det \Phi(\lambda)| > 0$ a.e. on R.

(b) Φ is called an *optimal function* in $L_\delta^{0+}(R)$, $0 < \delta \leqq \infty$, iff $\Phi \in L_\delta^{0+}(R)$, $\Phi^+(i) \geqq 0$ and $\Psi \in L_\delta^{0+}(R)$, $\Psi\Psi^* = \Phi\Phi^* \Rightarrow \{\Psi^+(i)(\Psi^+)^*(i)\}^{1/2} \leqq \Phi^+(i)$.

(c) The notion of optimality for a function Φ in $L_\delta^{0-}(R)$ may be introduced similarly.

If $q = 1$, a nonzero function Φ in $L_\delta^{0+}(R)$, $0 < \delta \leqq \infty$, is optimal iff

$$\Phi^+(i) = \exp \frac{1}{\pi} \int_{-\infty}^{\infty} \log|\Phi(\lambda)| \frac{d\lambda}{1+\lambda^2} > 0.$$

For $q > 1$ the following lemma yields a necessary and sufficient condition for optimality.

2.3 Lemma. (a) *Let* $\Phi \in L_\delta^{0+}(R)$. *Then the following statements are equivalent*:

(i) Φ *is of full-rank and optimal in* $L_\delta^{0+}(R)$,

(ii) $\Phi^+(i) \geqq 0$ *and* $\det \Phi$ *is a nonzero optimal function in* $L_{\delta/q}^{0+}(R)$.

(b) *Analogous results to* (a) *holds for a function in* $L_\delta^{0-}(R)$.

3. A general factorization algorithm for $q \times q$ matrix-valued functions on the real line R.

3.1 *Factorization Problem.* Given a $q \times q$ matrix-valued function F on the real line R such that $F \in L_1(R)$ and $\{\log|\det F(\lambda)|\}/(1+\lambda^2)$ is in $L_1(R)$, to find functions Φ_1, Φ_2 on R with the properties:

$$F(\lambda) = \Phi_1(\lambda)\Phi_2(\lambda) \text{ a.e. on } R,$$

$$\Phi_1 \in L_2^{0+}(R), \quad \Phi_2 \in L_2^{0-}(R),$$

$$|\det(\Phi_1)^+(i)| = |\det(\Phi_2)^-(-i)| = \exp \frac{1}{2\pi} \int_{-\infty}^{\infty} \log|\det F(\lambda)| \frac{d\lambda}{1+\lambda^2}.$$

We shall solve this problem under the following assumption.

3.2 ASSUMPTION. $\mu = \text{ess.l.u.b}_{-\infty<\lambda<\infty}|\pi(1+\lambda^2)\boldsymbol{F}(\lambda) - \boldsymbol{I}|_B < 1$. If we let $\boldsymbol{M}(\lambda) = \pi(1+\lambda^2)\boldsymbol{F}(\lambda) - \boldsymbol{I}$, then $\boldsymbol{F}(\lambda) = (\pi(1+\lambda^2))^{-1}\{\boldsymbol{I} + \boldsymbol{M}(\lambda)\}$, where

$$\mu = \underset{-\infty<\lambda<\infty}{\text{ess.l.u.b.}} |\boldsymbol{M}(\lambda)|_B < 1.$$

3.3 DEFINITION. We define two operators $\mathscr{P}_+$, $\mathscr{P}_-$ on $\boldsymbol{L}_2(R)$ by

$$\mathscr{P}_+(\boldsymbol{\Phi}) = (\boldsymbol{\Phi M})_+, \qquad \mathscr{P}_-(\boldsymbol{\Phi}) = (\boldsymbol{M\Phi})_-,$$

where the operations $(\ )_+$, $(\ )_-$ are the same as in §2.

Some easily established properties of these operators are stated in the next lemma.

3.4 LEMMA. (a) *$\mathscr{P}_+$ and $\mathscr{P}_-$ are bounded linear operators on $\boldsymbol{L}_2(R)$ into $\boldsymbol{L}_2^+(R)$ and $\boldsymbol{L}_2^-(R)$ respectively and $|\mathscr{P}_+|$, $|\mathscr{P}_-| \leqq \mu$.*

(b) *If $\mathscr{I}$ is the identity operator on $\boldsymbol{L}_2(R)$, then $\mathscr{I} + \mathscr{P}_+$ and $\mathscr{I} + \mathscr{P}_-$ are invertible and*

$$(\mathscr{I} + \mathscr{P}_\pm)^{-1} = \mathscr{I} - \mathscr{P}_\pm + \mathscr{P}_\pm^2 - \cdots,$$

where the series is absolutely convergent in the Banach algebra $\mathscr{A}$ of bounded linear operators on $\boldsymbol{L}_2(R)$.

