INTEGRAL GEOMETRY IN HOMOGENEOUS SPACES(¹)

BY
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1. **Introduction.** In [9] Federer proved an integralgeometric formula concerning the integral over the group of isometries of n dimensional Euclidean space of the $k+l-n$ dimensional Hausdorff measure $H^{k+l-n}[A \cap g(B)]$ of the intersection of a Hausdorff k rectifiable subset A with an isometric image of an l rectifiable subset B, $k+l \geqq n$. He also proved an integralgeometric formula concerning the integral over the space of l dimensional planes E of $H^{k+l-n}(A \cap E)$.

It is natural to seek analogous formulas for subsets of an n dimensional Riemannian manifold X with a transitive group of isometries G; partial results in this direction were obtained by Chern [4] and Kurita [20]. It suffices to consider proper submanifolds of class 1.

Let A_0, B_0 be proper k, l dimensional submanifolds of class 1 of X and $A \subset A_0$, $B \subset B_0$ be Borel sets. Assume that G acts transitively on the set of tangent spaces of A_0, and of B_0, respectively. Let Ψ be a (left invariant) Haar measure on G.

Suppose $k+l \geqq n$. There exists a constant α depending only upon A_0, B_0 and Ψ such that

$$(*) \qquad \int_G H^{k+l-n}[A \cap g(B)]\, d\Psi g = \alpha H^k(A) \int_B \Delta\, dH^l,$$

where Δ is a positive function induced on X by the modular function of G. $\alpha > 0$ if and only if for some $g \in G$ there exists $a \in A_0 \cap g(B_0)$ for which the union of the tangent space $T_a(A_0)$ of A_0 at a with $T_a[g(B_0)]$ spans $T_a(X)$.

Let $\mathscr{E}$ be a set of closed l dimensional submanifolds of X such that G acts transitively on $\mathscr{E}$ and if $E \in \mathscr{E}$, then $G \cap \{g : g(E) = E\}$ is transitive on E. Also assume that $\mathscr{E}$ has a G invariant measure Φ. Then

$$\int_{\mathscr{E}} H^{k+l-n}(A \cap E)\, d\Phi E = \beta\alpha H^k(A),$$

where β is a positive constant depending on the choice of Φ.

Suppose $k+l \leqq n$. Assume G to be compact, $\dim G = m+n$. If the metric on X is related in a certain way to a bi-invariant metric on G, then

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$$\int_G H^0[A \cap g(B)]dH^{k+l+m}g = \gamma H^k(A)H^l(B).$$

If instead of compact G, one has $X = \mathbf{R}^n$ and $G = \mathbf{R}^n \times O_n$, then

$$\int_{O_n}\int_{R^n} H^0(A \cap [z + g(B)])dH^{k+l}zd\Phi_n g = \gamma H^k(A)H^l(B),$$

where Φ_n is a Haar measure on O_n. γ is a constant depending only upon A_0 and B_0. $\gamma > 0$ if and only if for some $g \in G$ there exists $a \in A_0 \cap g(B_0)$ for which $T_a(A_0) \cap T_a[g(B_0)] = 0$. This theorem is a special case of 11.1 and is closely related to a result of Freilich [14]. In proving this theorem, Federer's coarea formula is generalized to give nontrivial results for maps $f: X \to Y$ such that $\operatorname{rank} f_{\#}(x) < \dim Y$ for $x \in X$.

If X has constant curvature, α and γ are independent of A_0, B_0.

Now suppose $k + l \geqq n$. In [12] Federer developed a theory which can be used to define in a natural manner the intersections of a normal k current with either the isometric images of a normal l current or the elements of $\mathscr{E}$. One may therefore seek integralgeometric formulas for normal currents.

In order to define these intersections, a "lifting" map $L_{\mathscr{B}}$ is defined and studied for fibre bundles $\mathscr{B}$ having coherently oriented fibres. $L_{\mathscr{B}}$ is a chainmonomorphism of the complex of normal currents in the base space into the complex of normal currents in the bundle space, continuous on N bounded sets.

Suppose X is oriented and has constant curvature, S is a normal k current in X and T is a normal l current in X. Then $S \cap g_{\#}(T)$ is a normal $k + l - n$ current for almost all $g \in G$ and there exists a constant $\delta > 0$, depending only upon k, l and Ψ, such that for each bounded Baire k form ϕ on X and each bounded Baire l form ψ on X,

$$\int_G S \cap g_{\#}(T)\ (*[*\phi \wedge *g^{-1\#}(\psi)])d\Psi g = \begin{cases} \delta S(\phi)T(\psi) & \text{if } k > n/2 \text{ or } l > n/2, \\ \delta[S(\phi)T(\psi) + S(*\phi)\,T(*\psi)] & \text{if } k = l = n/2. \end{cases}$$

Now assume that the elements of $\mathscr{E}$ are oriented and for $E \in \mathscr{E}$ $G \cap \{g : g_{\#}(E) = E\}$ acts transitively on E. Then $S \cap E$ is a normal $k + l - n$ current for almost all $E \in \mathscr{E}$, and there exists a constant $\varepsilon > 0$, depending only upon k, l and Φ, such that

$$\int_{\mathscr{E}} S \cap E[*(*\phi \wedge *\psi_E)]d\Phi E = \varepsilon S(\phi)$$

for each bounded Baire k form ϕ on X. [ψ_E is the covariant dual of the positively oriented unit l vector field on E.] These statements are also true for quasi-normal currents.

A corollary of this theorem generalizes to dimensions less than n a theorem proved independently by Krickeberg [19] and Federer [10]: Suppose $k + l > n$. A quasi-normal k current S is normal if and only if

$$\int_{\mathscr{E}} M[\partial(S \cap E)]d\Phi E < \infty .$$

If G acts transitively on the tangent k vector field of S, it is shown that for each Borel subset A of X,

$$\int_{\mathscr{E}} \| S \cap E \| (A) d\Phi E = \lambda \| S \| (A),$$

where $\| S \|$ is the variation measure of S and λ depends upon S and Φ. An analogous generalization of (*) is also obtained.

This paper is essentially a revision of the author's Ph.D. thesis [3]; however, many of the results have been extended, and some are altogether new.

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2. **Preliminaries.** The purpose of this section is to fix notations and terminology concerning certain well-known concepts; more details may be found in references such as [23], [21] and [7] regarding measure theory and the theory of Hausdorff measure, and [17], [5], [28] and [13] regarding differential geometry, Lie groups, differential forms and currents. The notation and terminology of [12] and [13] will be used in this paper.

2.1. *Tangent space.* If X is an n dimensional manifold of class 1 and $x \in X$, then $T_x(X)$ is the *n dimensional real vector space of tangent vectors of X at x.*

2.2. *Exterior algebra.* For each finite dimensional vector space V and $k = 0, 1, \cdots, \dim V$,

$$\textstyle\bigwedge_k(V) \quad \text{and} \quad \bigwedge^k(V)$$

are the associated spaces of *k vectors* and *k covectors* (contravariant and covariant skewsymmetric tensors of rank k). Furthermore,

$$\textstyle\bigwedge_*(V) = \bigoplus_{k=0}^{\dim V} \bigwedge_k(V) \quad \text{and} \quad \bigwedge^*(V) = \bigoplus_{k=0}^{\dim V} \bigwedge^k(V)$$

are the corresponding *exterior algebras*, with the exterior multiplication $\wedge$. For $v \in \bigwedge_k(V)$ and $\omega \in \bigwedge^k(V)$, ωv is the scalar product of v and ω.

Each inner product g of V, with the corresponding norm $|\ |$, induces inner products and norms on $\bigwedge_*(V)$ and $\bigwedge^*(V)$, also denoted by g and $|\ |$; dual orthonormal bases for $\bigwedge_*(V)$ and $\bigwedge^*(V)$ are obtained by exterior multiplication from dual orthonormal bases for V and $\bigwedge^1(V)$.

The vector spaces $\bigwedge_k(V)$ and $\bigwedge^k(V)$ are also dually paired with respect to the *mass* and *comass* norms $\| \ \|$ defined as follows:

If $v \in \bigwedge_k(V)$, then $\|v\| = \inf\{\sum_{\beta \in B}|\beta| : B$ is a finite set of simple k vectors and $v = \sum_{\beta \in B}\beta\}$.

If $\omega \in \bigwedge^k(V)$, then $\|\omega\| = \sup\{\omega\gamma : \gamma$ is a simple k vector and $|\gamma| \leqq 1\}$.

2.3. *Currents.* If X is a separable Riemannian manifold of class ∞, then $\boldsymbol{E}_k(X)$, $\boldsymbol{N}_k(X)$ and $\boldsymbol{I}_k(X)$ are respectively the vector spaces of k *currents, normal* k *currents* and *integral* k *currents having compact support in* X. $\boldsymbol{D}_k(X)$ is the space of k *currents in* X, as defined in [6].

$\boldsymbol{E}^k(X)$ is the vector space of *differential* k *forms of class* ∞ *on* X. $\boldsymbol{D}^k(X)$ is the subspace of $\boldsymbol{E}^k(X)$ consisting of all forms having compact support.

$T \in \boldsymbol{D}_k(X)$ is *locally finite* if $\boldsymbol{M}(T \wedge f) < \infty$ for each $f \in \boldsymbol{D}^0(X)$.

With T is associated the *variation measure* $\|T\|$ such that for $f \geqq 0$,

$$\|T\|(f) = \sup\{T(\phi) : \phi \in \boldsymbol{D}^k(X) \text{ and } \|\phi\| \leqq f\}.$$

$\vec{T}$ is the unit Baire k vector field on X characterized $\|T\|$ almost everywhere in X by the condition

$$T(\phi) = \int \phi \vec{T} d\|T\| \text{ for } \phi \in \boldsymbol{D}^k(X).$$

T is *locally normal* if T and ∂T are locally finite. Similarly, we define "*locally rectifiable*" and "*locally integral*". $\boldsymbol{N}_*^{\text{loc}}(X)$ is the space of locally normal currents in X.

2.4. *Differential.* Suppose X and Y are manifolds of class ∞ and $f: X \to Y$. If $x \in X$, $y = f(x)$ and f is differentiable at x, the *differential of* f *at* x is a linear transformation

$$f_\#(x): \boldsymbol{T}_x(X) \to \boldsymbol{T}_y(Y);$$

$f_\#(x)$ can be extended to unique algebra homomorphisms

$$f_\#(x): \bigwedge_*[\boldsymbol{T}_x(X)] \to \bigwedge_*[\boldsymbol{T}_y(Y)] \text{ and } f^\#(x): \bigwedge^*[\boldsymbol{T}_y(Y)] \to \bigwedge^*[\boldsymbol{T}_x(X)],$$

which induces continuous linear transformations

$$f^\#: \boldsymbol{E}^k(Y) \to \boldsymbol{E}^k(X) \text{ and } f_\#: \boldsymbol{D}_k(X) \to \boldsymbol{D}_k(Y).$$

2.5. *Adjoint map.* Let V be an n dimensional vector space with inner product and $k \leqq n$ a nonnegative integer. Choose a unit n vector e and denote by "$*$" the linear map $\bigwedge_k(V) \to \bigwedge_{n-k}(V)$ characterized by

$$v \wedge *v = |v|^2 e.$$

There is a unique extension of $*$ to $\bigwedge^k(V)$ such that

$$*\omega(*v) = \omega v \quad \text{for } \omega \in \bigwedge^k(V), \qquad v \in \bigwedge_k(V).$$

If $l \leqq n$, $k + l \geqq n$, $w \in \bigwedge_l(V)$, $*v \wedge *w \neq 0$ and $\zeta \in \bigwedge^l(V)$, define

$$v \cap w = (-1)^t \| *v \wedge *w \|^{-1} *(*v \wedge *w) \in \bigwedge_{k+l-n}(V)$$

and

$$\omega \cap \zeta = (-1)^t *(*\omega \wedge *\zeta), \qquad t = k + l + kl.$$

2.6. Definition. $\mathbf{R}^n$ is the *n dimensional Euclidean space*, consisting of all sequences $x = (x^1, \cdots, x^n)$ of real numbers, with the metric

$$x \cdot y = \sum_{i=1}^{n} x^i y^i \quad \text{for } x, y \in \mathbf{R}^n.$$

$e_1, \cdots, e_n$ are the *standard orthonormal basis vectors* of $\mathbf{R}^n$.

2.7. *Notation.* L_n is the Lebesgue measure over $\mathbf{R}^n$.

$$\alpha(n) = L_n(\mathbf{R}^n \cap \{x : |x| < 1\}).$$

2.8. Definition. H^k is the *k dimensional Hausdorff measure.* If A is a subset of a metric space X, then $H^k(A)$ equals the limit, as $r \to 0^+$, of the infimum of the sums

$$\sum 2^{-k} \alpha(k) \, (\text{diameter } f)^k, \quad f \in F$$

corresponding to all countable coverings F of A such that diameter $f < r$ for $f \in F$.

2.9. Remark. If $X = \mathbf{R}^n$, then $H^n = L_n$ (see [24]).

If S is a proper k-submanifold of class 1 of a separable Riemannian manifold X, then the restriction $H^k \cap S$ of H^k to S is the "volume element" of S induced by the metric of X.

If S is oriented, one also denotes by S the current which takes the value $\int_S \phi$ on each $\phi \in \mathbf{D}^k(X)$. It follows that $\vec{S}$ is the positively oriented unit k-vector field tangent to S, and

$$S(\phi) = \int_S \phi \vec{S} dH^k.$$

2.10. Remark. If X and Y are Riemannian manifolds of class 1 and $m = \dim X$, $n = \dim Y$, then

$$H^{m+n}(S) = H^m \times H^n(S) \qquad \text{for} \quad S \subset X \times Y,$$

but this is not generally true if $m < \dim X$ or $n < \dim Y$ (see [2]).

2.11. *Notation.* If G is a Lie group, one denotes by L_a and R_a the *left* and *right translations* of G by $a \in G$. Ad $a = L_a \circ R_a^{-1}$. e is the *identity* element of G.

2.12. *Notation.* O_n is the *orthogonal group* of $\boldsymbol{R}^n$; SO_n is the component of O_n containing e.

Φ_n is the *Haar measure* on SO_n for which $\Phi_n(SO_n) = 1$.

