PIERCING POINTS OF HOMEOMORPHISMS OF DIFFERENTIABLE MANIFOLDS

BY

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1. Introduction. In this paper we investigate the problem of relating homeomorphisms of euclidean n-space $E^n$ (resp. a differentiable manifold $M^n$) onto itself to diffeomorphisms of $E^n$ (resp. of $M^n$). Throughout this paper, "diffeomorphism" shall mean $C^p$-diffeomorphism, where $p > 0$. A discussion of these problems is facilitated by introducing the following equivalence relation $\sim$ on the set $H(E^n)$ of homeomorphisms of $E^n$ onto itself. If $F, G \in H(E^n)$, we say $F \sim G$ if there exist homeomorphisms $H_0 = F, H_1, \ldots, H_m = G$, where each $H_i \in H(E^n)$, and nonempty open sets $U_1, U_2, \ldots, U_m$ of $E^n$ such that $H_i|U_i = H_{i-1}|U_i$, $i = 1, 2, \ldots, m$. One asks, for example, whether a given homeomorphism $F \in H(E^n)$ is equivalent under $\sim$ to a diffeomorphism.

Fundamental in the study of this type of question is the notion of stable homeomorphisms of $E^n$ onto itself. Recall that a homeomorphism $H \in H(E^n)$ is called stable if there exist homeomorphisms $H_1, \ldots, H_m$, where each $H_i \in H(E^n)$ and nonempty open sets $U_1, \ldots, U_m$ of $E^n$ such that $H = H_1 H_2 \cdots H_m$, and $H_i|U_i = 1$, $i = 1, 2, \ldots, m$. All orientation-preserving diffeomorphisms of $E^n$ onto itself are stable. It is readily seen that if $F \sim G$, and $G$ is stable, then so is $F$. It also can be proved (cf. Theorem 5.4 of [1]) that if $F$ and $G$ are any two stable homeomorphisms, then $F \sim G$. It follows easily from these latter two statements that $F \sim G$ if and only if $G^{-1}F$ is a stable homeomorphism of $E^n$. Finally, the annulus conjecture is equivalent to the conjecture that all orientation-preserving homeomorphisms of $E^n$ onto itself are stable. This latter conjecture is known to be true for $n = 1, 2, 3$.

In an effort to relate homeomorphisms to diffeomorphisms, we define (cf. §3) the notion of a piercing point of a homeomorphism. In §§3–7 we develop some basic properties relating to piercing points. A proof is given in §8 of a result announced by the author in [2]. Theorem 1 of §9 relates the notion of piercing point to stability. The author thanks William Huebsch for many helpful conversations.

2. Notation. Let the points of $E^n$ be written $x = (x^1, \ldots, x^n)$, and provide $E^n$ with the usual euclidean norm and metric

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For $c$ a fixed point of $E^n$, and $r > 0$ a constant, we denote the $(n-1)$-sphere about $c$, of radius $r$, by

$$S^{n-1}(c,r) = \{ x \in E^n \mid d(x,c) = r \}.$$ 

We often delete the superscript $n-1$ when there is no danger of confusion. By a topological $(n-1)$-sphere $M$ in $E^n$ we mean the image in $E^n$ of $S^{n-1}(c,r)$ under some homeomorphism $h$. We say that $h$ defines $M$. If $M$ is a topological $(n-1)$-sphere in $E^n$, we denote the bounded component of $E^n - M$ by $JM$, and the closure of $JM$ in $E^n$ by $JM$.

A topological $(n-1)$-sphere $M$ in $E^n$ will be called elementary if some (hence every, as is readily proved) homeomorphism $h$ defining $M$ is extendable as a homeomorphism into $E^n$ of an open neighborhood $N$ of $S^{n-1}(c,r)$ relative to $E^n$. This is equivalent (cf. [3]) to requiring that $M$ is locally flat. A homeomorphism $h$ of an elementary $(n-1)$-sphere $M$ in $E^n$ which is extendable over a neighborhood of $M$ as a homeomorphism will itself be termed elementary.

If $M$ is a topological $(n-1)$-sphere in $E^n$, and $f: JM \to E^n$ is a homeomorphism of $JM$ into $E^n$, then

(\alpha) \quad f(JM) = JM(f(M)).

A proof of (\alpha) can be found, for example, in [4].

3. Piercing points. We assume in what follows that $n \geq 2$.

**Definition 1.** Let $f: U \to E^n$ be a homeomorphism of $U$ into $E^n$, where $U$ is an open subset of $E^n$. A point $x \in U$ is called a piercing point of $f$ if there exists a $C^p$-embedding ($p > 0$) $\sigma: [-1,1] \to U$, a diffeomorphism $H \in H(E^n)$, and an $(n-1)$-hyperplane $P$ in $E^n$ such that

(i) $\sigma(0) = x$,
(ii) $Hf\sigma([-1,1]) \cap P = Hf\sigma(0)$,
(iii) $Hf\sigma(-1)$ and $Hf\sigma(1)$ lie in opposite components of $E^n - P$.