(c) $\mathscr{P}_+^{n+1}(\boldsymbol{\Phi}) = (\mathscr{P}_+^n(\boldsymbol{\Phi})\boldsymbol{M})_+$, $\mathscr{P}_-^{n+1}(\boldsymbol{\Phi}) = (\boldsymbol{M}\mathscr{P}_-^n(\boldsymbol{\Phi}))_-$.

(d) $\mathscr{P}_+(l_0^-\boldsymbol{I}) = (l_0^+\boldsymbol{M})_+, \mathscr{P}_+^2(l_0^+\boldsymbol{I}) = ((l_0^+\boldsymbol{M})_+\boldsymbol{M})_+, \cdots \mathscr{P}_-(l_0^-\boldsymbol{I}) = (\boldsymbol{M}l_0)_-, \mathscr{P}_-^2(l_0^-\boldsymbol{I}) = (\boldsymbol{M}(\boldsymbol{M}l_0^-)_-)_-, \cdots$.

(e) $\|\mathscr{P}_+^n(l_0^+\boldsymbol{I})\|$, $\|\mathscr{P}_-^n(l_0^-\boldsymbol{I})\| \leqq (q\mu^n)^{1/2}$.

The following definition therefore makes sense.

3.5 DEFINITION. (a) $\boldsymbol{\Psi}_{0+} = (\mathscr{I} + \mathscr{P}_+)^{-1}(l_0^+\boldsymbol{I}), \boldsymbol{\Psi}_{0-} = (\mathscr{I} + \mathscr{P}_-)^{-1}(l_0^-\boldsymbol{I})$.

(b) $\boldsymbol{G} = (1/(l_0^- l_0^+))\boldsymbol{\Psi}_{0+}(\boldsymbol{I} + \boldsymbol{M})\boldsymbol{\Psi}_{0-}$.

We proceed to prove the crucial result that the function $\boldsymbol{G}$ is constant-valued, the constant being an invertible matrix. This will be done by considering the Laguerre coefficients of $l_0^- l_0^+\boldsymbol{G}$. We shall first show that $l_0^- l_0^+\boldsymbol{G} \in \boldsymbol{L}_1(R)$, and therefore has such a series.

3.6 LEMMA.

(a)
$$\boldsymbol{\Psi}_{0+} = l_0^+\boldsymbol{I} - \mathscr{P}_+(l_0^+\boldsymbol{I}) + \mathscr{P}_+^2(l_0^+\boldsymbol{I}) - \cdots \in \boldsymbol{L}_2^{0+}(R),$$
$$\boldsymbol{\Psi}_{0-} = l_0^-\boldsymbol{I} - \mathscr{P}_-(l_0^-\boldsymbol{I}) + \mathscr{P}_-^2(l_0^-\boldsymbol{I}) - \cdots \in \boldsymbol{L}_2^{0-}(R),$$

the series being absolutely convergent in the norm of $\boldsymbol{L}_2(R)$, and

(b)
$$l_0^- l_0^+\boldsymbol{G} \in \boldsymbol{L}_1(R).$$

Proof. (a) The series expansions obviously follow from the last definition and the expansions given in 3.4. Since the ranges of $\mathscr{P}_+$ and $\mathscr{P}_-$ are included in

$L_2^{0+}(R)$ and $L_2^{0-}(R)$ and these are closed subspaces of $L_2(R)$, it follows from the expansions that $\Psi_{0+} \in L_2^{0+}(R)$ and $\Psi_{0-} \in L_2^{0-}(R)$. Also these series converge in the $L_2(R)$ norm, since by 3.4 (e),

$$\sum_{n \geqq 0} \| \mathscr{P}_+^n(l_0^+ I) \| \text{ and } \sum_{n \geqq 0} \| \mathscr{P}_-^n(l_0^- I) \| < \sqrt{q} \sum_{n \geqq 0} \mu^n < \infty .$$

(b) follows from (a), since $l_0^- l_0^+ G = \Psi_{0+}(I+M)\Psi_{0-}$ and $I+M \in L_\infty(R)$. (Q.E.D.)

3.7 Theorem. *Let A_0^- be the $(0,-)$th Laguerre coefficient of $(M\Psi_{0-})$ and B_0^+ be the $(0,+)$th Laguerre coefficient of $(\Psi_{0+}M)$. Then*

(a) $(I+M)\Psi_{0-} = (I+A_0^-)l_0^- + (M\Psi_{0-})_{0+}$, $\Psi_{0+}(I+M) = (I+B_0^+)l_0^+ + (\Psi_{0+}M)_{0-}$.

(b) $G = \text{constant} = I + A_0^- = I + B_0^+$; $A_0^- = B_0^+$.

(c) $I+M, \Psi_{0+}, \Psi_{0-}$ *are invertible a.e. on R and* $(I+M)^{-1} \in L_\infty(R)$.