2.13. DEFINITION. If k, l, n are integers, $0 < k < n$, $0 < l < n$, $k + l \geqq n$, then

$$\gamma^2(n,k,l) = \binom{n}{k}^{-1} \binom{l}{k+l-n},$$

$$\gamma(n,k,l) = \gamma^2(n,k,l)\alpha(k)\alpha(l)\alpha(n)^{-1}\alpha(k+l-n)^{-1},$$

$$\gamma_c(n,k,l) = \frac{(n+1)(k+l-n+1)}{(l+1)\,(k+1)} \frac{\alpha(2n+2)\alpha[2(k+l-n+1)]}{\alpha[2(k+1)]\,\alpha[2(l+1)]}.$$

If k or l equals 0 or n, we define

$$\gamma(n,k,l) = \gamma^2(n,k,l) = \gamma_c(n,k,l) = 1.$$

3. **The lifting map.** Let G be a Lie group which acts effectively on a compact, oriented m dimensional Riemannian manifold Y of class ∞ as a group of orientation preserving transformations leaving H^m invariant. Assume $H^m(Y) = 1$, and let μ be the positively oriented unit m form on Y. We consider paracompact fibre bundles $\mathscr{B}$ of class ∞ with fibre Y and structure group G. We shall assume for convenience that the base space and bundle space of $\mathscr{B}$ are Riemannian manifolds. Basic information concerning fibre bundles can be found in [26].

3.1. LEMMA. *Let X and X' be Riemannian manifolds of class ∞. If $S \in \boldsymbol{E}_k(X)$, $T \in \boldsymbol{E}_l(X')$, $\boldsymbol{M}(S) < \infty$ and T is rectifiable, then*

$$\|S \times T\| = \|S\| \times \|T\|, \textit{ and } (S \times T)^{\rightarrow} = (\vec{S},0) \wedge (0,\vec{T})$$

$\|S \times T\|$ *almost everywhere in $X \times X'$.*

Proof. From the definition of $S \times T$, we infer that

$$(S \times T)^{\rightarrow} \|S \times T\| = (\vec{S},0) \wedge (0,\vec{T}) \; \|S\| \times \|T\|.$$

Furthermore,

$$\|(\vec{S}(x),0) \wedge (0,\vec{T}(y))\| \leqq \|\vec{S}(x)\| \; \|\vec{T}(y)\| = 1$$

for $\|S\| \times \|T\|$ almost all $(x,y) \in X \times X'$. On the other hand, by [13, 8.16] there exists a simple Baire l form ψ such that $\boldsymbol{M}(\psi) = 1$ and $T(\psi) = \boldsymbol{M}(T)$; consequently, if f is a nonnegative Baire function on X', then

$$\boldsymbol{M}(S)\boldsymbol{M}(T \wedge f) = \boldsymbol{M}(S)\, T \wedge f(\psi) \leqq \boldsymbol{M}(S \times T \wedge f)$$

and thus

$$\|S\| \times \|T\| \leqq \|S \times T\|.$$

3.2. LEMMA. *Let X and X' be Riemannian manifolds of class ∞ and $g: X \to G$, $h: X \to X'$ be of class ∞. Define $f: X \times Y \to X' \times Y$ by*

$$f(x, y) = (h(x), g(x)(y)).$$

If $T \in \boldsymbol{E}_k(X)$ and $M(T) < \infty$, then

$$f_\#(T \times Y) = h_\#(T) \times Y.$$

Proof. We can clearly assume that $X = X'$ and h is the identity map.

Let $p: X \times Y \to X$, $q: X \times Y \to Y$ be the projections and consider fixed forms $\phi \in \boldsymbol{E}^i(X)$, $\psi \in \boldsymbol{E}^j(Y)$, $i + j = k + m$.

$$f_\#(T \times Y)[p^\#(\phi) \wedge q^\#(\psi)] = T \times Y[p^\#(\phi) \wedge (q \circ f)^\#(\psi)],$$

and one uses 3.1 to show that this is zero unless $j = m$. Thus suppose $j = m$ and $\psi = \eta\mu$, $\eta \in E^0(Y)$. It is easy to verify that

$$(q \circ f)^\#(\psi) = Hq^\#(\mu) + \nu,$$

where $H(x, y) = \eta[g(x)(y)]$ and ν is the sum of terms of the form $I\omega \wedge \zeta$ with $I \in E^0(X \times Y), \omega \in p^\#[\boldsymbol{E}^r(X)]$, $\zeta \in q^\#[\boldsymbol{E}^r(Y)]$, $s < m$. Thus

$$T \times Y[p^\#(\phi) \wedge (q \circ f)^\#(\psi)] = T \times Y[Hp^\#(\phi) \wedge q^\#(\mu)],$$

which is equal to $T(\phi)Y(\psi)$ for constant g and consequently for g of class ∞.

3.3. THEOREM. *Let $\mathscr{B}$ be a fibre bundle with base space X and bundle space B. There exists a unique linear map*

$$L_{\mathscr{B}}: \boldsymbol{E}_*(X) \cap \{T: M(T) < \infty\} \to \boldsymbol{E}_*(B)$$

such that:

(1) *If $\mathscr{B}$ is the product bundle, $B = X \times Y$, then*

$$L_{\mathscr{B}}(T) = T \times Y \quad \textit{for } T \in \boldsymbol{E}_*(X).$$

(2) *L is natural with respect to bundle maps.*

Proof. Let $\{V_j : j \in J\}$ be a locally finite open covering of X by coordinate neighborhoods with a subordinate partition of unity $\{f_j : j \in J\}$. Let p the projection map of $\mathscr{B}$. If $j \in J$, $\phi_j: V_j \times Y \to p^{-1}(V_j)$ is a coordinate function associated with V_j.

If $T \in \boldsymbol{E}_k(X)$ and $M(T) < \infty$, $T \wedge f_j \times Y \in \boldsymbol{E}_{k+m}(V_j \times Y)$. Setting

$$T_j^* = \phi_{j\#}(T \wedge f_j \times Y),$$

we observe that if $L_{\mathscr{B}}$ exists, it is necessary by (1) and (2) that

$$L_{\mathscr{B}}(T) = \sum_{j \in J} T_j^*.$$

Thus define $L_{\mathscr{B}}(T) = \sum_{i \in J} T_j^*$. $L_{\mathscr{B}}$ is clearly linear. One uses 3.2 with $X = X'$, h equal to the identity map and g replaced by the coordinate transformations g_{ij} to show that $L_{\mathscr{B}}(T)$ is independent of the choice of V_j, f_j, ϕ_j. One also uses 3.2 to show that L is natural with respect to bundle maps. Consequently, $L_{\mathscr{B}}$ is the desired mapping.

3.4. REMARK. Let $\mathscr{B}$ be a fibre bundle with bundle space B. There exists $\omega \in \boldsymbol{E}^m(B)$ such that if $\xi: Y \to B$ is an admissible map, then $\xi^{\#}(\omega) = \mu$.

If $x \in B$, orient the fibre Y_x over x so that the restriction of ω to Y_x is positively oriented.

3.5. COROLLARY. *Let $\mathscr{B}$ be a fibre bundle with bundle space B, base space X and projection p.*

(1) *If $r \in \boldsymbol{R}$, then $L_{\mathscr{B}}$ is continuous on*

$$\boldsymbol{E}_k(X) \cap \{T: \boldsymbol{M}(T) \leqq r\}.$$

(2) *Let U be an open subset of B with compact closure such that for some $r > 0$, $H^m(Y_x \cap U) > r$ whenever $Y_x \cap U$ is not empty. There exist positive constants c_0, c_1 such that*

$$c_0 \boldsymbol{M}[T \cap p(U)] \leqq \boldsymbol{M}[L_{\mathscr{B}}(T) \cap U] \leqq c_1 \boldsymbol{M}[T \cap p(U)]$$

for $T \in \boldsymbol{E}_*(X)$, $\boldsymbol{M}(T) < \infty$.

(3) $\operatorname{Spt} L_{\mathscr{B}}(T) = p^{-1}(\operatorname{spt} T)$.

(4) *$L_{\mathscr{B}}: \boldsymbol{N}_*(X) \to \boldsymbol{N}_*(B)$ is a chainmap of degree m.*

(5) *If $T \in \boldsymbol{E}_k(X)$ and $\boldsymbol{M}(T) < \infty$, then there exists a Baire k vectorfield v on B and a positive Baire function h such that*

$$p_{\#}(b)[hv(b)] = \vec{T}[p(b)] \quad \text{and} \quad \overrightarrow{L_{\mathscr{B}}(T)}(b) = v(b) \wedge \vec{Y}_{p(b)}$$

for $\|L_{\mathscr{B}}(T)\|$ almost all $b \in B$.

(6) *If T is rectifiable, then $L_{\mathscr{B}}(T)$ is rectifiable; furthermore, for H^k almost all $x \in X$,*

$$\Theta^{k+m}(\|L_{\mathscr{B}}(T)\|, b) = \Theta^k(\|T\|, p(b))$$

for each $b \in Y_x$. In particular, if T is an oriented manifold of class 1, then $L_{\mathscr{B}}(T)$ is $p^{-1}(T)$ oriented as prescribed in (5).

Proof. (1) and (3) are immediate from our theorem. (2) follows from 3.1 for product bundles, and the theorem implies the general case.

For the proof of (4), we consider a fixed $T \in \boldsymbol{N}_k(X)$. By 3.3

$$\begin{aligned} L_{\mathscr{B}}(\partial T) - \partial L_{\mathscr{B}}(T) &= \sum_{j \in J} \phi_{j\#}(T \wedge df_j \times Y) \\ &= \sum_{j \in J} L_{\mathscr{B}}(T \wedge df_j) = 0. \end{aligned}$$

(5) follows from 3.1 and the observation that if $f:X \to X'$ is a diffeomorphism of class ∞, then

$$f_\#(\vec{T}) = \|f_\#(\vec{T})\| f_\#(T)^{\to}.$$

The rectifiability of $L_{\mathscr{B}}(T)$ follows from (2). As for the second assertion, if $T = hS$, where S is a k dimensional oriented submanifold of class 1 and h is a Baire function, then (6) follows immediately from our theorem. In general, we use [13, 8.16] to find k currents $T_1, T_2, \cdots$ such that $\lim_{i\to\infty} M(T - T_i) = 0$ and for each i, T_i is a finite sum of currents of the form hS with $h(y) \neq 0$ only if $y \in S$ and $\vec{T}(y) = \pm \vec{S}(y)$. Thus by (5), (2) and [13, 8.16(2)], for $\|L_{\mathscr{B}}(T)\|$ almost all $a \in B$

$$\begin{aligned} &\|L_{\mathscr{B}}(T)^{\to}(a)\| \; |\Theta^{k+m}(\|L_{\mathscr{B}}(T)\|, a) - \Theta^k(\|T_i\|, p(a))| \\ &\quad = \|L_{\mathscr{B}}(T)^{\to}(a)\Theta^{k+m}(\|L_{\mathscr{B}}(T)\|, a) - L_{\mathscr{B}}(T_i)^{\to}(a)\Theta^{k+m}(\|L_{\mathscr{B}}(T_i)\|, a)\| \\ &\quad = \Theta^{k+m}[\|L_{\mathscr{B}}(T - T_i)\|, a] \to 0 \qquad \text{as } i \to \infty \end{aligned}$$

Furthermore, if $p(b) = p(a)$ then

$$\Theta^{k+m}(\|L_{\mathscr{B}}(T)\|, b) = \Theta^{k+m}(\|L_{\mathscr{B}}(T)\|, a);$$

hence, by (2) and [13, 8.16(2)], for $\|T\|$ almost all $x \in X$

$$\Theta^{k+m}(\|L_{\mathscr{B}}(T)\|, a) = \Theta^k(\|T\|, x) \quad \text{for } p(a) = x,$$

and it is easy to obtain the assertion from this.

3.6. THEOREM. *Let $\mathscr{B}$ be a fibre bundle with bundle space B, base space X and projection p. Let $P: \mathbf{E}_*(B) \to \mathbf{E}_*(X)$ be the continuous linear map for which*

$$P(S) = (-1)^{mk} p_\#(S \wedge \omega) \quad \text{for } S \in \mathbf{E}_{k+m}(B).$$

Then:

(1) *$P \circ L_{\mathscr{B}}$ is the identity.*

(2) *If ψ is a bounded Baire l form on X with $l \leq k$, then*

$$P[S \wedge p^\#(\psi)] = P(S) \wedge \psi \textit{ for } S \in \mathbf{E}_{k+m}(B),$$

$$L_{\mathscr{B}}(T \wedge \psi) = L_{\mathscr{B}}(T) \wedge p^\#(\psi) \textit{ for } T \in \mathbf{E}_k(B).$$

Proof. Let $p': X \times Y \to X$ and $q: X \times Y \to Y$ be the projections. Let V_j, f_j and $\phi_j, j \in J$, be as in the proof of 3.3. To prove (1), we need to verify that

$$L_{\mathscr{B}}(T)[p^\#(\psi) \wedge \omega] = T(\psi)$$

for $\psi \in \mathbf{E}^k(X)$ and $T \in \mathbf{E}_k(X)$, $\mathbf{M}(T) < \infty$, and for this it suffices by 3.3 to show that for $j \in J$

$$T \wedge f_j \times Y[p'^\#(\psi) \wedge \phi_j^\#(\omega)] = T \wedge f_j(\psi).$$

But by 3.1, $T \wedge f_i \times Y(v) = 0$, where $v = p'^{\#}(\psi) \wedge [q^{\#}(\mu) - \phi_j^{\#}(\omega)]$.

The first statement of (2) can be verified directly, and the second follows from 3.3.

3.7. THEOREM. *Suppose $G = Y$ and $\mathscr{B}$ is a principal fibre bundle with base space X, bundle space B and projection p. Then if $T \in \mathbf{E}_{k+m}(X)$, $T \in \operatorname{image} L_{\mathscr{B}}$ if and only if $\mathbf{M}(T) < \infty$ and*

(1) *$g_{\#}(T) = \pm T$ for $g \in G$;*

(2) *For $\|T\|$ almost all $b \in B$, there exists $v \in \wedge_k[\mathbf{T}_b(B)]$ such that*

$$\vec{T}(b) = v \wedge \vec{Y}_{p(b)}(b).$$

Proof. If $T \in \operatorname{image} L_{\mathscr{B}}$, it follows from 3.3, 3.4(2) and 3.4(5) that T satisfies (1) and (2).