One verifies that every point of a diffeomorphism $H \in H(E^n)$ is a piercing point of $H$. On the other hand, we will prove (\S 8) that there exist homeomorphisms $F \in H(E^n)$ having a dense set of nonpiercing points.

**Proposition 1.** Let $U, V$ be open subsets of $E^n$, let $f: U \to E^n$ be a homeomorphism of $U$ into $E^n$, and let $g: V \to E^n$ be a diffeomorphism of $V$ into $E^n$. Then a point $x \in U \cap f^{-1}(V)$ [resp. $x \in U \cap g(V)$] is a piercing point of $f$ if, and only if, $x$ is a piercing point of $gf$ [resp. $g^{-1}(x)$ is a piercing point of $fg$].

**Proof.** In each situation considered, it can be assumed without loss of generality that $g$ is a diffeomorphism of $E^n$ onto itself (cf. [5, pp. 28–29]). Suppose that $x$ is a piercing point of $f$. Then, corresponding to $f$ and $x$, there exist $\sigma, H,$
and $P$ as in Definition 1. Then $\sigma, Hg^{-1}$, and $P$ [resp. $g^{-1}\sigma, H, P$] can be used to verify that $x$ is a piercing point of $gf$ [resp. $g^{-1}(x)$ is a piercing point of $fg$]. Conversely, suppose that $x$ is a piercing point of $H$ [resp. $g^{-1}$ is a piercing point of $fg$]. Then $\sigma, Hg$, and $P$ [resp. $g\sigma, H, P$] can be used to verify that $x$ is a piercing point of $f$.

With the aid of Proposition 1, the notion of piercing point may be extended, in the natural way, to homeomorphisms from one nonbounded differentiable $n$-manifold into another. More precisely, we make the following definition.

**Definition 1'**. Let $M_1^n, M_2^n$ be nonbound differentiable $n$-manifolds having differentiable structures $D_1, D_2$, respectively, both of class at least $p > 0$. Let $f: U \to M_2^n$ be a homeomorphism of $U$ into $M_1^n$, where $U$ is an open subset of $M_1^n$. Then $x$ is called a piercing point of $f$ if there exist coordinate systems $(U_1, h_1) \in D_1$, $(U_2, h_2) \in D_2$ at $x, f(x)$, respectively, such that $h_1(x)$ is a piercing point (in the sense of Definition 1) of the homeomorphism $h_2 f h_1^{-1} : h_1(U \cap U_1 \cap f^{-1}(U_2)) \to \mathbb{R}^n$.

It is easily verified, using Proposition 1 and its proof, that this definition is independent of the choice of coordinate systems at $x$ and $f(x)$.

4. Example 1. We now construct homeomorphisms $f, g \in H(E^2)$ such that $0 = (0,0)$ is a piercing point of $f$, $f(0) = 0$ is a piercing point of $g$, but $0$ is not a piercing point of the composition $gf$. Now it is easily seen that a point $x \in U \subset E^2$ is a piercing point of a homeomorphism $h: U \to E^2$ if, and only if, the point $h(x)$ is a piercing point of $h^{-1}: h(U) \to E^2$. Hence if $f, g$ are as above, we see that the homeomorphisms $gf, f = g^{-1}(gf), f^{-1} = (gf)^{-1} g$, show that all the conclusions of Proposition 1 can fail when the hypothesis is weakened to merely requiring, for example, that $g(x)$ be a piercing point of the homeomorphism $g$.

Let $x = (0,0) = 0 \in E^2$, and set $S = S'(0,1)$. Denote the line segments having one end point at $0$ and the other end point at $(0,1), (\sqrt{2}/2, \sqrt{2}/2), (\sqrt{2}/2, -\sqrt{2}/2), (-\sqrt{2}/2, \sqrt{2}/2), (-\sqrt{3}/2, -\sqrt{3}/2), (-\sqrt{3}/2, \sqrt{3}/2), (\sqrt{3}/2, \sqrt{3}/2)$, by $L_0, \ldots, L_6$, respectively. Let

$$
M_1 = \{y = (y_1, y_2) \in S | 0 \leq y_1 \leq \sqrt{2}/2, \sqrt{2}/2 \leq y_2 \leq 1\} \cup L_0 \cup L_1,
M_2 = \{y = (y_1, y_2) \in S | -\sqrt{2}/2 \leq y_1 \leq 0, \sqrt{2}/2 \leq y_2 \leq 1\} \cup L_0 \cup L_2,
$$

and

$$
M_3 = \{y = (y_1, y_2) \in S | -1 \leq y_1 \leq 1, -1 \leq y_2 \leq \sqrt{2}/2\} \cup L_1 \cup L_2.
$$