(d) G *is invertible.*

Proof. (a) Since $I+M \in L_\infty(R)$ and $\Psi_{0-} \in L_2(R)$, $(I+M)\Psi_{0-} \in L_2(R)$. Also by 3.3 and 3.5,

$$\begin{aligned}(I+M)\Psi_{0-} &= \Psi_{0-} + (M\Psi_{0-})_- + A_0^- l_0^- + (M\Psi_{0-})_{0+} \\ &= (\mathscr{I} + \mathscr{P}_-)(\Psi_{0-}) + A_0^- l_0^- + (M\Psi_{0-})_{0+} \\ &= l_0^- I + A_0^- l_0^- + (M\Psi_{0-})_{0+}.\end{aligned}$$

This gives the first relation in (a). The second is proved similarly.

(b) By 3.5 (b),

$$l_0^- l_0^+ G = \Psi_{0+}(I+M)\Psi_{0-} = \Psi_{0+}\{(I+A_0^-)l_0^+ + \text{a term in } L_2^{0+}(R)\},$$

hence for all $k \geqq 1$, the $(k,-)$th Laguerre coefficient of $l_0^- l_0^+ G$ is 0. But we also know that

$$\begin{aligned} l_0^- l_0^+ G &= \Psi_{0+}(I+M)\Psi_{0-} \\ &= \{(I+B_0^+)l_0^+ + \text{a term in } L_2^{0-}(R)\}\Psi_{0-},\end{aligned}$$

hence for all $k \geqq 1$, the $(k,+)$th Laguerre coefficient of $l_0^- l_0^+ G$ is $\mathbf{0}$. Thus for all $k \geqq 1$, the $(k,-)$th and $(k,+)$th Laguerre coefficients of $l_0^- l_0^+ G$ are $\mathbf{0}$. Therefore $l_0^-(\lambda) l_0^+(\lambda) G(\lambda) = C_0^- l_0^-(\lambda) + C_0^+ l_0^+(\lambda) = 2C_0^-/\pi(1+\lambda^2) + i(C_0 - C_0^-)/\pi^{1/2}(\lambda + i)$ a.e. Since we know $l_0^- l_0^+ G \in L_1(R)$, therefore $C_0^- = C_0^+ = C$ and hence

$$l_0^-(\lambda) l_0^+(\lambda) G(\lambda) = C\{l_0^-(\lambda) + l_0^+(\lambda)\} = 2\pi^{1/2} C l_0^-(\lambda) l_0^+(\lambda).$$

Thus $G(\lambda) = 2\pi^{1/2} C = \text{constant matrix}$.

The range of $\mathscr{P}_+$ is included in $L_2^+(R)$, and therefore by 3.6 (a), $\Psi_{0+} = l_0^+ I + \Psi_+$, where $\Psi_+ \in L_2^+(R)$. This together with the first equality in (a) entails that

$$l_0^- l_0^+ G = \Psi_{0+}(I+M)\Psi_{0-} = (l_0^+ I + \Psi_+)\{(I+A_0^-)l_0^- + (M\Psi_{0-})_{0+}\}$$
$$= l_0^- l_0^+(I+A_0) + l_0^+(M\Psi_{0-})_{0+} + \Psi_+(I+A_0^-)l_0^- + \Psi_+(M\Psi_{0-})_{0+}.$$

If we integrate both sides over R we get $G = I + A_0^-$.

The other expression for G is proved similarly.

(c) By Assumption 3.2, $I + M$ is in the Banach algebra $L_\infty(R)$ at a distance μ less than 1 from I. Hence it is invertible and $(I+M)^{-1} \in L_\infty(R)$. Next since $\Psi_{0+} \in L_2^{0+}(R)$ and $(\Psi_{0+})^+(i) = \pi^{-1/2}/2 > 0$, by 2.1 (b), $|\det \Psi_{0+}| > 0$ a.e.. Consequently Ψ_{0+} is invertible a.e. on R. We can similarly show that Ψ_{0-} is invertible a.e. on R.

(d) By 3.5 (b), $l_0^- l_0^+ G = \Psi_{0+}(I+M)\Psi_{0-}$. For almost all $\lambda \in R$, each of $\Psi_{0+}(\lambda)$, $I + M(\lambda)$ and $\Psi_{0-}(\lambda)$ is invertible. Therefore G is invertible. (Q.E.D.)

In view of 3.7 (c) we may invert the equation $l_0^- l_0^+ G = \Psi_{0+}(I+M)\Psi_{0-}$ to obtain

$$F = l_0^- l_0^+(I+M) = \{(l_0^+)^2 \Psi_{0+}^{-1}\} G \{(l_0^-)^2 \Psi_{0-}^{-1}\} \text{ a.e. on } R.$$

We shall now show that $(l_0^+)^2 \Psi_{0+}^{-1}$ and $(l_0^-)^2 \Psi_{0-}^{-1}$ are themselves in $L_2^{0+}(R)$ and $L_2^{0-}(R)$ respectively, so that we have a factorization of the desired kind.