Suppose $T \in \mathbf{E}_{k+m}(B)$, $\mathbf{M}(T) < \infty$ and T satisfies (1) and (2). Let $p' : X \times G \to X$, $q : X \times G \to G$ be the projections. Let V_j, f_j and ϕ_j be as in the proof of 3.3. Fix $\psi \in \mathbf{E}^k(X)$. For each j, set $T_j = \phi_j^{-1}{}_{\#}(T \wedge f_j \circ p)$. $T_j \in \mathbf{E}_{k+m}(X \times G)$ and $\gamma = q_{\#}[T_j \wedge p'^{\#}(\psi)] \wedge \mu$ is a right invariant finite measure on G. Thus $\|\gamma\|$ is a Haar measure on G, and it is not difficult to see that $\gamma = c_j H^m$ for some $c_j \in \mathbf{R}$. By (2), $T \wedge f_j \circ p[p^{\#}(\psi) \wedge v] = 0$, where $v = \omega - \phi_j^{-1\#} q^{\#}(\mu)$; therefore, by 3.6(2) we have

$$c_j = T \wedge f_j \circ p[p^{\#}(\psi) \wedge \omega] = P(T) \wedge f_j(\psi);$$

hence if $u \in \mathbf{E}^0(G)$, then

$$T_j[p'^{\#}(\psi) \wedge q^{\#}(u\mu)] = \gamma(u) = P(T) \wedge f_j(\psi)\, G(u\mu).$$

Finally, $\mathbf{M}[P(T)] < \infty$, and it follows from (2) that

$$T_j = P(T) \wedge f_j \times G$$

and from 3.3 that

$$T = L_{\mathscr{B}}[P(T)].$$

3.8. REMARKS. It is easy to find $T \in \mathbf{N}_{m+k}(B)$ for which (1) holds but (2) does not.

With slight modifications, the results of this section hold for fibre bundles with noncompact fibre. One defines the map $P : \mathbf{D}_*(G) \to \mathbf{D}_*(X)$ using the form ω_0 defined as follows: Choose $S \subset Y$ so that $H^m(S) = 1$; let q, V_j, f_j and ϕ_j, $j \in J$, be as in the proof of 3.3, and let

$$\omega_0 = \sum_{j \in J} f_j \circ p\; \phi_j^{-1\#} q^{\#}(\mu \mid S).$$

For 3.7 to hold, it is necessary to assume in addition that G is unimodular.

4. **Intersections of currents.** Here the intersection theory for currents in a connected, oriented, n dimensional homogeneous space X is developed.

Let G be a connected Lie group of transformations of X which acts transitively on X, $\dim G = n + m$. Fix $o \in X$ and define $\pi: G \to X$ by $\pi(g) = g(o)$. Let $\mathscr{G}$ be the principal fibre bundle with group $I = \pi^{-1}(o)$, bundle space G, base space X and projection π. $I_x = \pi^{-1}(x)$ for $x \in X$. Write L for $L_{\mathscr{G}}$.

Assume that I is unimodular and oriented and has a left invariant metric; we also denote by $\vec{I}$ the left invariant m vectorfield on G whose restriction to I is $\vec{I}$. Let μ be the covariant dual of $\vec{I}$; use μ as in 3.8 to define $P: \boldsymbol{D}_*(G) \to \boldsymbol{D}_*(X)$.

Let X have a Riemannian metric and v be the covariant dual of $\vec{X}$. Define $\omega = \pi^{\#}(v) \wedge \mu$, orient G so that ω is positively oriented, and assign a metric to G such that $|\vec{I}| = 1$ and $\pi_{\#}(g)$ is an orthogonal projection for $g \in G$. Clearly, $|\omega| = 1$. If X has a G-invariant measure, then we may assume v to be such that the Hausdorff measure H^{m+n} on G is left invariant; in all cases H^{m+n} is equivalent to a Haar measure Ψ on G.

f and p are the functions on $G \times G$ to G defined by $f(a,b) = ab^{-1}$ and $p(a,b) = a$.

If $S \in N_j(G \times G)$ and $j \geqq m + n$, then for Ψ almost all $g \in G$,

$$\langle S, f, g \rangle \in N^{\mathrm{loc}}_{j-m-n}(G \times G)$$

is the *slice of S by f over g* as characterized in [12].

4.1. LEMMA. *If u, v are left invariant vector fields on G, then for $(a,b) \in G \times G$*

$$(L_b \circ f)_{\#}(a,b)(u(a,), v(b)) = \operatorname{ad} b_{\#}(u - v)(bab^{-1}).$$

Proof. Define $\boldsymbol{I}$ on G by $\boldsymbol{I}(x) = x^{-1}$. Then $f(x,b) = R_b{}^{-1}(x)$ and $f(a,y) = L_a \circ \boldsymbol{I}(y)$, hence

$$(L_b \circ f)_{\#}(a,b)(u(a), v(b)) = \operatorname{ad} b_{\#}(u)(c) + (L_b \circ L_a \circ \boldsymbol{I})_{\#}(v)(c), \qquad c = bab^{-1}.$$

Furthermore, $\boldsymbol{I}_{\#}(v)(b^{-1}) = -R_b{}^{-1}{}_{\#}(e)[v(e)] = -(L_b \circ R_b{}^{-1})_{\#}(v)(b^{-1})$, hence

$$(L_b \circ L_a \circ \boldsymbol{I})_{\#}(v)(c) = -\operatorname{ad} b_{\#}(v)(c).$$

4.2. LEMMA. *Suppose $x \in X$, $g \in G$, $k > 0$, $l > 0$, $k + l \geqq n$, $\alpha \in \wedge_k[T_x(X)]$ and $\beta \in \wedge_l[T_y(X)]$, $y = g^{-1}(x)$. Fix $a \in I_x$ and let $i: T_x(X) \to T_a(G)$ be the transpose of $\pi_{\#}(a)$. Let $T_b(G)$, $a = g(b)$, have the metric for which $L_{g\#}(b)$ is an isometry. Let*

$$\alpha' = i(\alpha) \wedge \vec{I}(a),$$

$$\beta' = L_g{}^{-1}{}_{\#}(a) \circ i \circ g_{\#}(y)(\beta) \wedge \vec{I}(b),$$

and

$$w \in \wedge_{k+l-n+m}[\operatorname{kernel} f_{\#}(a,b)]$$

be the unique vector for which

$$\phi w = f^{\#}(\omega)(a,b) \wedge \phi[(\alpha', 0) \wedge (0, \beta')]$$

whenever $\phi \in \bigwedge^{k+l-n+m}[T_{(a,b)}(G \times G)]$. *Then*:

(1) $i(*\alpha) \wedge \vec{I}(a) = *i(\alpha)$.

(2) $(-1)^{m(n-k)} * [i(\alpha) \wedge \vec{I}(a)] = i(*\alpha)$.

(3)
$$\begin{aligned}(-1)^{m(k+l)} i[*(*\alpha \wedge *g_{\#}(y)(\beta))] \wedge \vec{I}(a) &= *[*\alpha' \wedge * L_{g\#}(b)(\beta')] \\ &= (-1)\, c^{\varepsilon-1} p_{\#}(a,b)(w),\end{aligned}$$

where $c\omega(a) = R_b^{-1\#}[(\omega)(a)]$ *and* $\varepsilon = (n-l)(k+l-n+m)+n-k$.

Proof. We can assume α and β to be simple unit vectors. To prove (1), observe that

$$\omega(a)[i(\alpha) \wedge (i(*\alpha) \wedge \vec{I}(a))] = v(x)(\alpha \wedge *\alpha) = 1.$$

To obtain (2), replace α by $*\alpha$ in (1).

The first equality of (3) follows from (1) and (2). Let A, B be respectively the $k+m, l+m$ dimensional linear subspaces of $T_a(G)$, $T_b(G)$ which contain α', β'. Suppose $\delta = \dim[A \cap L_{g\#}(b)(B)]$ and choose orthonormal bases $a_1, a_2, \cdots, a_{k+m}$ and $b_1, b_2, \cdots, b_{l+m}$ of A and B, respectively, such that

$$a_i = L_{g\#}(b)(b_i) \quad \text{for } i = 1, 2, \cdots, \delta,$$

$$\alpha' = \bigwedge_{i=1}^{k+m} a_i, \quad \beta' = \bigwedge_{i=1}^{l+m} b_i.$$

One can easily show that

$$(\alpha', 0) \wedge (0, \beta') = (-1)^{(\delta+1)(l+m)} v \wedge w_0,$$

where

$$w_0 = 2^{-\delta/2} \bigwedge_{i=1}^{\delta} (a_i, b_i)$$

and

$$v = 2^{-\delta/2} \bigwedge_{i=1}^{\delta} (a_i, -b_i) \wedge \bigwedge_{i=\delta+1}^{k+m} (a_i, 0) \wedge \bigwedge_{i=\delta+1}^{l+m} (0, -b_i).$$

From 4.1. we infer that we can assume $\delta = k+l-n+m$, and that $w = (-1)^{(\delta+1)(n-k)} K w_0$, where $K = f^{\#}(\omega)(a,b)(v)$, hence

$$p_{\#}(a,b)(w) = 2^{-\delta/2}(-1)^{(\delta+1)(n-k)} K \bigwedge_{i=1}^{\delta} a_i.$$

On the other hand

$$\begin{aligned}K &= \omega(g)[f_{\#}(a,b)(v)] \\ &= 2^{\delta/2} c\omega(a)\left[\alpha' \wedge \bigwedge_{i=\delta+1}^{l+m} L_{g\#}(b)(b_i)\right],\end{aligned}$$

and using this, one verifies directly that

$$\bigwedge_{i=1}^{\delta} a_i = (-1)^s 2^{\delta/2} c K^{-1} * [*\alpha' \wedge * L_{g\#}(b)(\beta')],$$

$s = (l+m)(n-l) + (k+m)(n-k)$, which completes the proof.

4.3. DEFINITION. Suppose $S \in N_k(X)$, $T \in N_l(X)$ and $k + l \geqq n$. Whenever $\langle L(S) \times L(T), f, g\rangle \in N^{\mathrm{loc}}_{k+l-n+m}(G \times G)$, we define

$$S \cap g_\#(T) = (-1)^t P \circ p_\#(\langle L(S) \times L(T), f, g\rangle), \quad t = nk + mk + mn.$$

4.4. THEOREM. (1) *If $S \cap g_\#(T)$ exists, then*

$$\operatorname{spt}[S \cap g_\#(T)] \subset (\operatorname{spt} S) \cap g(\operatorname{spt} T).$$

(2) *Let U be an open subset of X having compact closure, and $C \subset G$ be compact. There exists a constant c such that if $S \in N_k(U)$, $T \in N_l(U)$, then*

$$\int_C M[S \cap g_\#(T)] d\Psi g \leqq c M(S) M(T).$$

(3) *For each $\psi \in E^{k+l-n}(X)$ and $S \in N_k(X)$, $T \in N_l(X)$, define $\cap(S,T)(g) = S \cap g_\#(T)(\psi)$ whenever $S \cap g_\#(T)$ exists. Then*

$$\cap \mid N_k(X) \times N_l(X) \cap \{(S,T) : M(S) + M(T) < \infty\}$$

is continuous with respect to weak convergence in $N_k(X) \times N_l(X)$, and in $D_o(G)$.

(4) *For Ψ almost all $g \in G$, the following are true:*

(i) *$S \cap g_\#(T) \in N_{k+l-n}(X)$.*

(ii) *For $\|S \cap g_\#(T)\|$ almost all $x \in X$,*

$$[S \cap g_\#(T)]^{\rightarrow}(x) = \vec{S}(x) \cap g_\#(\vec{T})(x).$$

(iii) *If S and T are oriented proper submanifolds of class 1, then $S \cap g(T) = A \cup B$ where A is an orientable $k + l - n$ proper submanifold of class 1 and $H^{k+l-n}(B) = 0$; if A is oriented according to* (ii), *then $A = S \cap g_\#(T)$.*

Proof. (1) follows from 3.5(3) and [12, 3.5].

To prove (2), first choose an open subset V of G having compact closure such that U is contained in the interior of $p(V)$ and for some $r > 0, H^m(I_x \cap V) > r$ whenever $I_x \cap V$ is not empty. Let $W = V \cup C^{-1}V$ and note that if $g \in C$, then

$$V \subset V \cap g(W)$$

and

$$U \cap g(U) \subset \pi[V \cap g(W)].$$

Now observe that by [12, 3.6(4)] and 3.5(2) there exists c_0 depending only on V and C such that

$$\int_C M[p_\#(\langle L(S) \times L(T), f, g\rangle) \cap V \cap g(W)] d\Psi g$$
$$= \int_C M[p_\#(\langle L(S) \cap V \times L(T) \cap W, f, g\rangle)] d\Psi g$$
$$\leqq c_0 M(S) M(T).$$

Finally, use 3.7 to show that $p_\#(\langle L(S) \times L(T), f, g\rangle) \in \operatorname{image} L$ and apply 3.5(2) and (1) to infer the existence of $c_1 > 0$ such that the first number is not less than

$$c_1 \int_C M[S \cap g_\#(T)] d\Psi g.$$

(3) follows from 3.5(1) and [12, 3.6(6)].

(4)(i) follows from 3.5(4), [12, 3.5] and the definition of P.

Now consider a fixed $x \in X$ for which $\vec{S}(x), \vec{T}[g^{-1}(x)]$ and $(S \times T)^{\to}(x, g^{-1}(x))$ exist. Choose $a \in I_x$, and observe that

$$[L(S) \times L(T)]^{\to}(a, b) = c(L(S)^{\to}(a), 0) \wedge (0, L(T)^{\to}(b)), \quad b = g^{-1}a.$$

Furthermore, if w is defined as in 4.2, then it follows as in [12, 3.13(3)] that we can assume g to be such that

$$w = \langle L(S) \times L(T), f, g\rangle^{\to}(a, b).$$

On the other hand, by 3.7

$$\mathscr{I}_g = (-1)^t p_\#(\langle L(S) \times L(T), f, g\rangle) \in \operatorname{image} L$$

and in view of 3.5(5), we can apply 4.2 to obtain (ii).

As for (iii), 3.5(6), [11, 3.1] and [12, 3.13(3)] imply that we can assume g to be such that $\mathscr{I}_g = A'$ and $\pi^{-1}[S \cap g(T)] = A' \cup B'$, where $\pi^{-1}[\pi(A')] = A'$, $\pi^{-1}[\pi(B')] = B'$, A' is a proper $k + l - n + m$ dimensional submanifold of class 1 of G oriented according to 4.2 and $H^{k+l-n+m}(B') = 0$. Thus $A = \pi(A')$ is a proper $k + l - n$ submanifold of class 1, and if we set $B = \pi(B')$, then $H^{k+l-n}(B) = 0$ by [9, 3.2]. Finally, we infer from 4.2 and 3.5(6) that if we orient A according to (ii), then $L(A) = A'$ and consequently $P(A') = A$ by 3.6(1).

4.5. REMARKS. 4.4(3) and (4) and [13, 7.4] show that the intersections $S \cap g_\#(T)$ are intrinsically determined by the action of G and, in particular, are independent of the metric on X.