Let

$$
\{y_i\} = \{1/i(\sqrt{2}/2, \sqrt{2}/2)\}, \quad \{z_i\} = \{1/i(-\sqrt{2}/2, \sqrt{2}/2)\}, \quad i = 1, 2, \ldots.
$$

We now define $f$ on $M_1$ as follows. Set $f(w) = w$ for $w = (w^1, w^2) \in L_0$, and

$$
f(w) = (w^1 \cos(5\pi \sqrt{2}/12w^1) + w^2 \sin(5\pi \sqrt{2}/12w^1),
-w^1 \sin(5\pi \sqrt{2}/12w^1) + w^2 \cos(5\pi \sqrt{2}/12w^1))
$$
for \( w \in M_1 \cap S \). We complete the definition of \( f \) on \( M_1 \) as follows. Set \( f(y_{2i-1}) = 1/i(\sqrt{3}/2, -1/2) \), and \( f(y_{2i}) = 1/i(-1/2, -\sqrt{3}/2) \), \( i = 1, 2, \ldots \). Then let \( f \) map the segment between \( y_i \) and \( y_{i+1} \) in the manner indicated in Figure 1 below.

Define \( f \) on \( M_2 \) by letting
\[
f((w_1, w_2)) = (-f^1(-w_1, w_2), f^2(-w_1, w_2)) \text{ for } w \in L_0 \cup (M_2 \cap S) \cup \bigcup_{i=1}^{\infty} z_i.
\]
Complete the definition of \( f \) on \( M_2 \) by mapping the segment between \( z_i \) and \( z_{i+1} \) in the manner indicated in Figure 1.

To define \( f \) on \( M_3 \), note that \( f \) is defined on \((M_3 - S) \cup \{\sqrt{2}/2, -\sqrt{2}/2\} \cup \{(-\sqrt{2}/2, \sqrt{2}/2)\} \). We complete the definition of \( f \) on \( M_3 \) by mapping \( M_3 -(L_1 \cup L_2) \) homeomorphically onto \( S - \{f(M_1 \cup M_2)\} \).

We now have defined \( f \) on \( M_1 \cup M_2 \cup M_3 \) consistently as a homeomorphism. Since every homeomorphism of a topological 1-sphere \( M \) in \( E^2 \) admits an extension as a homeomorphism over \( JM \) (cf. [6]), we may extend \( f| M_i \) to a homeomorphism of \( JM_i \) into \( E^2 \), \( i = 1, 2, 3 \). Actually, it is clear that \( f| M_i \) is elementary, and hence we could use the Schoenflies extension theorem (cf. [7] or [8]) to get the extension of \( f| M_i \) over \( JM_i \). This latter method of obtaining the extension is employed when modifying the example to dimensions greater than 2. Now since \( fJM_i = fM_i \) (cf. (a) of §2), we obtain a homeomorphism \( f \) of \( JS \) into \( E^n \). For convenience, we apply the Schoenflies extension theorem again to assume that \( f \in H(E^2) \). Note that \( 0 \) is a piercing point of \( f \).

We now consider a homeomorphism \( g \in H(E^2) \) with the following properties (cf. Figure 2). Let \( L_3, L_4, L_5, \) and \( L_6 \) be as in Figure 1. Denote the line segments having one end point at \( 0 \) and the other end point at \((-\frac{1}{2}, \sqrt{3}/2), (-\sqrt{3}/2, \frac{1}{2}), \sqrt{3}/2, \frac{1}{2}, (\frac{1}{2}, \sqrt{3}/2), \) by \( L_7, L_8, L_9, L_{10}, \) respectively. We suppose \( g(L_i) = L_{i+4}, \)
We further suppose that $g$ is the identity on the $w^2$-axis. A homeomorphism $g \in H(E^2)$ with these properties is easily constructed. Note that $0$ is a piercing point of $g$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Figure 2}
\end{figure}

It is readily seen that $0$ is not a piercing point of $gf$.

**Remarks.** An analogous construction can be carried out for any $n \geq 2$. Also, one verifies with the aid of Theorem 4.1 of [5] (only really needed for $f$), that $f$ and $g$ can be constructed so as to be diffeomorphisms on $E^n - 0$.

5. **One-sided piercing points.** Note that the homeomorphism $gf$ of Example 1, while not having $0$ as a piercing point, nevertheless leaves the segment from $(0,1)$ to $0$ pointwise invariant. Hence $0$ could be called a "one-sided" piercing point of $gf$. We make this concept precise in the following definition.