3.8 Lemma. *$(l_0^+)^2 \Psi_{0+}^{-1} \in L_2^{0+}(R)$ and $(l_0^-)^2 \Psi_{0-}^{-1} \in L_2^{0-}(R)$.*

Proof. Let A be the $(0,-)$th Laguerre coefficient of $M\Psi_{0-}$ or equivalently the $(0,+)$th Laguerre coefficient of $\Psi_{0+}M$ (cf. 3.7 (b)). By 3.7 (a) and 3.5 (b), $\Psi_{0+}\{(I+A)l_0^- + (M\Psi_{0-})_{0+}\}G^{-1} = \Psi_{0+}(I+M)\Psi_{0-}G^{-1} = l_0^- l_0^+ I$. Therefore $\Psi_{0+}^{-1} = \{(I+A)l_0^- + (M\Psi_{0-})\}G^{-1}$. Since G^{-1} is a constant, it easily follows that $(l_0^+)\Psi_{0+}^{-1} \in L_2^{0+}(R)$.

(b) is proved similarly. (Q.E.D.)

3.9 Lemma. (a) *Ψ_{0+} is a full-rank optimal function in $L_2^{0+}(R)$ and Ψ_{0-} is a full-rank optimal function in $L_2^{0-}(R)$.*

(b) $\exp(\pi^{-1}\int_{-\infty}^{\infty} \log|\det(I+M(\lambda))|(1+\lambda^2)^{-1}d\lambda = |\det G|$.

Proof. By 3.6 (a), $\Psi_{0+} \in L_2^{0+}(R)$ and $(\Psi_{0+})^+(i) = \pi^{-1/2}/2I > 0$. Also by 3.7 (c), $|\det \Psi_{0+}| > 0$ a.e., and by 3.8, $(l_0^+)^2\Psi_{0+}^{-1} \in L_2^{0+}(R)$. Therefore by 2.3, Ψ_{0+} is a full-rank optimal function in $L_2^{0+}(R)$.

The results for Ψ_{0-} can be proved similarly.

(b) By 3.5 (b), $l_0^- l_0^+ G = \Psi_{0+}(I+M)\Psi_{0-}$. Hence $\log|\det G| + q\log(l_0^- l_0^+) = \log|\det \Psi_{0+}| + \log|\det \Psi_{0-}| + \log|(I+M)|$. Multiplying both sides by $l_0^- l_0^+$ and integrating over R, by (a) and 2.3 the result follows. (Q.E.D.)

To sum up, we have proved the following theorem.

3.10 Theorem. *If* (i) *the function $M \in L_\infty(R)$ and*

$$\mu = \operatorname*{ess.l.u.b.}_{-\infty<\lambda<\infty} |M(\lambda)|_B < 1,$$

(ii) $\Psi_{0+} = l_0^+ I - (l_0^+ M)_+ + ((l_0^+ M)_+ M)_+ - \cdots, \Psi_{0-} = l_0^- I - (l_0^- M)_- + (M(l_0^- M)_-)_- - \cdots$, *and*

(iii) $G = \Psi_{0+}(I+M)\Psi_{0-}/(l_0^- l_0^+)$.

Then (a) *the* $(0,-)$*th Laguerre coefficient of* $M\Psi_{0-}$ *is equal to the* $(0,+)$*th Laguerre coefficient of* $\Psi_{0+}M$.

(b) Ψ_{0+} *and* $(l_0^+)^2\Psi_{0+}^{-1}$ *are of full-rank optimal functions in* $L_2^{0+}(R)$ *and* $(\Psi_{0+})^+(i) = (\pi^{-1/2}/2)I$.

(c) Ψ_{0-} *and* $(l_0^-)^2\Psi_{0-}^{-1}$ *are of full-rank optimal functions in* $L_2^{0-}(R)$ *and* $(\Psi_{0-})^-(-i) = (\pi^{-1/2}/2)I$.

(d) $(I+M) = l_0^- l_0^+ \Psi_{0+}^{-1} G \Psi_{0-}^{-1}$ *and consequently*

$$F = l_0^- l_0^+ (I+M) = \{(l_0^+)^2\Psi_{0+}^{-1}\}\, G\{(l_0^-)^2\Psi_{0-}^{-1}\}.$$

(e) $|\det G| = \exp(1/\pi)\int_{-\infty}^{\infty} \log|\det(I+M(\lambda))|(1+\lambda^2)^{-1}d\lambda$.

Now let $\sqrt{G}$ *be any square root of* G. *Then setting*

$$\Phi_1 = (l_0^+)^2\Psi_{0+}^{-1}\sqrt{G}, \qquad \Phi_2 = (l_0^-)^2\sqrt{G}\Psi_{0-}^{-1},$$

we obtain a solution of the Factorization Problem 3.1 *under the Assumption* 3.2.