In order to be consistent with 4.4, we are led to define

$$g_\#(S) \cap T = (-1)^{(n-k)(n-l)} T \cap g_\#(S)$$

whenever $T \cap g_\#(S) \in N_{k+l-n}(X)$.

If I is compact, spt $\cap(S, T)$ is compact; however, if I is not compact, then it is not always true that $\cap(S, T) \in L^1(G)$. Examples may be found in R^2, if we take G to be the affine group.

4.6. Corollary. *If* $S \in N_k(X)$ *and* $T \in N_l(X)$, *then*:

(1) *If* $k+l>n$ *and* $S \cap g_\#(T)$ *exists, then*

$$\partial[S \cap g_\#(T)] = \partial S \cap g_\#(T) + (-1)^{n-k} S \cap g_\#(\partial T).$$

(2) *If* $R \in N_j(X)$ *and* $j+k, k+l \geqq n$, *then for* $\Psi \times \Psi$ *almost all* $(g,h) \in G \times G$,

$$[g_\#(R) \cap S] \cap h_\#(T) = g_\#(R) \cap [S \cap h_\#(T)].$$

(3) *Whenever* $\phi \in E^i(X)$, $\psi \in E^j(X)$, $i \leqq k$, $j \leqq l$, $i+j \leqq k+l-n$, *and* $S \cap g_\#(T)$ *exists, then so does*

$$S \wedge \phi \cap g_\#(T \wedge \psi) = (-1)^{j(k+n)}[S \cap g_\#(T)] \wedge \phi \wedge g^{-1\#}(\psi).$$

Proof. Since $p_\#(\langle L(S) \times L(T), f, g\rangle) \in \operatorname{image} L$ by 3.7, (1) follows from 3.5(4), [12, 3.6(1)] and 3.6(1).

For oriented manifolds, (2) follows from 4.5 and 4.4(4), and reference to [13, 7.4] completes the proof.

One verifies (3) directly from the definition, using the second statement of 3.6(2), [12, 3.6(2)] and the first statement of 3.6(2).

4.7. Corollary. *Suppose* $S \in \boldsymbol{I}_k(X)$ *and* $T \in \boldsymbol{I}_l(X)$. *For almost all* $g \in G$, *the following are true*:

(1) $S \cap g_\#(T) \in \boldsymbol{I}_{k+l-n}(X)$.

(2) *For* H^{k+l-n} *almost all* $x \in X$, $\Theta^k(\|S\|, x)$ *and* $\Theta^l(\|T\|, g^{-1}(x))$ *are integers with*

$$\Theta^{k+l-n}(\|S \cap g_\#(T)\|, x) = \Theta^k(\|S\|, x)\,\Theta^l(\|T\|, g^{-1}(x)).$$

Proof. (1) follows from 4.4(2) and 4.6(1).

The proof of (2) is similar to that of 3.5(6): The assertion for $S = h_0 Y_0$, $T = hY$, where Y_0, Y are k, l submanifolds of class 1 and h_0, h are Baire functions, follows from 4.4(4) and (see 4.9) 4.6(3). For the general case, we proceed as in 3.5(6), using 4.4(4) instead of 3.5(5) and 4.4(2) instead of 3.5(2).

4.8. Remark. If X is not orientable, $S \cap g_\#(T)$ cannot be defined. However, if I has a finite number s of components, we can define $\|S \cap g_\#(T)\|$ as follows: If I_0 is the component of I containing e, then $Y = G/I_0$ is orientable and there exists a covering map $\sigma: Y \to X$. Let Y have the metric for which σ is locally an isometry. If $\mathscr{Y}$ is the bundle with bundle space Y, base space X and fibre I/I_0, then we define

$$\|S \cap g_\#(T)\| = s\sigma_\#(\|L_{\mathscr{Y}}(S) \cap g_\#[L_{\mathscr{Y}}(T)]\|).$$

It is easy to see that this definition is independent of the choice of σ.

4.9. Remark. The constructions and results of this section remain valid if S and T are locally normal.

Using 4.4(2), we can also define $S \cap g_{\#}(T)$ for S and T quasi-normal (mass limits of normal currents). $S \cap g_{\#}(T)$ is quasi-normal for Ψ almost all $g \in G$.

The following theorem generalizes 4.4(3). Let $\mathbf{QN}_k(X)$ denote the set of quasi-normal k currents in X.

4.10. THEOREM. *Suppose $\phi \in E^i(X)$ and $\psi \in E^j(X)$.*

(1) *Suppose $i+j = k+l \geqq n$. If $S \in \mathbf{QN}_k(X)$ and $T \in \mathbf{QN}_l(X)$, define*

$$F(S,T)(g) = S \cap g_{\#}(T)[\phi \cap g^{-1\#}(\psi)]$$

whenever $S \cap g_{\#}(T)$ exists. Then

$$F \,|\, \mathbf{QN}_k(X) \times \mathbf{QN}_l(X) \cap \{(S,T): \mathbf{M}(S) + \mathbf{M}(T) < \infty\}$$

is continuous with respect to weak convergence in $\mathbf{E}_k(S) \times \mathbf{E}_l(X)$, and in $\mathbf{D}_o(G)$.

(2) *Suppose $i+j = k+l-1 \geqq n$. If $S \in \mathbf{QN}_k(X)$ and $T \in \mathbf{QN}_l(X)$, define*

$$F(S,T)(g) = \partial[S \cap g_{\#}(T)][\phi \cap g^{-1\#}(\psi)]$$

whenever $S \cap g_{\#}(T)$ exists. Then

$$F \,|\, \mathbf{QN}_k(X) \times \mathbf{QN}_l(X) \cap \{(S,T): \mathbf{M}(S) + \mathbf{M}(T) < \infty\}$$

is continuous with respect to weak convergence in $\mathbf{E}_k(X) \times \mathbf{E}_l(X)$, and in $\mathbf{D}_o(G)$.

Proof. We shall prove (1); the proof of (2) is similar. Consider fixed $S \in \mathbf{QN}_k(X)$, $T \in \mathbf{QN}_l(X)$ and $g \in G$ such that $S \cap g_{\#}(T)$ exists. Use $p^{\#}(\omega) \wedge q^{\#}(\omega)$ to orient $G \times G$. We first show that

$$\begin{aligned}(-1)^s S \cap g_{\#}(T)[\phi \cap g^{-1\#}(\psi)] \\ = \langle L(S) \times L(T), f, g \rangle (*[p^{\#}(*[\pi^{\#}(\phi) \wedge \mu]) \wedge *q^{\#}[\pi^{\#}(\psi) \wedge \mu]]),\end{aligned}$$

where $s = m + n + ln + km + mn$ and $q(a,b) = b$, $(a,b) \in G \times G$. We next observe that [12, 3.6(3)] holds for quasi-normal currents, and apply this proposition together with 3.5(1) to complete the proof.

To verify the formula, first apply 4.2(1) and (2) to show that

$$\pi^{\#}[\phi \cap g^{-1\#}(\psi)] \wedge \mu = (-1)^{m(k+l)} * (*[\pi^{\#}(\phi) \wedge \mu] \wedge *L_g^{-1\#}[\pi^{\#}(\psi) \wedge \mu]).$$

Next observe that

$$p^{\#}(*\xi) = (-1)^{\tau} * [p^{\#}(\xi) \wedge q^{\#}(\omega)] \quad \text{for } \xi \in E^k(G),$$

where $\tau = (m+n+k)(m+n)$, hence

$$\begin{aligned}p^{\#}(\pi^{\#}[\phi \cap g^{-1\#}(\psi)] \wedge \mu) \\ = (-1)^r * [p^{\#}(*[\pi^{\#}(\phi) \wedge \mu]) \wedge * p^{\#} L_g^{-1\#}[\pi^{\#}(\psi) \wedge \mu]],\end{aligned}$$

where $r = n(k+l+1) + m$. Furthermore, this form coincides on $f^{-1}(g)$ with

$$(-1)^r * [p^\#(*[\pi^\#(\phi) \wedge \mu]) \wedge * q^\#[\pi^\#(\psi) \wedge \mu]],$$

and the formula follows from this.

5. **The principal integralgeometric formula.** Here a general formula which is valid in any Riemannian homogeneous space is derived.

Let G be a connected group of isometries of an n dimensional Riemannian manifold X which acts transitively on X. Let $\dim G = m + n$, define π, f, I, I_x as in §4, orient $T_o(X)$, and let G be provided with a left invariant metric such that $H^m(I) = 1$ and $\pi_\#(g)$ is an orthogonal projection for $g \in G$.

5.1. REMARK. It is easy to see that $|\det(\operatorname{ad} g_\#)|$ is independent of $g \in I_x$ for $x \in X$. Consequently, there exists a positive analytic function Δ_π on X such that $\Delta_\pi(x) = |\det(\operatorname{ad} g_\#)|$ for $x \in X$, $g \in I_x$.

5.2. DEFINITION. Ψ_π is the Haar measure on G having H^n as its π image.

5.3. DEFINITION. If $x \in X$, then Φ_x is the unique Carathéodory measure on I_x such that Φ_x is invariant under right translation by elements of I and $\Phi_x(I_x) = 1$.

5.4. REMARK. If G is unimodular, $\Psi = \Psi_\pi$ is independent of choice of π.

5.5. THEOREM. *Suppose $S \in N_k(X)$ and $T \in N_l(X)$ with $k + l \geqq n$, and F is a bounded Baire function on $G \times G$. For each $(x, y) \in X \times X$, define*

$$\mathscr{I}(x,y) = \int_{I_x \times I_y} F(a,b) \| * a^{-1}{}_\#(\vec{S}) \wedge * b^{-1}{}_\#(\vec{T}) \| (o) \, d\Phi_x \times \Phi_y(a,b).$$

Then

$$\int_G \int_X \int_I F(a, g^{-1}a) d\Phi_x a \, d\| S \cap g_\#(T) \| x d\Psi_\pi g$$
$$= \int_{X+X} \mathscr{I}(x,y) \Delta_\pi(y) \, d \| S \| \times \| T \| (x,y).$$

Proof. In view of 4.8, we can clearly assume X to be oriented. The proof is divided into three parts. Define ω as in §4.

Part 1. Let h be a bounded Baire function on G. Then

$$\| L(S) \| (h) = \int_X \int_{I_x} h d\Phi_x d \| S \| x.$$

Proof. Clearly, we may assume $\operatorname{spt} S$ to lie in a coordinate neighborhood N, with associated coordinate function $\phi: N \times I \to G$. If we denote the right member of our conclusion by $\lambda(h)$, 5.3 implies that $\lambda = \phi_\#(\| S \| \times \| I \|)$. On the other hand, 3.5(5) implies that

$$L(S)^{\rightarrow}(b) = [\operatorname{transpose} \pi_\#(b)(\vec{S}[\pi(b)])] \wedge \vec{I}(b) = \phi_\#(\vec{S}, 0) \wedge \vec{I}(b)$$

for $\| L(S) \|$ almost all $b \in G$, hence by 3.3 and 3.1,

$$L(S) = \phi_\#(S \times I) = \phi_\#(\vec{S}, 0) \wedge \vec{I} \lambda = L(S)^{\rightarrow} \lambda.$$

Part 2.

$$\int_G \int_X \int_{I_x} F(a, g^{-1}a)\, d\Phi_x a\, d\, \| S \cap g_\#(T) \| x\, d\Psi_\pi g$$

$$= 2^{-\delta} \| L(S) \times L(T) \wedge f^\#(\omega) \| (F), \qquad \delta = \tfrac{1}{2}(k + l - n + m).$$

Proof. We use 3.7 and Part 1 to infer that the left member is equal to

$$2^{-\delta} \int_G \| \langle L(S) \times L(T), f, g \rangle \| (F)\, d\Psi_\pi g,$$

and apply [12, 3.6(5)] to complete the proof.

Part 3.

$$2^{-\delta} \| L(S) \times L(T) \wedge f^\#(\omega) \| (F) = \int_{\times XX} \mathscr{I}(x, y) \Delta_\pi(y)\, d\, \| S \| \times \| T \| (x, y).$$

Proof. $L(S) \times L(T) = (L(S)^{\rightarrow}, 0) \wedge (0, L(T)^{\rightarrow}) \, \| L(S) \| \times \| L(T) \|$, hence

$$\| L(S) \times L(T) \wedge f^\#(\omega) \| = \| w \| \, \| L(S) \| \times \| L(T) \|,$$

where w is the Baire $k + l - n + m$ vectorfield defined, for $\| L(S) \| \times \| L(T) \|$ almost all $(a, b) \in G \times G$, in the same way as the vector w in the hypothesis of 4.2. Our assertion now follows from Part 1 and 4.2(3).

5.6. Definition. Suppose v, w to be k, l vectorfields of unit mass, each defined on some subset of X, such that if $v(x)$ and $v(y)$ exist, then there exists $g \in G$ such that

$$g_\#(x)[v(x)] = \pm v(y),$$

and similarly with respect to w. Choose $g_x, g_y \in G$ such that $g_x(o) = x, g_y(o) = y$. Define

$$\gamma(v, w) = \int_I \| * g_{x\#}^{-1} \quad (v) \wedge * h g_{y\#}^{-1} \quad (w) \| (o)\, d\Phi_0 h.$$

Clearly, $\gamma(v, w)$ depends on v and w only.

If v and w are simple and $A(x)$ is the k dimensional subspace of $T_x(X)$ containing $v(x)$, and similarly for $B(y)$, then for convenience we also denote $\gamma(v, w)$ by $\gamma(A, B)$.

5.7. Corollary. *Suppose that $\vec{S}$ and $\vec{T}$ have the same property as v and w in* 5.6. *Then, whenever C, D are Borel subsets of X,*

$$\int_G \| S \cap g_\#(T) \| [C \cap g(D)]\, d\Psi_\pi g = \gamma(\vec{S}, \vec{T}) \| S \| (C) \int_D \Delta_\pi d \| T \|.$$

Proof. Follows from our theorem with $F(a, b) = 1$ if $\pi(a) \in C$ and $\pi(b) \in D$, and $F(a, b) = 0$ otherwise.

5.8. COROLLARY. *Let A, B be proper k, l dimensional submanifolds of class 1 of X such that G acts transitively on the sets $T(A)$ and $T(B)$ of tangent spaces of A, and of B, respectively. Then, whenever C is a Borel subset of A and D is a Borel subset of B,*

$$\int_G H^{k+l-n}[C \cap g(D)]d\Psi_n g = \gamma[T(A), T(B)]H^k(C) \int_D \Delta_n dH^l.$$

Proof. Follows from 5.7 and 4.4(4).