**Definition 2.** Let $f: U \to E^n$ be a homeomorphism of $U$ into $E^n$, where $U$ is an open subset of $E^n$. A point $x \in U$ is called a one-sided piercing point of $f$ if there exists $C^p$-imbedding $(p > 0)$ $\sigma: [0,1] \to U$, a diffeomorphism $H \in H(E^n)$ and an $(n - 1)$-hyperplane $P$ in $E^n$ such that

1. $\sigma(1) = x$,
2. $H \sigma([0,1]) \cap P = H \sigma(1)$.

A point $x \in U$ which is not a one-sided piercing point will be called a spiral point of $f$.

We see, then, that a spiral point $x \in U$ of $f$ has the following characteristic property: given any $C^p$-imbedding $(p > 0)$ $\sigma: [0,1] \to U$, any diffeomorphism $H \in H(E^n)$, and any $(n - 1)$-hyperplane $P$ in $E^n$, there exists a sequence of points $t_1 \in [0,1]$ converging to 1 and such that $H \sigma(t_1) \in P$. Clearly, a spiral point of $f$ is a nonpiercing point of $f$, but the converse does not hold as the homeomorphism $gf$ of Example 1 shows.

**Proposition 2.** Let $f: U \to E^n$ be a homeomorphism of $U$ into $E^n$, where $U$ is
an open subset of $E^n$. Then the set of one-sided piercing points of $f$ is dense in $U$.

In fact, if $\sigma : [0,1] \to U$ is any $C^p$-imbedding, $(p > 0)$, then there exists a $t \in (0,1)$ such that $\sigma(t)$ is a one-sided piercing point of $f$.

Proof. Let $\sigma : [0,1] \to U$ be a $C^p$-imbedding, and let $P$ be any $(n - 1)$-hyperplane in $E^n$ such that $f\sigma(0)$ and $f\sigma(1)$ lie in opposite components of $E^n - P$. Since $f\sigma([0,1]) \cap P \neq \emptyset$, there exists a (unique) $t \in (0,1)$ such that $f\sigma([0,t]) \cap P = \sigma(t)$. Hence $\sigma(t)$ is a one-sided piercing point of $f$.

Remark. The notion of "one-sided" piercing point extends, in the natural way (cf. Definition 1'), to differentiable manifolds. Proposition 2 holds with this notion so extended.

6. An alternative definition. Suppose we altered Definition 1 by requiring that $H$ be the identity diffeomorphism of $E^n$ onto itself. The question arises as to whether every piercing point in the old sense would also be one in this new more restrictive sense. Example 2 below answers this latter question in the negative.

Example 2. Let $S = S^1(0,1)$, and let $f : E^2 \to E^2$ be a homeomorphism of $E^2$ onto itself having the following properties. First $f(x) = x$ for $x \in C \cup L$, where $L$ is the radius segment of $S$ joining $0 = (0,0)$ to $(0,1)$. Secondly, the image under $f$ of every radius segment of $S$ makes an angle of $0^\circ$ with $L$ at $0$ (cf. Figure 3). Such homeomorphism clearly exists (in fact, $f$ may be required to be a diffeomorphism on $E^2 - 0$). Note that $0$ is not a piercing point of $f$ if $H$ is required to be the identity map of $E^n$.

We now construct a diffeomorphism $H \in H(E^n)$ which will show that $0$ is a piercing point (in the sense of Definition 1) of $f$.

We can assume $f$ is so constructed that there exists a $C^\infty$-imbedding $\tau : (L \cup L') \to E^2$ such that $\tau(L) = f(L')$ and $\tau|L' = 1$, where $L, L'$ are the radius segments of $S$ which go through $(\sqrt{2}/2, \sqrt{2}/2), (0, -1)$, respectively. Let $H$ be a $C^\infty$-diffeomorphism of $E^2$ onto itself such that $H(f(L') \cup L') = \tau^{-1}$. Let $\sigma : [-1,1] \to E^2$ be the linear imbedding defined by $\sigma(-1) = (0, -\frac{1}{2}), \sigma(1) = (0, \frac{1}{2})$. Finally, let $P$ be the $x^2$ axis. One verifies that $\sigma, H$, and $P$ satisfy the conditions of Definition 1 with respect to $f$ and $0$.

Remark. Example 2 also shows that Proposition 1 would not be satisfied if our definition of piercing point would have been the more restrictive one. This example can be modified to yield the corresponding results for $n \geq 2$.