3.11 Remark. By a simple calculation, we get

$$(\Phi_1)^+(i) = (\Phi_2)^-(-i) = \frac{(G/\pi)^{1/2}}{2}$$

and

$$|\det(\Phi_1)^+(i)| = |\det(\Phi_2)^-(-i)| = \exp(1/2\pi)\int_{-\infty}^{\infty}\log|\det F(\lambda)|(1+\lambda^2)^{-1}d\lambda.$$

Since G may not be nonnegative hermitian, Φ_1 and Φ_2 are not optimal in $L_2^{0+}(R)$ and $L_2^{0-}(R)$. However taking polar decomposition of $(\Phi_1)^+(i)$ and $(\Phi_2)^-(-i)$ (cf. [2, §83]) we have $(\Phi_1)^+(i) = (\Phi_2)^-(-i) = P_0U_0$, where $P_0 > 0$ and U_0 is a unitary matrix. We see that the functions

$$\Phi_1^0 = \Phi_1 U_0^{-1}, \quad \Phi_2^0 = \Phi_2 U_0^{-1}$$

are optimal and of full-rank. Moreover

$$F = \Phi_1\Phi_2 = \Phi_1^0(U_0\Phi_2^0U_0).$$

The second factor will not in general be optimal in $L_2^{0-}(R)$, since

$$(U_0\Phi_2^0U_0)^-(-i) = U_0P_0U_0$$

which is not nonnegative hermitian. However when F is hermitian-valued, we shall show that $\Phi_2 = \Phi_1^*$, that G is hermitian, and that both factors can be taken to be optimal.

4. **A factorization algorithm for nonnegative hermitian $q \times q$ matrix-valued functions on the real line R.** We shall apply the algorithm obtained in §3 to the case in which $\boldsymbol{F}$ is hermitian-valued. In this case our restriction on $\boldsymbol{F}$ can be weakened and is stated in terms of the eigenvalues of $\boldsymbol{F}(\lambda)$, $\lambda \in R$. The final results are also stronger than those obtained in §3.

4.1 Assumption. $\boldsymbol{F}$ satisfies the following conditions:

(i) $\boldsymbol{F}$ is nonnegative, hermitian-valued on R such that $\boldsymbol{F} \in \boldsymbol{L}_1(R)$ and $\{\log\det \boldsymbol{F}(\lambda)\}/(1+\lambda^2) \in L_1(R)$.

(ii) There exist nonnegative measurable functions $g(\lambda)$ and $h(\lambda)$ such that $g\boldsymbol{I} \leqq \boldsymbol{F} \leqq h\boldsymbol{I}$, where $\delta = \text{ess.l.u.b.}_{-\infty<\lambda<\infty}\{h(\lambda)/g(\lambda)\} < \infty$.

It easily follows that $|[2/(g(\lambda)+h(\lambda))]\boldsymbol{F}(\lambda) - \boldsymbol{I}|_B \leqq (h(\lambda)-g(\lambda))/(h(\lambda)+g(\lambda)) \leqq \delta/(\delta+1) < 1$. Letting $\boldsymbol{M} = 2\boldsymbol{F}/(g+h) - \boldsymbol{I}$, we have

$$\boldsymbol{F} = (1/2)(g+h)(\boldsymbol{I}+\boldsymbol{M}) \,\&\, \underset{-\infty<\lambda<\infty}{\text{ess.l.u.b.}} \ |\boldsymbol{M}(\lambda)|_B \leqq \delta/(\delta+1) < 1.$$

We next state the following lemma.

4.2 Lemma. (a) *g, h and $(g+h) \in L_1(R)$.*

(b) *$\log g(\lambda)/(1+\lambda^2)$, $\log h(\lambda)/(1+\lambda^2)$ and $\log\{g(\lambda)+h(\lambda)\}/(1+\lambda^2)$ are in $L_1(R)$.*

By (a) and (b) there exists a full-rank optimal function ϕ in $L_2^{0+}(R)$ such that

$$(g+h)/2 = |\phi|^2.$$

Because $(g+h)/2$ is a scalar-valued function we can determine the $(k,+)$th Laguerre coefficient a_k of ϕ (cf. [1, §XII]) and may write

$$\phi = \sum_{k=0}^{\infty} a_k l_k^+.$$

It remains to obtain a factorization for $(\boldsymbol{I}+\boldsymbol{M})$. However in this case because $\boldsymbol{M}$ is hermitian-valued, $\mathscr{P}_-$ is expressible in terms of $\mathscr{P}_+$ by means of the adjoint operator and consequently the general algorithm given in §3 can be considerably simplified. More fully since $\boldsymbol{M}^* = \boldsymbol{M}$, by 3.3 for all $\boldsymbol{\Phi} \in \boldsymbol{L}_2(R)$ we have

$$(\mathscr{P}_+(\boldsymbol{\Phi}))^* = ((\boldsymbol{\Phi M}))_+^* = ((\boldsymbol{\Phi M})^*)_- = \mathscr{P}_-(\boldsymbol{\Phi}^*).$$