5.9. REMARKS. If $\dim G = n(n+1)/2$, then I acts on $T_o(X)$ as either the orthogonal group or the special orthogonal group, and it follows from [9, 6.2] that

$$\gamma[T(A), T(B)] = \gamma(n,k,l).$$

In 8.6 we indicate how this can be derived from 5.8.

5.8 can be proved directly by use of the coarea formula [11, 3.1] in place of [12, 3.6(5)].

Clearly, we can have $\gamma[T(A), T(B)] = 0$. It is, however, possible that for some $r < n$, the intersections $A \cap g(B)$ are almost always of dimension $k+l-r > k+l-n$, hence one might expect that for a suitable metric on G,

$$\int_G H^{k+l-r}[C \cap g(D)]\, dH^{r+m} g = \delta H^k(C) \int_D \Delta_n dH^l,$$

where δ is a positive constant depending on A and B. This is discussed in §11.

5.10. REMARK. If G is either compact, semisimple or nilpotent, then G is unimodular (see [16]).

Suppose G is connected. If K is a connected, unimodular Lie subgroup of G which is transitive on X, then G is unimodular [18, p. 69]. Suppose G is the maximal connected isometry group of X and for $x \in X$, $G \cap \{g : g(x) = x\}$ acts irreducibly on $T_x(X)$. It is proved in [22, p. 54] that if G is not compact, then X is Riemannian symmetric and either G is semisimple or $X = \mathbf{R}^n$.

On the other hand, there is an example in [25] of a bounded domain D in $\mathbf{C}^5$ which admits a transitive connected Lie group G of complex analytic homeomorphisms but which is not symmetric. By a theorem of Hano [15, p. 886], G is not unimodular. The Bergman metric on D is invariant under the action of G.

5.11. EXAMPLE. Let $\mathbf{C}$ denote the complex numbers, $\mathbf{C}^* = \mathbf{C} - \{0\}$, $\mathbf{R}_+ = \mathbf{R} \cap \{x : x > 0\}$, and $G_0 = \mathbf{C}^* \times \mathbf{C}$. We shall regard G_0 as a subgroup of the affine group of $\mathbf{C}$,

$$(a,b)(z) = az + b \quad \text{for } (a,b) \in G_0,\ z \in \mathbf{C}.$$

G_0 is not unimodular; in fact, if $(a,b) \in G_0$,

$$|\det[\operatorname{ad}(a,b)_\#]| = |a|^2.$$

G_0 acts transitively and effectively on $X_0 = \boldsymbol{R}_+ \times \boldsymbol{C}$ in such a way that

$$(a,b)(x,z) = (|a|x,\ (a,b)(z)) \qquad \text{for } (x,z) \in X_0.$$

Let $I = G_0 \cap \{g: g(1,0) = (1,0)\}$. $I = \boldsymbol{C} \cap \{a: |a| = 1\} \times \{0\}$, and the Euclidean metric on $\boldsymbol{T}_{(1,0)}(X_0)$ is invariant under the action of I; extend this by the action of G_0 to a metric on X_0.

An example which is more interesting from the standpoint of our theorems is the space $X = X_0 \times X_0$ with the group of isometries $G = G_0 \times G_0$, where $(u,v)(x,y) = (u(x), v(y))$ for $(u,v) \in G, (x,y) \in X$.

5.12. Definition. A subset A of a Riemannian manifold Y of class 1 is k *rectifiable* if and only if $0 < k < n$ and there exists a Lipschitzian function on $\boldsymbol{R}^k$ to Y which maps some bounded subset of $\boldsymbol{R}^k$ onto A.

5.13. Definition. A subset A of a Riemannian manifold Y of class 1 is *Hausdorff k rectifiable* if and only if

$$H^k(A) < \infty$$

and there exist k rectifiable subsets $B_0, B_1, \cdots$ of Y such that

$$H^k\left(A - \bigcup_{i=0}^{\infty} B_i\right) = 0.$$

5.14. Remark. Since any bounded L_k measurable subset B of $\boldsymbol{R}^k$ can be regarded as being a rectifiable k current, to each H^k measurable Hausdorff k rectifiable subset A of X there corresponds a locally rectifiable k current T such that

$$\|T\| = H^k \cap A.$$

By [13, 8.16], for H^k almost all $x \in A$ the k vector $\vec{T}(x)$ is simple; let $\boldsymbol{T}_x(A)$ denote the k dimensional linear subspace of $\boldsymbol{T}_x(X)$ which contains $\vec{T}(x)$. Using [13, 8.16], one easily sees that $\boldsymbol{T}_x(A)$ depends only on A and $x \in A$.

5.15. Theorem. *Let A be an H^k measurable subset of X and B be an H^l measurable subset of X such that A is Hausdorff k rectifiable and B is l rectifiable, $k + l \geq n$. Further, suppose that G acts transitively on $\boldsymbol{T}(A)$ and on $\boldsymbol{T}(B)$. Then*

$$\int_G H^{k+l-n}[A \cap g(B)]\, d\Psi_\pi g = \gamma[\boldsymbol{T}(A), \boldsymbol{T}(B)]\, H^k(A) \int_B \Delta_\pi dH^l.$$

Proof. Using 5.14 and [13, 8.16], we choose sets A_0, A_1, B_0, B_1 such that $A = A_0 \cup A_1$, $B = B_0 \cup B_1$, $H^k(A_0) = H^l(B_0) = 0, A_1$ is H^k measurable, B_1 is H^l measurable, A_1 is contained in the union of countably many proper k-submanifolds of class 1, and B_1 is contained in the union of countably many proper l-submanifolds of class 1.

For A_1, B_1 the assertion follows from 5.8; to complete the proof, we show that for Ψ_π almost all $g \in G$,

$$H^{k+l-n}([A_0 \cap g(B)] \cup [A_1 \cap g(B_0)]) = 0.$$

Applying [9, 3.2] to f and using [9, 4.1], we infer the existence of numbers c_0, c_1, c_2 such that

$$\begin{aligned}
&\int_G^* H^{k+l-n}([A_0 \cap g(B)] \cup [A_1 \cap g(B_0)])d\Psi_\pi g \\
&\leqq c_0 \int_G^* H^{k+l-n+m}[(\pi^{-1}(A_0) \cap L_g[\pi^{-1}(B)]) \\
&\qquad\qquad \cup (\pi^{-1}(A_1) \cap L_g[\pi^{-1}(B_0)])] \, d\Psi_\pi g \\
&\leqq c_1 H^{k+l+2m}[\pi^{-1}(A_0) \times \pi^{-1}(B) \cup \pi^{-1}(A_1) \times \pi^{-1}(B_0)] \\
&\leqq c_2 (H^{k+m}[\pi^{-1}(A_0)] H^{l+m}[\pi^{-1}(B)] \\
&\qquad\qquad + H^{k+m}[\pi^{-1}(A_1)] H^{l+m}[\pi^{-1}(B_0)]) = 0.
\end{aligned}$$

6. **The integral geometry of oriented "domains of integration"**. In this section an integralgeometric formula concerning the action of currents on forms is established for spaces having constant curvature.

Let X be a connected Riemannian homogeneous space whose group of isometries G has dimension $n(n+1)/2$. Suppose G to be connected.

6.1. Definition. If k, l are nonnegative integers such that $k \leqq n$, $l \leqq n$, $k + l \geqq n$, then

$$\Gamma^2(n,k,l) = \int_{SO_n} \left| e_1 \wedge \cdots \wedge e_{n-k} \wedge R(e_1 \wedge \cdots \wedge e_{n-l}) \right|^2 d\Phi_n R.$$

In 9 we show that $\Gamma^2(n,k,l) = \gamma^2(n,k,l)$.

6.2. Theorem. *Suppose* $r + s \leqq n$, $v_0 \in \bigwedge_r(\mathbf{R}^n)$, $v \in \bigwedge_r(\mathbf{R}^n)$, $w_0 \in \bigwedge_s(\mathbf{R}^n)$, $w \in \bigwedge_s(\mathbf{R}^n)$, *and*

$$N(v, v_0, w, w_0) = \int_{SO_n} [v_0 \wedge R(w_0)] \cdot [v \wedge R(w)] d\Phi_n R.$$

(1) *If* $r \neq n/2$ *or* $s \neq n/2$, *then*

$$N(v, v_0, w, w_0) = \Gamma^2(n, n-r, n-s)(v_0 \cdot v)(w_0 \cdot w).$$

(2) *If* $r = s = n/2$, *then*

$$N(v, v_0, w, w_0) = \Gamma^2(n, n-r, n-s)[(v_0 \cdot v)(w_0 \cdot w) + (*v_0 \cdot v)(*w_0 \cdot w)].$$

Proof. We identify the elements of SO_n with their matrix representations. For each pair of positive integers $\alpha < \beta \leqq n$, $I_{\alpha\beta}$ is the linear transformation of $\mathbf{R}^n$ carrying e_i to e_i for $i \neq \alpha$, $i \neq \beta$, and carrying e_α to $-e_\alpha$, e_β to $-e_\beta$. I_n is the linear transformation of $\mathbf{R}^n$ carrying e_i to e_i for $i < n$ and e_n to $-e_n$.

We divide the proof into two steps; the assertions are easy consequences of Step 2.

Step 1. Suppose f to be such a continuous real valued function on SO_n that, for some pair of distinct positive integers $\alpha \leqq n$, $\beta \leqq n$, $f(R)$ is independent of the βth column of R for $R \in SO_n$ and is linear and homogeneous in the elements of either the αth column of R or the βth row of R. Then

$$\int_{SO_n} f d\Phi_n = 0.$$

Proof. In the first case, if $\alpha < \beta$,

$$\int f d\Phi_n = \int f(RI_{\alpha\beta}) d\Phi_n R = -\int f d\Phi_n.$$

In the second case, choose a left and right invariant metric on O_n for which $H^m(SO_n) = 1$, $m = n(n-1)/2$. Since $\operatorname{ad} I_n(SO_n) = SO_n$,

$$\int_{SO_n} f d\Phi_n = \int_{SO_n} f dH^m = \int_{SO_n} f \circ \operatorname{ad}(I_{\beta n} I_n) dH^m = -\int_{SO_n} f d\Phi_n.$$

Step 2. Suppose $v_0 = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r} \neq 0$, $v = e_{\beta_1} \wedge \cdots \wedge e_{\beta r} \neq 0$, $w_0 = e_{\mu_1} \wedge \cdots \wedge e_{\mu_s} \neq 0$, and $w = e_{\nu_1} \wedge \cdots \wedge e_{\nu_s} \neq 0$. If $v_0 \cdot v = 0$, then

(1) $r \neq n/2$ or $s \neq n/2$ implies that $N(v, v_0, w, w_0) = 0$,

(2) $r = s = n/2$ implies that $N(v, v_0, w, w_0) = 0$ unless $v_0 = \pm *v$ and $w_0 = \pm *w$.

Proof. For the proof of (1), choose ν so that $\beta_\nu \neq \alpha_i$ for $i = 1, 2, \cdots, r$. For $R \in SO_n$, $[v_0 \wedge R(w_0)] \cdot [v \wedge R(w)]$ is the sum of terms of the form

$$(*) \qquad [v_0 \cdot v' \wedge R(w')][v'' \wedge R(w'') \cdot R(w_0)]$$

where v', v'', w', w'' are standard basis multi-vectors, $v = v' \wedge v''$, $w = w' \wedge w''$, and $v'' = e_{\delta_1} \wedge \cdots \wedge e_{\delta_p}$ with $p > 0$ and, for some μ, $\delta_\mu = \beta_\nu$. Furthermore, $(*)$ is the sum of terms of the form

$$(v_0' \cdot v')[v_0'' \cdot R(w')][v'' \cdot R(w_0')](w'' \cdot w_0'')$$

where v_0', v_0'', w_0', w_0'' are standard basis multi-vectors and $v_0 = v_0' \wedge v_0''$, $w_0 = w_0' \wedge w_0''$. It therefore suffices to show that

$$\int_{SO_n} [R(v_0'') \cdot w'][R(v'') \cdot w_0'] d\Phi_n R = 0.$$

But $\deg v'' + \deg v_0'' < n$, hence $[R(v_0'') \cdot w'][R(v'') \cdot w_0']$ does not involve some column of R and is linear and homogeneous in the elements of the δ_μth column of R, and Step 1 can be applied.

To prove (2), observe that if $v_0 \neq \pm *v$, the first part of Step 1 implies that $N(v, v_0, w, w_0) = 0$. Furthermore, if $w_0 \neq \pm * w$, then there exists α such that $[* v \wedge R(w)_0] \cdot [v \wedge R(w)]$ is independent of the elements of the αth column of R, and thus $N(v, *v, w, w_0) = 0$ by the second part of Step 1.

6.3. REMARK. G is unimodular; an elementary proof of this fact can be given with little difficulty. Let Ψ be the Haar measure on G characterized in 5.4.

6.4. THEOREM. *Suppose X is oriented, $S \in N_k(X)$ and $T \in N_l(X)$, $k+l \geqq n$. Then, for each bounded Baire k form ϕ on X and each bounded Baire l form ψ on X,*

$$\int_G S \cap g_\#(T)[\phi \cap g^{-1\#}(\psi)]\, d\Psi g$$
$$= \begin{cases} \Gamma^2(n,k,l)\, S(\phi)\, T(\psi), & k > n/2 \text{ or } l > n/2, \\ \Gamma^2(n,k,l)[S(\phi)\, T(\psi) + S(*\phi)\, T(*\psi)], & k = l = n/2. \end{cases}$$

Proof. Fix ϕ and ψ, and for each $(x,y) \in X \times X$ such that $\|\vec{S}(x)\| = \|\vec{T}(y)\| = 1$ define

$$P(x,y) = \begin{cases} [\phi\vec{S}(x)][\psi\vec{T}(y)], & k > n/2 \text{ or } l > n/2. \\ [\phi\vec{S}(x)][\psi\vec{T}(y)] + [*\phi\vec{S}(x)][*\psi\vec{T}(y)], & k = l = n/2. \end{cases}$$

Fix $o \in X$ and define $\pi: G \to X$ by $\pi(g) = g(o)$. If $(a,b) \in G \times G$ is such that $\|\vec{S} \cap ab^{-1}{}_\#(\vec{T})[\pi(a)]\| = 1$, define

$$F(a,b) = [\phi \cap ba^{-1\#}(\psi)][\vec{S} \cap ab^{-1}{}_\#(\vec{T})][\pi(a)];$$

otherwise, set $F(a,b) = 0$. F is a bounded Baire function and we can apply 5.5 to F.