7. Property $Q$. We now introduce a property which concerns the existence of piercing points of a uniform type.

Definition 3. Let $f : U \to E^n$ be a homeomorphism of $U$ into $E^n$, where $U$ is an open set in $E^n$. We say that $f$ has property $Q$ at a point $x_0 \in U$ if there exists a diffeomorphism $L$ of $E^n$ onto itself such that the following three conditions are satisfied:

(i) $L^{-1}(x_0)$ is a piercing point of $fL$, i.e. there exists $\sigma, H,$ and $P$ satisfying (i), (ii), and (iii) of Definition 1 relative to $fL$ and $L^{-1}(x_0)$,
(ii) $\sigma$ is linear, i.e. $\sigma([-1,1])$ is a straight line segment,
(iii) there exists an open set $W$ in $P$ such that for all $x \in L^{-1}f^{-1}H^{-1}(W)$, $H,P,$
and $\sigma_x$ defined by $\sigma_x(t) = \sigma(t) + x - \sigma(0)$ satisfy conditions (i), (ii), and (iii) of
Definition 1 with respect to $fL$ and $x$.

Figure 3

Remarks. Again, one verifies that if $f: U \to \mathbb{E}^n$ is a $C^p$-imbedding, ($p > 0$),
then $f$ has property $Q$ at each point of its domain. Also, it is readily seen that the
definition of "property $Q$" can be extended in the natural way (cf. Definition 1')
to differentiable manifolds. One verifies that if $M^*_1,M^*_2$ are nonbounded differ-
entiable $n$-manifolds, and if $f: U \to M^*_2$ is a $C^p$-imbedding of $U$ into $M^*_2$, where $U$
is an open subset of $M^*_1$, then $f$ has property $Q$ at each point of $U$.

8. A homeomorphism having a dense set of nonpiercing points. We now construct
a homeomorphism $F$ of $\mathbb{E}^n$ onto itself having a dense set of nonpiercing points.
For $c = (c^1,c^2,\cdots,c^n)$, $x = (x^1,x^2,\cdots,x^n)$, and $r > 0$, we define the homeomorphism
$F_{c,r,0}, i = 2,\cdots,n,$ of $\mathbb{E}^n$ onto itself, as follows:

$F_{c,r,0}(x) = x,$

$F_{c,r,i}(x) = ((x^1-c^i)\cos(x)-(x^{i-1}-c^i)\sin(x)) + c^i, x^2,\cdots,x^{i-1},$

$\quad (x^1-c^1)\sin(x) + (x^{i-1}-c^i)\cos(x) + c^i, x^{i+1},\cdots,x^n)$

$[x \in JS(c,r) - JS(c,r/2)]$ where $\alpha(x) = 4\pi \left(\frac{r-\|x-c\|}{r}\right)$.

In the above formula, and throughout the following, $CX$ denotes the complement
in $\mathbb{E}^n$ of $X$. We then define the homeomorphism $F_{c,r}$ of $\mathbb{E}^n$ onto itself by setting
$F_{c,r} = F_{c,r,0}F_{c,r,3} \cdots F_{c,r,n}$. We then choose a constant $\delta > 0$ with the following
property ($\beta$).
(β) If $S(x_1, r_1), \ldots, S(x_m, r_m)$ are $(n - 1)$-spheres in $E^n$ such that $JS(x_i, r_i) \cap JS(x_j, r_j) = \emptyset$, $i \neq j$, $1 \leq i, j \leq n$, and if $H = F_{x_n, r_n} F_{x_{n-1}, r_{n-1}} \cdots F_{x_1, r_1}$ (note that $H$ is independent of the order of the factors), then $d(H(x), H(y)) \geq \delta d(x, y)$ for all $x, y \in E^n$.

Such a $\delta$ clearly exists. We now define, inductively, a sequence of homeomorphisms $\{F_i\}$ of $E^n$ into itself, and set $F = \lim_{i \to \infty} F_i$.

Let $X$ be a countable dense subset in $E^n$ of distinct points $x_i$, $i = 1, 2, \ldots$. Set $F_0 = 1$. Select a positive constant $r_{11} < \frac{1}{2}$ such that $S(x_1, r_{11}) \cap X = \emptyset$. Set $F_1 = F_{x_1, r_{11}} = F_{11}$. Now consider $F_1(x_2)$. By our choice of $S(x_1, r_{11})$, and since $F_1|S(x_1, r_{11}) = 1$, we have $F_1(x_2) \notin S(x_1, r_{11})$. We have two cases.

Case 1. $F_1(x_2) \in JS(x_1, r_{11})$. Then select positive constants $r_{12}, r_{22}$ such that:

1. $r_{12} < \frac{1}{4} r_{11}$,
2. $JS(F_1(x_2), r_{22}) = JS(x_1, r_{11})$,
3. $JS(F_1(x_1), r_{12}) \cap JS(F_1(x_2), r_{22}) = \emptyset$,
4. $JS(F_1(x_1), r_{12}) \cup JS(F_1(x_2), r_{22}) \cup S(F_1(x_1), r_{12}/2) \cup S(F_1(x_2), r_{22}/2)$