By induction, it readily follows that

$$(\mathscr{P}_+^n(l_0^+\boldsymbol{I}))^* = \mathscr{P}_-^n(l_0^-\boldsymbol{I}).$$

Hence by 3.6 (a), $\boldsymbol{\Psi}_{0+}^* = \boldsymbol{\Psi}_{0-}$ and therefore

$$\boldsymbol{G} = \boldsymbol{\Psi}_{0+}(\boldsymbol{I}+\boldsymbol{M})\boldsymbol{\Psi}_{0+}^*/(l_0^- l_0^+)$$

is nonnegative hermitian, in fact positive definite, since it is invertible. Letting

$\boldsymbol{\Phi}_1 = (l_0^+)^2\boldsymbol{\Psi}_{0+}^{-1}\sqrt{\boldsymbol{G}}$, where $\sqrt{\boldsymbol{G}}$ is now the unique positive definite square root of $\boldsymbol{G}$, then the equality in 3.10 (d) becomes

$$(\boldsymbol{I} + \boldsymbol{M}) = (\boldsymbol{\Phi}_1\boldsymbol{\Phi}_1^*)/(l_0^- l_0^+).$$

Since we have already seen (cf. 3.10 (b)) that $\boldsymbol{\Phi}_1$ and $(l_0^+)^2\boldsymbol{\Phi}_1^{-1} = \sqrt{\boldsymbol{G}^{-1}}\boldsymbol{\Psi}_{0+}$ are optimal functions of full-rank in $\boldsymbol{L}_2^{0+}(R)$, we have proved the following theorem.

4.3 THEOREM. *Let* (i) $\boldsymbol{F}$ *satisfy Assumption* 4.1,
(ii) $\boldsymbol{M} = \{2/(g+h)\}\boldsymbol{F} - \boldsymbol{I}$,
(iii) $\boldsymbol{\Psi}_{0+} = l_0^+\boldsymbol{I} - (l_0^+\boldsymbol{M})_+ + ((l_0^+\boldsymbol{M})_+\boldsymbol{M})_+ - \cdots$,
(iv) $\boldsymbol{G} = \boldsymbol{\Psi}_{0+}(\boldsymbol{I}+\boldsymbol{M})\boldsymbol{\Psi}_{0+}^*/(l_0^- l_0^+)$,
(v) $\boldsymbol{\Phi}_1 = (l_0^+)^2\boldsymbol{\Psi}_{0+}^{-1}\sqrt{\boldsymbol{G}}$.
Then (a) $\boldsymbol{\Phi}_1$ *and* $(l_0^+)^2\boldsymbol{\Phi}_1^{-1}$ *are full-rank optimal functions in* $\boldsymbol{L}_2^{0+}(R)$.
(b) $\boldsymbol{F} = |\phi|^2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_1^*/(l_0^- l_0^+)$,
where ϕ *is the full-rank optimal factor of the function* $(g+h)/2$.

Letting $\boldsymbol{\Phi} = \phi\boldsymbol{\Phi}_1/l_0^+$ we have

$$\boldsymbol{F} = \boldsymbol{\Phi}\boldsymbol{\Phi}^*.$$

In the following lemma we show that $\boldsymbol{\Phi}$ is the desired factor.

4.4 LEMMA. $\boldsymbol{\Phi} = \phi\boldsymbol{\Phi}_1/l_0^+$ *is a full-rank optimal function in* $\boldsymbol{L}_2^{0+}(R)$.

Proof. Since $\phi \in L_2^{0+}(R)$ and $\boldsymbol{\Phi}_1$ is in $\boldsymbol{L}_2^{0+}(R)$, $\phi = \sum_{k=0}^{\infty} a_k l_k^+$ and

$$\boldsymbol{\Phi}_1 = \sum_{k=0}^{\infty} \boldsymbol{A}_k l_k^+.$$

Since $(\boldsymbol{I}+\boldsymbol{M}) = \boldsymbol{\Phi}_1\boldsymbol{\Phi}_1^*/(l_0^- l_0^+)$,

(1) $$|\boldsymbol{\Phi}_1/l_0^+|_E^2 = \operatorname{tr}(\boldsymbol{\Phi}_1\boldsymbol{\Phi}_1^*/(l_0^+ l_0^-)) = \operatorname{tr}(\boldsymbol{I}+\boldsymbol{M}) \leqq 2q.$$

Since $\phi \in \boldsymbol{L}_2^{0+}(R)$, it follows by (1) that $\boldsymbol{\Phi} \in L_2(R)$. It is easy to see that for all $n \geqq 0$, the $(n,-)$th Laguerre coefficient of $\boldsymbol{\Phi}$ is 0 and the $(n,+)$th Laguerre coefficient of $\boldsymbol{\Phi}$ is $\sum_{k=0}^{\infty} a_k \boldsymbol{A}_{n-k}$. Hence $\boldsymbol{\Phi} = \sum_{n=0}^{\infty}\{\sum_{k=0}^{\infty} a_k\boldsymbol{A}_{n-k}\} \in \boldsymbol{L}_2^{0+}(R)$. (Q.E.D.)