To evaluate $\mathscr{I}$, consider a fixed $(x,y) \in X \times X$ such that $\|\vec{S}(x)\| = \|\vec{T}(y)\| = 1$, and choose $h \in I_x$, $h' \in I_y$. We have

$$\mathscr{I}(x,y) = \int_{I_x \times I_y} [a^\#(*\phi) \wedge b^\#(*\psi)][a^{-1}{}_\#(*\vec{S}) \wedge b^{-1}{}_\#(*\vec{T})](o)\, d\Phi_x \times \Phi_y(a,b)$$
$$= \int_I [h^\#(*\phi) \wedge g^\# h'^\#(*\psi)][h^{-1}{}_\#(*\vec{S}) \wedge g^{-1}{}_\# h'^{-1}{}_\#(*\vec{T})](o)\, d\Phi_0 g;$$

consequently, 6.2 implies that $\mathscr{I}(x,y) = \Gamma^2(n,k,l)P(x,y)$. Therefore, application of 5.5 yields

$$\int_G S \cap g_\#(T)[\phi \cap g^{-1\#}(\psi)]\, d\Psi g = \Gamma^2(n,k,l) \int_{X \times X} P\, d\|S\| \times \|T\|,$$

which completes the proof.

6.5. REMARK. If X is not orientable, then it is well known that n is even and X is isometric to the n dimensional real projective space P_n. See, for instance, [17, p. 308]. If σ is a covering map of the oriented unit n-sphere S_n onto P_n and $L': \boldsymbol{E}_*(P_n) \cap \{T: \boldsymbol{M}(T) < \infty\} \to \boldsymbol{E}_*(S_n)$ is the associated lifting map, then

$$\int_G L'(S) \cap g_\#[L'(T)][\sigma^\#(\phi) \cap g^{-1\#}\sigma^\#(\psi)]\, d\Psi g = \tfrac{1}{2}\Gamma^2(n,k\,l)\, S(\phi)\, T(\psi).$$

6.6. THEOREM. *If $S \in N_k(X)$ and $T \in N_l(X)$, $k+l \geqq n$, then*

$$\Gamma^2(n,k,l)M(S)M(T) \leqq \int_G M[S \cap g_\#(T)]\,d\Psi g$$
$$\leqq \gamma(n,k,l)M(S)M(T);$$

the second inequality is strict unless $\vec{S}$ and $\vec{T}$ are simple.

Proof. By 5.7, we only need show that if $v \in \bigwedge_k(\mathbf{R}^n)$, $w \in \bigwedge_l(\mathbf{R}^n)$ and $\|v\| = \|w\| = 1$, then

$$\Gamma^2(n,k,l) \leqq \gamma(v,w) \leqq \gamma(n,k,l),$$

the second inequality being strict unless v and w are simple.

The first inequality follows from 5.6 and 6.2. To verify the second, use [13, 2.2] to obtain finite sets V, W of simple k, l vectors such that

$$v = \sum_{\alpha \in V} \alpha,\ 1 = \sum_{\alpha \in V} |\alpha|$$

and

$$w = \sum_{\beta \in W} \beta,\ 1 = \sum_{\beta \in W} |\beta|.$$

Then by 5.9,

$$\gamma(v,w) \leqq \sum_{(\alpha,\beta) \in V\times W} \int_{SO_n} \|\alpha \wedge g(\beta)\|\,d\Phi_n g$$
$$= \gamma(n,k,l) \sum_{(\alpha,\beta) \in V\times W} |\alpha|\,|\beta| = \gamma(n,k,l).$$

Finally, note that the inequality is strict unless v and w are simple.

7. **The Cauchy-Crofton formulas**. The formula derived by Chern in [4] is generalized to currents. A formula is obtained for the oriented case also. The proofs were suggested by the proof of [11, 6.13].

Let G and X be as in 5. Suppose $\mathscr{E}$ to be a set of closed, oriented l dimensional submanifolds of X such that G acts transitively on $\mathscr{E}$, the action preserving the orientation of the elements of $\mathscr{E}$, and such that if $E \in \mathscr{E}$, then $G \cap \{g : g_\#(E) = E\}$ is transitive on E. Also assume $\mathscr{E}$ to have a G invariant Haar measure.

Fix $Y \in \mathscr{E}$, define $K = G \cap \{g : g_\#(Y) = Y\}$, and choosing $o \in Y$, define $\pi : G \to X$ by $\pi(g) = g(o)$. Let $I = \pi^{-1}(o)$. There exists a left invariant metric on G such that $H^m(I) = 1$ and

$$\pi_\#(a) \mid \mathbf{T}_a(G), \qquad \pi_\#(z) \mid \mathbf{T}_z(K)$$

are orthogonal projections whenever $a \in G$, $z \in K$. Let $\dim K = l + \lambda$.

Letting $gK \in G/K$ correspond to $g(Y) = \pi(gK)$, we can identify $\mathscr{E}$ with G/K. By [27, 9] there exists a Haar measure Φ on $\mathscr{E}$ such that for each Borel set $S \subset G$,

$$\int_{\mathscr{E}} H^{l+\lambda}(S \cap E) d\Phi E = H^{\lambda}(K \cap I) H^{m+n}(S).$$

It follows from 4.9 that if X is oriented and $S \in N_k(X)$, $k+l \geqq n$, then $S \cap E \in N_{k+l-n}(X)$ for Φ almost all $E \in \mathscr{E}$.

7.1. Theorem. *Suppose $S \in N_k(X)$, $k+l \geqq n$, to be such that whenever $\vec{S}(x)$ and $\vec{S}(y)$ exist, there exists $g \in G$ such that $g_{\#}(x)[\vec{S}(x)] = \pm \vec{S}(y)$. Then, for each Borel subset B of X,*

$$\int_{\mathscr{E}} \| S \cap E \| (B) d\Phi E = \gamma(\vec{S}, \vec{Y}) \| S \| (B).$$

Proof. If $g \in G$, define $f(g) = \det(\operatorname{ad} g_{\#})$. The proof is divided into three parts.

Part 1. Let U be a compact neighborhood of o,

$$A = I[\pi^{-1}(U) \cup [\pi^{-1}(U)]^{-1}]I$$

and, for $t > 1$,

$$A_t = A \cap \{g : t^{-1} \leqq f(g) \leqq t\}.$$

Then

$$A_t = \pi^{-1}[\pi(A_t)] = A_t^{-1} \text{ and } \lim_{t \to 1^+} H^{m+n}(A_t)^{-1} \int_{A_t} f dH^{m+n} = 1.$$

Proof. This is proved by means of a straightforward application of the coarea formula [11, 3.1] to f.

Part 2. Let A be a compact subset of K such that $A^{-1} = A$ and $A = \pi^{-1}[\pi(A)] \cap K$. Then

$$H^{l+\lambda}(A) \int_{\mathscr{E}} \| S \cap E \| (B) d\Phi E = \gamma(\vec{S}, \vec{Y}) \int_A f dH^{l+\lambda} \| S \| (B).$$

Proof. Let η and ζ denote the characteristic functions of $\pi(A)$ and of B, respectively. By [11, 3.1] and 5.1,

$$\int_A f dH^{l+\lambda} = H^{\lambda}(K \cap I) \int_Y \eta \Delta_\pi dH^l.$$

Applying 5.7 and recalling 5.2, we have

$$\begin{aligned} &\gamma(\vec{S}, \vec{Y}) \| S \| (B) \int_A f dH^{l+\lambda} \\ &= H^{\lambda}(K \cap I) \int_G \int_X \zeta(\eta \circ g^{-1}) d \| S \cap g_{\#}(Y) \| dH^{m+n} g \\ &= \int_{G/K} \left\{ \int_F \int_X \zeta(\eta \circ g^{-1}) d \| S \cap \pi(F) \| dH^{l+\lambda} g \right\} d\Phi F. \end{aligned}$$

since $g_{\#}(Y) = \pi(F)$ for $g \in F$. In order to compute the expression $\mathscr{I}(F)$ in braces, we use 4.4(4) to find $F = g_F(K)$ such that $S \cap \pi F \in N_{k+l-n}(X)$, and let $\mu = \| g_{F\#}^{-1}\ (S) \cap Y \|$. The left invariance of the metric on G implies that

$$\mathscr{I}(F) = \int_K \int_X (\zeta \circ g_F)(\eta \circ z^{-1})\, d\mu dH^{l+\lambda} z$$
$$= \int_X \zeta \circ g_F(x) \int_K \eta \circ z^{-1}(x) dH^{l+\lambda} z d\mu x.$$

Consider a fixed $x \in Y$, $x = z_0(o)$, where $z_0 \in K$. For $z \in K, \eta \circ z^{-1} z_0(o) = 1$ if and only if $z \in z_0 A$; consequently,

$$\int_K \eta \circ z^{-1}(x) dH^{l+\lambda} z = H^{l+\lambda}(z_0 A) = H^{l+\lambda}(A),$$

hence

$$\mathscr{I}(F) = H^{l+\lambda}(A) \| S \cap g_{F\#}(Y) \| (B),$$

and the remainder of the proof follows easily from this.

Part 3. $\int_{\mathscr{E}} \| S \cap E \| (B) d\Phi E = \gamma(\vec{S}, \vec{Y}) \| S \| (B)$.

Proof. Use [27, 9] and Part 1 with G replaced by K and X replaced by Y to obtain compact subsets A_t of K such that $A_t^{-1} = A_t$, $A_t = \pi^{-1}[\pi(A_t)] \cap K$, and

$$\lim_{t \to 1^+} H^{l+\lambda}(A_t)^{-1} \int_{A_t} f dH^{l+\lambda} = 1.$$

Now apply Part 2 with $A = A_t$, divide by $H^{l+\lambda}(A_t)$, and let $t \to 1^+$.

7.2. Remark. For the following theorem, we need make no assumption concerning the orientability of the elements of $\mathscr{E}$.

7.3. Theorem. *Let A be an H^k measurable subset of X, $k + l \geqq n$, such that A is Hausdorff k rectifiable. Further, suppose that G acts transitively on $\mathbf{T}(A)$. Then*

$$\int_{\mathscr{E}} H^{k+l-n}(A \cap E) d\Phi E = \gamma[\mathbf{T}(A), \mathbf{T}(Y)] H^k(A).$$

Proof. Using 5.15 instead of 5.7, one proceeds in the same manner as in the proof of 7.1.

7.4. Definition. If $T \in E_l(X)$ and $M(T) < \infty$, then ψ_T is the Baire l form characterized by the following conditions: $\psi_T \vec{T} = 1$, $\psi_T v = 0$ if v is an l vectorfield such that $v \cdot \vec{T} = 0$, and $\psi_T(x) = 0$ if $\vec{T}(x)$ is not defined.

7.5. Theorem. *Suppose X is oriented and* $\dim G = n(n+1)/2$. *If $S \in N_k(X)$, $k + l \geqq n$, then, for each bounded Baire k form ϕ,*

$$\int_{\mathscr{E}} S \cap E(\phi \cap \psi_E) d\Phi E = \Gamma^2(n, k, l) S(\phi).$$

Proof. The proof is analogous to the proof of 7.1. Choose A and η as in the proof of 7.1, Part 2. Applying 6.4 with $T = Y$ and $\psi = \eta\psi_Y$, we have

$$\Gamma^2(n,k,l)\,S(\phi)H^{l+\lambda}(A)$$

$$= H^\lambda(I \cap K)\int_G S \cap g_\#(Y)[\phi \cap g^{-1\#}(\eta\psi_Y)]\,dH^{m+n}g$$

$$= \int_{G/K}\int_F S \cap \pi(F)[\phi \cap g^{-1\#}(\eta\psi_Y)]\,dH^{l+\lambda}g\,d\Phi F.$$

Use 4.4(4) to find $F = g_F(K)$ such that $S \cap \pi(F) \in N_{k+l-n}(X)$, and define

$$\mu = \|g_F{}^{-1}{}_\#(S) \cap Y\|,$$

$$\zeta = g_F{}^\#(\phi) \cap \psi_Y[g_F{}^{-1}{}_\#(S) \cap Y]^{\rightarrow} \circ g_F{}^{-1}.$$

We proceed in the same way as in the proof of 7.1, Part 2 to show that the inner integral is equal to

$$\int_K\int_X (\zeta \circ g_F)(\eta \circ z^{-1})\,d\mu\,dH^{l+\lambda}z$$

$$= H^{l+\lambda}(A)S \cap g_{F\#}(Y)[\phi \cap g_F{}^{-1\#}(\psi_Y)]$$

$$= H^{l+\lambda}(A)\,S \cap \pi(F)(\phi \cap \psi_{\pi(F)}),$$

which completes the proof.

7.6. Remark. If $X = P_n$, σ and L' are as in 6.5, and $\mathscr{E}$ is the set of isometric embeddings of S_l in S_n, with Φ defined relative to S_n, then

$$\int_{\mathscr{E}} L'(S) \cap E[\sigma^\#(\phi) \cap \psi_E]\,d\Phi E = \Gamma^2(n,k,l)\,S(\phi).$$

7.7. Remarks. It follows from 4.4(2) that if S is quasi-normal (see 4.9), then 7.1 and 7.5 are true for S.

The following theorem becomes, for $k = n$, $X = \mathbf{R}^n$, and $\mathscr{E}$ the set of oriented lines in $\mathbf{R}^n$, the theorem proved independently by Krickeberg [19, Theorem 5.5, p. 120] and Federer [10].

7.8. Theorem. *Suppose X is oriented and $\dim G = n(n+1)/2$. Let S be a quasi-normal k current in X, $k + l > n$. Then $S \in N_k(X)$ if and only if*

$$\int_{\mathscr{E}} M[\partial(S \cap E)]\,d\Phi E < \infty.$$

Proof. If $\phi \in \mathbf{E}^{k-1}(X)$, we infer from 7.5, 4.10(2), the definition of Φ and 4.6(1) that

$$\int_{\mathscr{E}} \partial(S \cap E)(\phi \cap \psi_E) d\Phi E \;=\; \Gamma^2(n, k-1, l)\, \partial S(\phi),$$

and the sufficiency of our condition follows from this.

Necessity follows from 7.1.

8. **Evaluation of $\gamma(v, w)$ for the complex case.** Let X be an n dimensional complex Riemannian manifold such that there exists a group H of holomorphic isometries of X which is transitive on X. Suppose also that for $x \in X$,

$$\{h_{\#}(x): h \in H \text{ and } h(x) = x\}$$

is the unitary group of $T_x(X)$. We evaluate $\gamma(A, B)$ for A, B which have as values complex subspaces of the tangent spaces of X. We need only consider the case where X is the n dimensional complex projective space CP^n.