Then set $F_2 = F_{F_1(x_1), r_{11}}, F_2 = F_{F_1(x_2), r_{22}}$, and

$$F_2 = F_{22} F_{12} F_{11} = F_{22} F_{12} F_1.$$  

Case 2. $F_1(x_2) \in CJS(x_1, r_{11})$. Then select positive constants $r_{12}, r_{22}$ such that (2.1), (2.2), (2.4) and (2.5) hold, together with the following relation analogous to (2.3).

$$JS(F_1(x_2), r_{22}) \subseteq CJS(x_1, r_{11}).$$

Then construct $F_2$ as in (2.6). Note that the following relations are satisfied.

$$F_2(x_1) = F_1(x_1) = x_1, \quad F_2(x_2) = F_1(x_2).$$

Suppose, inductively, that positive constants $r_{ij}$, $i = 1, 2, \ldots, k - 1$, $i \leq j \leq k - 1$ have been chosen, together with homeomorphisms $F_0 = 1, F_1, \ldots, F_{k-1}$ of $E^n$ onto itself, such that the following conditions are satisfied. First, for $1 \leq m \leq k - 1$, $F_m = F_{mm} F_{m-1,m} \cdots F_{1m} F_{m-1}$, where $F_{ij} = F_{F_{j-1}(x_i), r_{ij}}, i = 1, 2, \ldots, k - 1, i \leq j \leq k - 1$, and, moreover:
\[(k-1.1)\] \[r_{ij} < \frac{1}{2} r_{ij-1} < \cdots < \frac{1}{2} r_{il},\]
\[(k-1.2)\] \[\max(r_{ij}) < \delta^{1/2^j},\]
\[(k-1.3)\] \[JS(F_{j-1}(x_i), r_{ij}) \cap JS(F_{j-1}(x_m), r_{mj}) = \emptyset \quad [l \neq m, 1 \leq j \leq k-1, l, m \leq j],\]
\[(k-1.4)\] \[\bigcup_{i=1, \ldots, k-1; i \leq j \leq k-1} S(F_{j-1}(x_i), r_{ij}) \cup S(F_{j-1}(x_i), r_{ij}/2) \cap \{X \cup F_1(X) \cup \cdots \cup F_{k-1}(X)\} = \emptyset,\]
\[(k-1.5)\] \[F_{j-1}(x_j) \in JS(F_{j-1}(x_i), r_{ip}) \Rightarrow JS(F_{j-1}(x_j), r_{jj}) \subset JS(F_{j-1}(x_i), r_{ip}) \quad [j = 1, \ldots, k-1; 1 \leq p \leq j - 1],\]
\[(k-1.6)\] \[F_{j-1}(x_j) \in CJS(F_{j-1}(x_i), r_{ip}) \Rightarrow JS(F_{j-1}(x_j), r_{jj}) \subset CJS(F_{j-1}(x_i), r_{ip}) \quad [j = 1, \ldots, k-1; 1 \leq l \leq p \leq j - 1].\]

Note by \((k-1.4)\) that \((k-1.5)\) and \((k-1.6)\) cover the possible locations of \(F_{j-1}(x_j)\). Note also that the following relations are necessarily satisfied:

\[(k-1.7)\] \[F_j(x_i) = F_{j-1}(x_i) \quad [j = 1, \ldots, k-1, 1 \leq l \leq j],\]
\[(k-1.8)\] \[F_{j-1}(y) \in JS(F_{j-1}(x_i), r_{ip}) \Rightarrow F_q(y) \in JS(F_{j-1}(x_i), r_{ip}) \quad [j = 1, \ldots, k-1, 1 \leq l \leq p \leq j - 1 \leq q \leq k - 1, y \in E^n],\]
\[(k-1.9)\] \[F_{j-1}(y) \in CJS(F_{j-1}(x_i), r_{ip}) \Rightarrow F_q(y) \in CJS(F_{j-1}(x_i), r_{ip}) \quad [j = 1, \ldots, k-1, 1 \leq l \leq p \leq j - 1 \leq q \leq k - 1, y \in E^n],\]
\[(k-10)\] \[d(F_j(x), F_j(y)) \geq \delta^{1/2^j}d(x, y) \quad [x, y \in E^n, j = 1, \ldots, k-1],\]
\[(k-11)\] \[d(F_j(x), F_{j-1}(x)) < \delta^{1/2^j} < 1/2^j \quad [j = 1, \ldots, k-1].\]

Clearly, we may choose positive constants \(r_{ik}, i = 1, \ldots, k,\) and define \(F_k = F_{k+1}F_{k+1} \cdots F_{k+1}F_k\) so that the relations \((k.1)-(k.11)\) analogous to \((k-1.1)-(k-1.11)\) hold. Hereafter, we understand \((m.1)\) to mean the relation in stage \(m\) of our construction analogous to \((k-1.1)\) above. Set \(F = \lim_{k \to \infty} F_k\). Then using \((k.11), F\) is a continuous mapping of \(E^n\) onto itself. Hence to show that \(F\) is a homeomorphism, it suffices to show that \(F\) is biunique. To verify the biuniqueness of \(F\), let \(x, y\) be distinct points of \(E^n\). We have the following two cases.