Since each factor $(1/l_0^+)$, ϕ and $\boldsymbol{\Phi}_1$ is of full-rank so is the function

$$\boldsymbol{\Phi} = (1/l_0^+)\phi\boldsymbol{\Phi}_1.$$

Also since $l_0^+(i), \phi^+(i)$ are positive and $\boldsymbol{\Phi}_0^+(i) > \boldsymbol{0}$,

(2) $$\boldsymbol{\Phi}^+(i) = \{1/l_0^+(i)\}\phi^+(i)\boldsymbol{\Phi}_1^+(i) > \boldsymbol{0}.$$

By 2.3 it easily follows that

(3) $$\det\boldsymbol{\Phi}^+(i) = \exp \pi^{-1}\int_{-\infty}^{\infty} \log|\det\boldsymbol{\Phi}|(1+\lambda^2)^{-1}\,d\lambda > 0.$$

We have already seen that $\boldsymbol{\Phi}$ is a full-rank function in $L_2^{0+}(R)$ and by (3), det $\boldsymbol{\Phi}$ is a nonzero optimal function in $L_{2/q}^{0+}(R)$, therefore by (2) and 2.3, $\boldsymbol{\Phi}$ is a full-rank optimal function in $L_2^{0+}(R)$. (Q.E.D.)

4.5 Remark. Since $\boldsymbol{\Psi}_{0+}$ and $(l_0^+)^2\boldsymbol{\Psi}_{0+}^{-1} \in L_2^{0+}(R)$,

$$\boldsymbol{\Psi}_{0+} = \sum_{n\geq 0} \boldsymbol{A}_n l_n^+ \ \& \ (l_0^+)^2\boldsymbol{\Psi}_{0+}^{-1} = \sum_{n\geq 0} \boldsymbol{B}_n l_n^+. \tag{1}$$

From 3.6 (a), we find that $\boldsymbol{A}_0 = \boldsymbol{I}$ and for all $n \geqq 1$,

$$\boldsymbol{A}_n = -\boldsymbol{\Gamma}_n + \sum_{k=1}^{\infty} \boldsymbol{\Gamma}_k\boldsymbol{\Gamma}_{n-k} - \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \boldsymbol{\Gamma}_k\boldsymbol{\Gamma}_{m-k}\boldsymbol{\Gamma}_{n-m} + \cdots,$$

where $\boldsymbol{\Gamma}_k$ is the $(k, +)$th Lagurre coefficient of $(l_0^+\boldsymbol{M})$. The coefficients $\boldsymbol{A}_k$ are thus determinable. By the identity

$$\sum_{k=0}^{\infty} \boldsymbol{A}_k l_k^+ \cdot \sum_{k=0}^{\infty} \boldsymbol{B}_k l_k^+ = (l_0^+)^2\boldsymbol{I}.$$

The coefficients $\boldsymbol{B}_k$ can be found from the recurrence relations

$$\begin{aligned} \boldsymbol{A}_0\boldsymbol{B}_0 &= \boldsymbol{I} = \boldsymbol{B}_0\boldsymbol{A}_0, \\ \boldsymbol{A}_0\boldsymbol{B}_1 + \boldsymbol{A}_1\boldsymbol{B}_0 &= \boldsymbol{0} = \boldsymbol{B}_0\boldsymbol{A}_1 + \boldsymbol{B}_1\boldsymbol{A}_0, \\ \boldsymbol{A}_0\boldsymbol{B}_2 + \boldsymbol{A}_1\boldsymbol{B}_1 + \boldsymbol{A}_2\boldsymbol{B}_0 &= \boldsymbol{0} = \boldsymbol{B}_0\boldsymbol{A}_2 + \boldsymbol{B}_1\boldsymbol{A}_1 + \boldsymbol{B}_2\boldsymbol{A}_0. \end{aligned}$$

Since $\boldsymbol{A}_0 = \boldsymbol{I}$, matrix inversion will not be encountered in finding the $\boldsymbol{B}_k$'s. From $\boldsymbol{\Phi}_1 = (l_0^+)^2\boldsymbol{\Psi}_{0+}^{-1}\sqrt{\boldsymbol{G}}$, by (1), it follows that

$$\boldsymbol{\Phi}_1 = \left\{\sum_{k=0}^{\infty} \boldsymbol{B}_k l_k^+\right\}\sqrt{\boldsymbol{G}} = \sum_{k=0}^{\infty} (\boldsymbol{B}_k\sqrt{\boldsymbol{G}}) l_k^+.$$

From the proof of 4.4, we see that

$$\boldsymbol{\Phi} = \sum_{n\geq 0}\sum_{k\geq 0} a_k\boldsymbol{B}_{n-k} \sqrt{\boldsymbol{G}} l_n^+, \text{[2]}$$

and hence$(\sum_{k\geq 0} a_k\boldsymbol{B}_{n-k})\boldsymbol{G}^{1/2}$, the $(n, +)$th Laguerre coefficient of $\boldsymbol{\Phi}$, is determinable.