8.1. Definitions. $\boldsymbol{C}^n$ is the *n dimensional complex vector space* of all n-tuples $(z^1, \cdots, z^n)$ of complex numbers. $\boldsymbol{C}^n$ has the *standard Hermitian metric* $\boldsymbol{h}_0$ defined by

$$\boldsymbol{h}_0(v, w) \;=\; \sum_{j=1}^{n} v^j \bar{w}^j .$$

CP^n is the *n dimensional complex projective space* with homogeneous coordinates $[z^1, \cdots, z^{n+1}]$.

U_n is the *unitary group* of linear transformations h of $\boldsymbol{C}^n$ for which $h^{\#}(\boldsymbol{h}_0) = \boldsymbol{h}_0$.

8.2. Remark. One introduces coordinates and a Riemannian metric in $\boldsymbol{C}^n$ by means of the map $c: \boldsymbol{R}^{2n} \to \boldsymbol{C}^n$ for which

$$c(x^1, y^1, \cdots, x^n, y^n) \;=\; (x^1 + iy^1, \cdots, x^n + iy^n).$$

$\boldsymbol{h}_0(v, w) = c^{-1}(v) \cdot c^{-1}(w)$, and if $F: \boldsymbol{C}^n \to \boldsymbol{C}^n$ is a differentiable map for which $F^{\#}(\boldsymbol{h}_0) = \boldsymbol{h}_0$, then F is an isometry.

CP^n is a compact, complex manifold; the complex structure on CP^n is obtained by requiring the map $\sigma': \boldsymbol{C}^{n+1} \to CP^n$, $\sigma'(z) = [z]$, to be holomorphic. S_1 acts freely on $S_{2n+1} = \boldsymbol{C}^{n+1} \cap \{z: |z| = 1\}$ as a group of isometries by means of the map $S_{2n+1} \times S_1 \to S_{2n+1}$ defined by $(z, \lambda) \to \lambda z$. $\sigma = \sigma' | S_{2n+1}$ is a principal fibration of S_{2n+1} over CP^n with structure group S_1.

There therefore exists a Riemannian metric on CP^n such that $\sigma_{\#}(z)$ is an orthogonal projection for $z \in S_{2n+1}$. Furthermore, the action of U_n on S_{2n+1} commutes with the action of S_1 and, consequently, if $Z_n = U_{n+1} \cap \{\lambda e: \lambda \in S_1\}$, then

$$G_n \;=\; U_{n+1}/Z_n$$

acts transitively on CP^n as a group of holomorphic isometries.

8.3. Lemma. $H^{2n}(CP^n) = \pi^{-1}(n+1)\,\alpha(2n+2)$.

Proof. If $x \in CP^n$, then $\sigma^{-1}(x)$ is isometric with S_1. The formula follows from application of the coarea formula [11, 3.1] to σ.

8.4. Theorem. *If $x \in CP^n$, then*

$$\{h_\#(x) : h \in G_n \text{ and } h(x) = x\}$$

is the unitary group of $T_x(CP^n)$.

Proof. Let $\pi: U_{n+1} \to S_{2n+1}$ be the projection defined by $\pi(h) = h[c(e_{2n+2})]$, and let $\tau = \sigma \circ \pi$. We can assume that $x = \tau(e)$. $j: \boldsymbol{C}^n \to \boldsymbol{C}^{n+1}$ is the isometry defined by $j(z) = (z, 0)$. j induces a monomorphic embedding j_* of the direct product $U_n \times S_1$ into U_{n+1} such that

$$j_*(h, \lambda)(j(z), w) = (j[h(z)], \lambda w)$$

for $(h, \lambda) \in U_n \times S_1$, $z \in \boldsymbol{C}^n$, $w \in \boldsymbol{C}$. It is clear that $\tau^{-1}[\tau(e)] = \text{image } j_*$, and that

$$j_*[U_n \times \{1\}]/Z_n \cap j_*[U_n \times \{1\}] = \text{image} j_*/Z_n$$

acts on $T_x(CP^n)$ as the unitary group.

8.5. Theorem. *Suppose $x \in CP^n$ and P, Q are respectively k, l (complex) dimensional complex subspaces of $T_x(CP^n)$, $k + l \geqq n$. Then*

$$\gamma(P, Q) = \gamma_c(2n, 2k, 2l).$$

Proof. We may assume that $0 < k < n$, $0 < l < n$, $x = [0, \cdots, 0, 1]$, and $P = T_x(A)$ and $Q = T_x(B)$, where

$$A = CP^n \cap \{[z] : z^j = 0,\ j = 1, \cdots, n-k\} \text{ and}$$

$$B = CP^n \cap \{[z] : z^j = 0,\ j = n-k+1, \cdots, 2n-k-l\}.$$

$$A \cap B = CP^n \cap \{[z] : z^j = 0,\ j = 1, \cdots, 2n-k-l\},$$

and it is easy to see that A, B, $A \cap B$ are isometric embeddings of CP^k, CP^l, CP^{k+l-n} in CP^n, and that for $g \in G_n$,

$$A \cap g(B) = g'(A \cap B)$$

for some $g' \in G_n$. Recalling 5.2, we apply 5.8 and 8.3 to obtain

$$\gamma(P, Q) = H^{2k}(CP^k)^{-1} H^{2l}(CP^l)^{-1} H^{2(k+l-n)}(CP^{k+l-n}) H^{2n}(CP^n) = \gamma_c(2n, 2k, 2l).$$

8.6. Remark. We can readily evaluate the constant $\gamma(n, k, l)$ by using S_n and isometric embeddings of S_k and S_l in S_n, and proceeding as in the proof of 8.5.

9. **Evaluation of $\Gamma^2(n, k, l)$.**

9.1. Definitions. With $z \in \boldsymbol{R}^n$ one associates the *translation* $T_z: \boldsymbol{R}^n \to \boldsymbol{R}^n$, $T_z(x) = z + x$ for $x \in \boldsymbol{R}^n$.

With $g \in SO^n$ and $w \in \boldsymbol{R}^{n-m}$ one associates the *oriented m dimensional plane*

$$\lambda_n^m(g,w) = g_\#(R_n \cap \{x : x^i = w^i \text{ for } i = 1,\cdots,n-m\}),$$

where $\lambda_n^m(e,w)$ has $e_{n-m+1} \wedge \cdots \wedge e_n$ as a positively oriented tangent m vector-field.

G_n is the group of *orientation preserving isometries* of $\boldsymbol{R}^n$. Under the map which associates with $(z,g) \in \boldsymbol{R}^n \times SO_n$ the isometry $T_z \circ g$ of $\boldsymbol{R}^n$, the image of $L_n \times \Phi_n$ is the Haar measure Ψ characterized in 5.2 and 5.4.

Under the map λ_n^m, the image of the measure $\Phi_n \times L_{n-m}$ is a G_n invariant measure for the space $\mathscr{E}_m$ of oriented m dimensional planes in $\boldsymbol{R}^n$.

9.2. Remark. Identify G_n with $\boldsymbol{R}^n \times SO_n$ and define the projection $\pi : G_n \to \boldsymbol{R}^n$, $\pi(z,h) = z$. Let $Y = \lambda_n^m(e,0)$ and $K = G_n \cap \{(x,g) : x \in Y, g_\#(Y) = Y\}$.

By evaluating for $S = \boldsymbol{R}^n \cap \{z : |z| < 1\} \times SO_n$ the formula in §7 which characterizes the measure Φ on $\mathscr{E}_m$, we see that Φ is equal to the λ_n^m image of $\Phi_n \times L_{n-m}$.

9.3. Theorem. $\Gamma^2(n,k,l) = \gamma^2(n,k,l)$.

Proof. The proof is divided into three steps; the formula follows upon comparison of Step 3 with 7.5. Let $A = \lambda_n^k(e,0) \cap \{z : |z| < 1\}$.

Step 1.

$$\Gamma^2(n,k,n-k) = \binom{n}{k}^{-1}.$$

Proof. Let Λ be the set of increasing functions

$$\lambda : \{1,\cdots,k\} \to \{1,\cdots,n\}$$

and λ_0 be the inclusion. If $\lambda \in \Lambda$, then $j_\lambda : \boldsymbol{R}^k \to \boldsymbol{R}^n$ is the linear map for which $j_\lambda(e_i) = e_{\lambda(i)}$, $i = 1,\cdots,k$, and p_λ is the orthogonal projection which is the transpose of j_λ. For each $\lambda \in \Lambda$, we have

$$\Gamma^2(n,k,n-k) = \int_{SO_n} (\det p_\lambda \circ g \circ j_{\lambda_0})^2 d\Phi_n g.$$

On the other hand, if "J" denotes "Jacobian", then

$$1 = [J(g \circ j_{\lambda_0})]^2 = \sum_{\lambda \in \Lambda} (\det p_\lambda \circ g \circ j_{\lambda_0})^2.$$

Step 2. If $\int_A f dH^k < \infty$, then

$$\int_{SO_n \times \boldsymbol{R}^k} \int_{A \cap \lambda_n^{n-k}(g,w)} f \left| \psi_A \wedge g^{-1\#}(dx^{k+1} \wedge \cdots \wedge dx^n) \right| dH^0 d\Phi_n \times L_k(g,w)$$

$$= \binom{n}{k}^{-1} \int_A f dH^k.$$

Proof. Keeping 9.1, 9.2 and 4.4(4) (ii) in mind, one applies 7.5 with $\phi = f\psi_A$.

Step 3. If $0 < m < k$ and $\phi = dx^{k-m+1} \wedge \cdots \wedge dx^n$, then

$$\int_{SO_n \times \mathbf{R}^{k-m}} A \cap \lambda_n^{n-k+m}(g,w)[\psi_A \cap g^{-1\#}(\phi)]\, d\Phi_n \times L_{k-m}(g,w)$$

$$= \binom{n-k+m}{m}\binom{n}{k}^{-1} H^k(A).$$

Proof. The method of proof is exactly the same as for the proof of [8, 8]; one uses Step 2 in place of [8, 5].

10. **A generalization of the coarea formula.** Federer's coarea formula [11, 3.1] is generalized to give nontrivial results for maps $f: X \to Y$ such that rank $f_\#(x) < \dim Y$ for $x \in X$.

10.1. LEMMA. *Suppose p, q, r, s are nonnegative integers, $p > 0$, $q > 0$, $r \leqq n$, $0 < s \leqq n$ and $S_{p/q}$ is the linear transformation of $\mathbf{R}^n$ such that*

$$S_{p/q}(e_i) = e_i, \quad i = 1, \cdots, r,$$

and

$$S_{p/q}(e_i) = pq^{-1}e_i, \quad i = r+1, \cdots, n.$$

If $A \subset \mathbf{R}^n$, then

$$2^{2s}p^{r-s}q^{n-r}H^s[S_{p/q}(A)] \geqq 2^s n^{-s/2} H^s(A)$$
$$\geqq n^{-s}p^{r-n}q^{s-r}H^s[S_{p/q}(A)].$$

Proof. If $\alpha > 0$ and $A \subset \mathbf{R}^n$, define $c_\alpha(A)$ to be the infimum of the sums

$$\Sigma(\text{edge } f)^s, \qquad f \in F,$$

corresponding to all countable coverings F of A by cubes of the form

$$(*) \qquad \mathbf{R}^n \cap \{x: |x^i - a^i| \leqq \beta,\ i = 1, \cdots, n\}$$

for $\beta < \alpha$, $a \in \mathbf{R}^n$. Define $d_\alpha(A)$ to be the infimum of the sums

$$\Sigma(\text{diameter } f)^s, \quad f \in F,$$

corresponding to all countable coverings F of A such that diameter $f < \alpha$ for $f \in F$. It is easy to see that

$$(**) \qquad 2^s d_{\alpha/2}(A) \geqq c_\alpha(A) \geqq n^{-s/2} d_{\alpha n^{1/2}}(A).$$

Now fix $\alpha > 0$, $A \subset \mathbf{R}^n$ and consider a countable covering F of A by cubes of the form (*). Observing that each $S_p(f)$, $f \in F$, is the union of p^{n-r} translates of f, we conclude that

$$c_\alpha(A) \geqq p^{r-n} c_\alpha[S_p(A)].$$

Furthermore, let S, S^p be the linear transformations of $\mathbf{R}^n$ for which $S(x) = p^{-1}x$ and

$$S^p(e_i) = pe_i, \; i = 1, \cdots, r,$$

and

$$S^p(e_i) = e_i, \; i = r+1, \cdots, n.$$

Since $S_p^{-1} = S^p \circ S$, it follows that

$$p^{-s} c_{p\alpha}[S_p(A)] = c_\alpha[S \circ S_p(A)] \geqq p^{-r} c_\alpha(A).$$

Similarly, we have

$$c_\alpha[S_{p/q}(A)] \geqq q^{r-n} c_\alpha[S_p(A)]$$

and

$$q^{-s} c_{q\alpha}[S_p(A)] \geqq q^{-r} c_\alpha[S_{p/q}(A)].$$

It follows that

$$q^{n-r} p^{r-s} c_{pq\alpha}[S_{p/q}(A)] \geqq c_{q\alpha}(A) \geqq p^{r-n} q^{s-r} c_\alpha[S_{p/q}(A)],$$

and from this we obtain the conclusion by applying (∗∗), letting $\alpha \to 0^+$ and referring to 2.8.

10.2. Definition. If X and Y are Riemannian manifolds of class 1, $f: X \to Y$, f is differentiable at x and r is a positive integer, then

$$J_r f(x) = \sup\{f_\#(x)(v) : v \in \textstyle\bigwedge_r[T_x(X)], \; |v| = 1\}.$$

10.3. Theorem. *If X and Y are separable Riemannian manifolds of class* 1 *with* $\dim X = m$, $f: X \to Y$ *is a Lipschitzian map, and*

$$r = \sup\{\operatorname{rank} f_\#(x) : x \in X\},$$

then

$$\int_A J_r f \, dH^m = \int_Y H^{m-r}[A \cap f^{-1}(y)] \, dH^r y$$

whenever A is an H^m measurable subset of X, and consequently

$$\int_X g J_r f \, dH^m = \int_Y \int_{f^{-1}(y)} g \, dH^{m-r} dH^r y$$

whenever g is an H^m integrable function on X(2).

Proof. The proof is a modification of the proof of [11, 3.1]. Assume that $r < k = \dim Y$. Referring to the proof of Part 7 of [11, 3.1], we see that we can assume f to be continuously differentiable.

(2) In research to be announced in Bull. Amer. Math. Soc., H. Federer has shown that $Y \cap \{y : H^{m-r}[f^{-1}(y)] > 0\}$ is Hausdorff r rectifiable.