Case 1. There exists a sequence \(k_1 < k_2 < \cdots\) such that, for example, \(F_{k_1}(x) \neq F_{k_1}(x)\).

Then \(F_{k_1}(x) \in JS(F_{k_1}(x), r_{ik_1-1})\) for some \(l_i \leq k_{i-1}\). Using \((k-5), (k-6)\) and \((k-8)\), we see that \(F(x) \in JS(F_{k_1}(x), r_{ik_1-1})\), \(i = 1,2, \ldots\). Set \(\zeta = d(x, y)\), and choose \(p = k_j\) when \(j\) is so large that \(1/2^{p-1} < \zeta/3\). Now \(F(x) \in JS(F_{p-1}(x), r_{ip-1})\), and by \((k.10)\), we have

\[(\xi)\] \[d(F_{p-1}(x), F_{p-1}(y)) \geq \delta^{p-1}d(x, y) = \delta^{p-1}\zeta.\]

Now by \((k.2), (\beta)\), and our choice of \(p\), we have
Then $F_{p-1}(y) \in CJS(F_{p-1}(x_{i_1}), r_{i_1}, p^{-1})$, and using (k.9), $F(y) \notin JS(F_{p-1}(x_{i_1}), r_{i_1}, p^{-1})$. Hence $F(x) \neq F(y)$. A similar proof holds when there exists a sequence $m_1 < m_2 < \cdots$ such that $F_m(y) \neq F_{m-1}(y)$.

Case 2. There exist integers $M(x), N(y)$ such that $F_t(x) = F_{M(x)}(x)$, $t \geq M(x)$, and $F_t(y) = F_{N(y)}(y)$, $t \geq N(y)$.

Then $F(y) = F_{N(y)}(y)$ and $F(x) = F_{M(x)}(x)$. Suppose $M(x) = N(y)$. Then since $F_{M(x)} = F_{N(y)}$ is a homeomorphism, we have $F(x) = F_{M(x)}(x) = F_{N(y)}(y)$.

Now if $M(x) < N(y)$, then $F(x) = F_{M(x)}(x) = F_{N(y)}(y)$, and since $F_{N(y)}$ is a homeomorphism, we have $F(x) = F_{N(y)}(y) = F(y)$. A similar proof holds when $N(y) < M(x)$.

This completes the proof of the biuniqueness of $F$, and therefore $F$ is a homeomorphism of $E^n$ onto itself.

Claim. $X$ consists of nonpiercing points of $F$. We show first that $x_1$ is a nonpiercing point of $F$. Figure 4 illustrates the situation when $n = 2$. The situation for general $n$ is analogous.

Clearly, $x_1$ is a nonpiercing point of $F$, in fact, $x_1$ is a spiral point. Now consider $x_j$, where $j > 1$. Construct a new sequence of homeomorphisms $\{G_{i,j}\}$, $i = 1, 2, \ldots$, where $G_{i,j}$ is obtained from $F_i$ by deleting all factors in $F_i$ which are of the form $F_{k,l}$, where $k < j$ (with $G_{1,j} = G_{2,j} = \cdots = G_{j-1,j} = 1$). Setting $G_j = \lim_{i \to \infty} G_{i,j}$, one verifies, in a manner analogous to that concerning $F$ and $x_1$, that $G_j$ is a homeomorphism of $E^n$ onto itself having $F_{j-1}(x_j)$ as a nonpiercing (spiral) point. Setting $V^j = F_{j-1}^{-1}(S(F_{j-1}(x_j), r_{j,j}))$, we see that $V^j$ is a neighborhood of $x_j$, and $F|V^j = G_j|F_{j-1}|V$. Moreover, using (j.4) and the fact that $F_{c,r}|E^n - \{S(c, r) \cup S(c, r/2)\}$ is a $C^\infty$-diffeomorphism, there exists an open neighborhood $U^j$ of $x_j$ in $V^j$ such that $F_{j-1}|U^j$ is a $C^\infty$-diffeomorphism. It follows from Proposition 1 that $x_j$ is a nonpiercing point of $G_j|F_{j-1}$, and hence $x_j$ is a nonpiercing point of $F$. This completes the proof of the claim.