4.6 Remark Condition (ii) of Assumption 4.1 may equivalently be stated in terms of the eigenvalues of $\boldsymbol{F}(\lambda)$. In general these eigenvalues are not known and it is easier to work with any known pair of functions g and h satisfying condition (ii).

5. **A factorization algorithm for nonnegative hermitian rational** $q \times q$ **matrix-valued functions on the real line** R. In this section we consider nonnegative hermitian $q \times q$ matrix-valued functions $\boldsymbol{F}$ which satisfy Assumption 4.1 and in addition are rational. To simplify our work it is assumed that the poles of $l_0^+(g+h)\boldsymbol{F}^{-1}$ are simple in Δ^-. However the results are true for more general situations.

[2] $a_k, k \geqq 0$, is the $(k, +)$th Laguerre coefficient of the optimal factor ϕ of the scalar-valued function $(g+h)/2$ (cf. 4.2).

By 3.7 (a), $\Psi_{0+}(I+M) = (I+B_0^+)l_0^+ + (\Psi_{0+}M)_{0-}$, where B_0^+ is the $(0,+)$th Laguerre coefficient of $\Psi_{0+}M$. Since $(I+M)$ is a rational function, its inverse exists everywhere except possibly at finitely many points so that

$$\Psi_{0+} = \{I + B_0^+ + (\Psi_{0+}M)_{0-}/l_0^+\}\,(I+M)^{-1} \text{ a.e. on } R.$$

$\{I + B_0^+ + (\Psi_{0+}M)_{0-}/l_0^+\}$ admits an analytic extension $\{I + B_0^+ + (\Psi_{0+}M)_{0-}/l_0^+\}^-$ to Δ^- and $(I+M)^{-1}$, l_0^+ may be extended to meromorphic functions $(I+M)^{-1}(w)$, $l_0^+(w)$ to Δ^-. It easily follows that

$$\Psi_{0+}(\lambda) = \sum_{j=1}^{n} \frac{C_j}{\lambda - \sigma_j} \text{ a.e. on } R,$$

where the σ_j, $1 \leqq j \leqq n$, are the poles of $l_0^+(g+h)F^{-1}$ in Δ^- and the C_j, $1 \leqq j \leqq n$, are constant matrices.

To sum up we have proved the following theorem.

5.1 Theorem. *Let* (i) *F be a rational $q \times q$ matrix-valued function on the real line R satisfying Assumption* 4.1.

(ii)(3) *$l_0^+(g+h)F^{-1}$ have simple poles $\sigma_1, \cdots, \sigma_n$ in Δ^-. Then there exist constant matrices $C_1, C_2, \cdots, C_n$ such that*

$$\Psi_{0+}(\lambda) = \sum_{j=1}^{n} \frac{C_j}{\lambda - \sigma_j} \text{ a.e. on } R$$

We shall now indicate how the C_j's may be obtained. We know that $\Psi_{0+} = \sum_{k=0}^{\infty} A_k l_0^+$, where A_k are given by Remark 4.5. Having obtained $A_0, \cdots, A_{n-1}$, we assert the following corollary.

5.2 Corollary. *Let $A_k = [a_k^{r,s}]$ and $C_j = [c_j^{r,s}]$, $1 \leqq r, s \leqq q$. Then*

$$c_j^{r,s} = -\frac{1}{2\sqrt{\pi}} \frac{\det \Delta_j^{r,s}}{\det \Delta},$$

where

$$\Delta = \left[\frac{(\sigma_j + i)^k}{(\sigma_j - i)^{k+1}}\right], \quad 1 \leqq j \leqq n; \quad 0 \leqq k \leqq n-1;$$

and $\Delta_j^{r,s}$ is obtained from Δ by replacing the jth column of Δ by the transpose of $(a_0^{r,s}, a_1^{r,s}, \cdots, a_{n-1}^{r,s})$.

5.3 Remark. The later A_k $(k \geqq n)$ need not be computed from the relations given in Remark 4.5, but directly from $C_1, \cdots, C_n$ as indicated in the next corollary.

5.4 Corollary. *For all $k \geqq 0$, A_k are given by*

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(3) We may choose g and h to be rational. E.g. $1/\mathrm{tr}\, F^{-1}$ and $\mathrm{tr}\, F$ are such a pair of functions.

$$A_k = -2\sqrt{\pi} \sum_{j=1}^{n} C_j \frac{(\sigma_j + i)^k}{(\sigma_j - i)^{k+1}}.$$

In general Φ is given by an expression containing infinitely many terms. However for this case a closed-form expression for the factor is obtainable.

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