Define

$$B = \{x: \operatorname{rank} f_{\#}(x) = r\}.$$

Since B is open, $f(B)$ is the union of countably many proper r-submanifolds of class 1 of Y, hence we can apply [11, 3.1] to obtain the formulas for $f|B$. To complete the proof, it suffices to show that

$$(*) \qquad \int_Y H^{m-r}[f^{-1}(y) \cap (X - B)]\, dH^r y = 0.$$

Suppose M to be a Lipschitz constant for f and if $A \subset X$, define

$$v_A(y) = H^{m-r}[A \cap f^{-1}(y)] \quad \text{for } y \in Y.$$

Let μ be the measure on X such that

$$\mu(A) = \int_Y^* v_A\, dH^r \quad \text{for } A \subset X,$$

where "$\int^*$" means "upper integral". For $a \in X$, let

$$K(a, \rho) = X \cap \{x : \operatorname{distance}(x, a) < \rho\}$$

whenever $\rho > 0$, and let

$$\mu'(a) = \lim_{\rho \to 0+} \mu[K(a, \rho)]/H^m[K(a, \rho)].$$

The remainder of the argument is divided into four parts; $(*)$ follows from Parts 3 and 4.

Part 1. If $A \subset X$, then

$$\mu(A) \leqq M^r \alpha(m)^{-1} \alpha(r)\, \alpha(m - r) H^m(A)$$

and

$$Y \cap \{y: v_A(y) > 0\}$$

is the union of countably many sets of finite H^r measure.

Proof. The first statement follows from the second and [9, 3.2]. Suppose $H^m(A) < \infty$. Let $c = M^r \alpha(m)^{-1} \alpha(r) \alpha(m - r)$ and

$$C_s = Y \cap \{y: v_A(y) > s\} \text{ for } s \geqq 0.$$

If C_0 is not the union of countably many sets of finite H^r measure, there exists $s > 0$ such that $H^r(C_s) = \infty$. [1] asserts the existence of $D \subset C_s$ such that $s^{-1} c H^m(A) < H^r(D) < \infty$ and we infer from [9, 3.2] that

$$cH^m(A) < \int_D^* v_A\, dH^r \leqq cH^m(A).$$

Part 2. If A is an H^m measurable subset of X, then v_A is an H^r measurable function.

Proof. The proof of Part 2 of [11, 3.1] extends easily to the present situation.

Part 3. If A is an H^m measurable subset of X, then

$$\mu(A) = \int_Y H^{m-r}[A \cap f^{-1}(y)]\, dH^r y = \int_A \mu' dH^m.$$

Proof. The proof of Part 3 of [11, 3.1] extends easily to the present situation.

Part 4. If $a \in X = \boldsymbol{R}^m$, $Y = \boldsymbol{R}^k$ and $J_r f(a) = 0$, then $\mu'(a) = 0$.

Proof. Let j be a positive integer. Since rank $f_\#(a) < r$, there exists an orthogonal projection $q: \boldsymbol{R}^k \to \boldsymbol{R}^{k-r+1}$ such that $q \circ f_\#(a) = 0$. The continuity of $f_\#$ at a implies the existence of a convex neighborhood U of a such that

$$|q \circ f_\#(x)| \leqq j^{-1}M \quad \text{for } x \in U,$$

whence

$$|q \circ f(x) - q \circ f(z)| \leqq j^{-1}M|x - z| \quad \text{for } x, z \in U.$$

Furthermore, since μ is invariant under rotations of $\boldsymbol{R}^k$, one may assume that

$$q(e_i) = e_i \quad \text{for } i = 1, \cdots, k - r + 1.$$

It follows that if S is the endomorphism of $\boldsymbol{R}^k$ such that

$$S(y) = (jy^1, \cdots, jy^{k-r+1}, y^{k-r+2}, \cdots, y^k)$$

for $y \in \boldsymbol{R}^k$, then

$$|S \circ f(x) - S \circ f(z)| \leqq 2M|x - z| \quad \text{for } x, z \in U.$$

Furthermore, application of 10.1 to S yields

$$H^r(C) \leqq j^{-1}2^r k^{r/2} H^r[S(C)] \quad \text{for } C \subset \boldsymbol{R}^k.$$

Applying Part 3 to f and $S \circ f$ and Part 1 to $S \circ f$, one concludes that if A is an H^m measurable subset of U, then

$$\begin{aligned}\mu(A) &= \int_{\boldsymbol{R}^k} H^{m-r}[A \cap (S \circ f)^{-1}[S(y)]]\, dH^r y \\ &\leqq j^{-1}2^r k^{r/2} \int_{\boldsymbol{R}^k} H^{m-r}[A \cap (S \circ f)^{-1}(w)]\, dH^r w \\ &\leqq j^{-1}2^{2r} M^r k^{r/2} \alpha(m)^{-1} \alpha(r) \alpha(m-r) H^m(A).\end{aligned}$$

11. **Sets having high dimensional intersections.** The situation hypothesized in 5.9 is discussed, and in certain spaces a more general version of the integralgeometric formula derived in 5.8 is obtained.

The notation used in §5 will be readopted. In addition, suppose there exist closed subgroups J, K of G such that:

(a) I is the direct product of J and $L = K \cap I$.

(b) Each $g \in G$ has a unique representation $g = \tau\rho$, $\tau \in K$, $\rho \in J$.

(c) K is normal in G and has a bi-invariant metric such that $\pi | K_{\#}(a)$ is an orthogonal projection for $a \in K$.

Note that one may have $X = \mathbf{R}^n$ or X compact. Let $\dim L = j$ and Φ be the Haar measure on J for which $\Phi(J)H^j(L) = 1$.

11.1. THEOREM. *Let A be an H^k measurable subset of X and B be an H^l measurable subset of X such that A is Hausdorff k rectifiable and B is l rectifiable, and let (see 5.14)*

$$r = \sup\{\dim(\mathbf{T}_x(A) + \mathbf{T}_x[g(B)]) : g \in G,\ x \in A \cap g(B)\}.$$

Suppose also that G acts transitively on $\mathbf{T}(A)$ and on $\mathbf{T}(B)$.

There exists $\beta > 0$ depending only upon $\mathbf{T}(A)$, $\mathbf{T}(B)$ such that

$$\int_J \int_K H^{k+l-r}[A \cap \tau\rho(B)]\, dH^{r+j}\tau d\Phi\rho = \beta H^k(A) H^l(B).$$

Furthermore, if $r = k + l$ and each tangent space of X is oriented, then

$$\beta = \gamma[* \mathbf{T}(A), * \mathbf{T}(B)].$$

Proof. Define $f: K \times K \to K$ by $f(a,b) = ab^{-1}$. If $S \subset K \times K$ and $H^{k+l+2j}(S) = 0$, one proceeds as in the proof of Part 1 of 10.3 to show that

$$\int_K H^{k+l-r+j}[S \cap f^{-1}(g)]\, dH^{r+j}g = 0.$$

Using this in place of [9, 3.2] and proceeding as in the proof of 5.15, we see that it is only necessary to prove the assertions for $A \subset A_0$, $B \subset B_0$, where A_0, B_0 are proper k, l-submanifolds of class 1 of X.

If $S \subset X$, let $S' = \pi^{-1}(S) \cap K$. Define ξ as follows: Let M, N be k, l dimensional linear subspaces of $\mathbf{T}_0(X)$. $\xi(M,N) = 0$ if $\dim(M + N) \neq r$; otherwise, choose simple unit vectors $v \in \bigwedge_{r-l}(M)$, $w \in \bigwedge_{r-k}(N)$ so that v, w lie in the orthogonal complement of $M \cap N$ and define

$$\xi(M,N) = |v \wedge w|.$$

If $(x,y) \in A \times B$, define for each $\rho \in J$,

$$F_\rho(x,y) = \int^*_{\{x\}' \times \{\rho(y)\}'} \xi[a^{-1}{}_{\#}(x)[\mathbf{T}_x(A_0)], b^{-1}\rho_{\#}(y)[\mathbf{T}_y(B_0)]]\, dH^{2j}(a,b).$$

The remainder of the argument is divided into two steps.

Step 1. For each $\rho \in J$,

$$H^j(L) \int_K H^{k+l-r}[A \cap \tau\rho(B)]\, dH^{r+j}\tau = \int_{A \times B} F_\rho\, dH^{k+l}.$$

Proof. It is easy to see that $\dim K = n + j$. Consider a fixed $\rho \in J$ and define $f: A_0' \times \rho(B_0)' \to K$ by $f(a,b) = ab^{-1}$. Application of 10.3 to f yields

$$\int_K H^{k+l-r+j}[f^{-1}(\tau) \cap A' \times \rho(B)']dH^{r+j}\tau$$

$$= \int_{A'\times\rho(B)'} J_{r+j}f\,dH^{k+l+2j}$$

$$= \int_{A\times B} \int_{\{x\}'\times\{\rho(y)\}'} J_{r+j}f\,dH^{2j}dH^{k+l}(x,y).$$

Furthermore, the first integral is equal to

$$2^{\delta/2}H^j(L) \int_K H^{k+l-r}[A \cap \tau\rho(B)]dH^{r+j}\tau, \qquad \delta = k+l-r+j,$$

because by 10.3, H^δ almost all of $f^{-1}(\tau)$ is a δ-submanifold of class 1 for H^{r+j} almost all $\tau \in K$, and the projection $p: f^{-1}(\tau) \to K$, $p(a, \tau^{-1}a) = a$, has Jacobian $2^{-\delta/2}$ at each manifold point of $f^{-1}(\tau)$. Finally, if $(a,b) \in A' \times \rho(B)'$, one uses 4.1 and the bi-invariance of the metric on K to show that

$$J_{r+j}f(a,b) = 2^{\delta/2}\xi[a^{-1}{}_\#(x)[T_x(A_0)], b^{-1}\rho_\#(y)[T_y(B_0)]],$$

$$x = a(o), \qquad y = \rho^{-1}b(o).$$

Step 2. There exists $\beta > 0$ depending only on $T(A)$, $T(B)$ such that

$$\int_J \int_{A\times B} F_\rho dH^{k+l}\, d\Phi\rho = \beta H^j(L)H^k(A)H^l(B).$$

If $r = k+l$, then $\beta = \gamma[*\,T(A),\ *\,T(B)]$.

Proof. There exist $M \subset T_o(X)$, $N \subset T_o(X)$ such that for H^{k+l} almost all $(x,y) \in A \times B$, there exist $g_x \in K$, $h_y \in K$, $\rho_x \in J$, $\sigma_y \in J$ for which

$$T_x(A_0) = g_x\rho_{x\#}(o)(M), \qquad T_y(B_0) = h_y\sigma_{y\#}(o)(N).$$

Since K is normal in G and I is the direct product of J and L, we have for each $\rho \in J$,

$$F_\rho(x,y) = \int_{L\times L} \xi[(g_xu)^{-1}{}_\#(x)[T_x(A_0)],\ (\rho h_y\rho^{-1}v)^{-1}\rho_\#(y)[T_y(B_0)]]\,dH^{2j}(u,v)$$

$$= H^j(L) \int_L \xi[M, \lambda\rho_x{}^{-1}\rho\sigma_{y\#}(o)(N)]\,dH^j\lambda.$$

It is clear that ξ is continuous on the product of the space of k planes in $T_o(X)$ with the space of l planes in $T_o(X)$. Furthermore, Step 1 implies that F_ρ is H^{k+l} measurable, hence for the function F defined on $J \times A \times B$ by $F(\rho,x,y) = F_\rho(x,y)$ is $\Phi \times H^{k+l}$ measurable and we can apply the Fubini theorem to obtain

$$H^j(L)^{-1} \int_J \int_{A\times B} F_\rho dH^{k+l} d\Phi\rho$$

$$= \int_{A\times B} \int_L \int_J \xi[M, \lambda\rho_x{}^{-1}\rho\sigma_{y\#}(o)(N)] d\Phi\rho dH^j\lambda dH^{k+l}(x,y)$$

$$= H^k(A)H^l(B) \int_{L\times J} \xi[M, \lambda\rho_\#(o)(N)] dH^j \times \Phi(\lambda,\rho).$$

Finally, the integral

$$\int_{L\times J} \xi[M, \lambda\rho_\#(o)(N)] dH^j \times \Phi(\lambda,\rho)$$

is clearly positive; if $r = k + l$, then it is equal to $\gamma[*\boldsymbol{T}(A), *\boldsymbol{T}(B)]$, since $H^j \times \Phi$ is a Haar measure on $I = L \times J$.

11.2. Remark. Examination of the proof of Step 2 reveals that if $K = G$, then for $r = k + l$ it is not necessary to assume that $\pi_\#(a)$ is an orthogonal projection for $a \in K$. (This assumption is, of course, necessary for the formula for β to be valid.)

If $k + l \leqq n$, then the formula obtained by replacing r by $k + l$ is true regardless of the value of r. However, $\beta = 0$ unless $r = k + l$.

11.3. Remark. Suppose $X = \boldsymbol{R}^n$ and $\mathscr{E}_l, G_n$ are as in 9.1. Let A be an H^k measurable, Hausdorff k rectifiable subset of X, $k + l \leqq n$.

Set $Y = \lambda_n^l(e, 0)$ and define $H_g = G_n \cap \{(z,g)\colon [z + g(Y)] \cap A \text{ is not empty}\}$ for each $g \in SO_n$. One uses [9, 4.1] to show that H_g is Hausdorff $k + l$ rectifiable. Defining $\sigma\colon G_n \to SO_n \times \boldsymbol{R}^{n-l}$ by $\sigma(z,g) = (g, p \circ g^{-1}(z))$, where p is the orthogonal projection of $\boldsymbol{R}^n$ on the orthogonal complement of Y, one easily sees that $\sigma_\#(h) | \boldsymbol{T}_h(H_g)$ is an orthogonal projection for H^{k+l} almost all $h \in H_g$. Application of Part 1 of 10.3 to $\sigma | H_0$, where $H_0 \subset H_g$ and $H^{k+l}(H_0) = 0$, shows that 10.3 can be applied to $\sigma | H_g$ to yield

$$\int_{R^{n-l}} H^l(S \cap \lambda_n^l(g,w) \times \{g\}) dH^k w = H^{k+l}(S)$$

for each Borel subset S of H_g.

Using 11.1 instead of 5.15, one proceeds as in the proof of 7.3 to show that

$$\int_{SO_n} \int_{R^{n-l}} H^0[A \cap \lambda_n^l(g,w)] dH^k w d\Phi_n g = \gamma(n, n-k, n-l) H^k(A).$$

It follows from [7, 9.7] that this is the formula derived by Freilich in [14, 3.1].

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