Remarks. Now $F|S(x_1, r_{11}) = 1$, and hence we see that $F$ has property $Q$ at every point of $S(x_1, r_{11})$. Also, $F$ is stable, since it is equivalent under $\sim$ to the stable homeomorphism $\tilde{F} \in H(E^n)$ defined by $\tilde{F}|CJS(x_1, r_{11}) = F|CJS(x_1, r_{11})$, and $\tilde{F}|JS(x_1, r_{11}) = 1$. Now if $T$ is any triangulation of $E^n$, we see that $F$ could have been constructed to have a dense set of nonpiercing points, and, moreover, reduce to the identity on the $(n-1)$-skeleton $T^{n-1}$ of $T$. We merely take our countable dense set $X$, and all the spheres entering into our construction, to be disjoint from $T^{n-1}$. In particular, if $P$ is an $(n-1)$-hyperplane in $E^n$, then $F$ could have been required to reduce to the identity on $P$ (and hence have property $Q$ at each point of $P$).

Theorem 1. If $M^n$ is any nonbounded differentiable $n$-manifold, then there exists a homeomorphism $F$ of $M^n$ onto itself having a dense set of nonpiercing points (cf. Definition 1').
Proof. We take a countable covering \((U_i, h_i)\) of \(M^n\) by coordinate systems. Let \(f_1 : h_1(U_1) \to h_1(U_1)\) be a homeomorphism of \(h_1(U_1)\) onto itself having a dense set of nonpiercing points, and such that \(d(x, f_1(x)) \to 0\) as \(x\) approaches the boundary of \(h_1(U_1)\). Then let \(F_1 : M^n \to M^n\) be the homeomorphism of \(M^n\) onto itself defined by \(F_1|U_1 = h_1^{-1}f_1h_1\), and \(F_1|(M^n - U_1) = 1\). Note that there is a dense subset of \(U_1\) consisting of nonpiercing points of \(F_1\). Let \(f_2 : h_2(U_2 - \bar{U}_1) \to h_2(U_2 - \bar{U}_1)\) be a homeomorphism of \(h_2(U_2 - \bar{U}_1)\) onto itself having a dense set of nonpiercing points, and such that \(d(x, f_2(x)) \to 0\) as \(x\) approaches the boundary of \(h_2(U_2 - \bar{U}_1)\). Then let \(F_2 : M^n \to M^n\) be the homeomorphism of \(M^n\) onto itself defined by \(F_2|U_1 = F_1|U_1, F_2|(U_2 - \bar{U}_1) = h_2^{-1}f_2h_2\), and \(F_2|(M^n - (U_1 \cup U_2)) = 1\). Inductively, we construct homeomorphisms \(F_i : M^n \to M^n\) of \(M^n\) onto itself such that \(F_i|(U_1 \cup \cdots \cup U_{i-1}) = F_{i-1}|(U_1 \cup \cdots \cup U_{i-1}), F_i|(M^n - (U_1 \cup \cdots \cup U_{i})) = 1\), and there exists a dense subset of \(U_1 \cup \cdots \cup U_i\) consisting of nonpiercing points of \(F_i\). Then set \(F = \lim_{i \to \infty} F_i\). It is readily seen that \(F\) is a homeomorphism of \(M^n\) onto itself having a dense set (in \(M^n\)) of nonpiercing points.

9. Applications. We now prove a theorem which relates the notion of piercing point to stability of homeomorphisms.
Theorem 2. For any integer $n$, if every homeomorphism $H \in H(E^k)$, $2 \leq k \leq n$, is such that $H \sim G$, where $G$ has property $Q$ at some point, then all orientation preserving homeomorphisms of $E^n$ onto itself are stable.

We first prove the following lemma.

Lemma 1. Suppose, for some $n$, that all orientation-preserving homeomorphisms of $E^{n-1}$ onto itself are stable. If $F \in H(E^n)$ is orientation-preserving, and if there exist $(n-1)$-hyperplanes $P, P'$ in $E^n$, and an open set $U$ in $P'$, such that $f(U) \subset P$, then $F$ is stable.

Proof of Lemma 1. Choose a point $x \in U$, and let $JS^{n-2}(x, r) = B$ be a closed $(n-1)$-ball in $P' \cap U$. Let $M$ be an elementary topological $(n-1)$-sphere in $E^n$ such that $M \cap P' = B$. Since we have assumed that all orientation-preserving homeomorphisms of $E^{n-1}$ onto itself are stable, and since $F(B) \subset P$, we can modify $F|B$ (cf. Theorem 5.4 of [1]) to obtain a homeomorphism $\tilde{F} : B \to E^n$ such that $\tilde{F}|JS^{n-2}(x, r) = F|JS^{n-2}(x, r)$, $\tilde{F}(B) \subset P$, and $\tilde{F}|JS^{n-2}(x, s)$ is a $C^p$-imbedding ($p > 0$) for some $s < r$. Note that $\tilde{F}(B) = F(B)$ (cf. (a) of §2). We then obtain a homeomorphism $\tilde{F} : M \to E^n$ by setting

\[
\tilde{F}(x) = F(x), \quad [x \in M \setminus JS^{n-2}(x, r)],
\]

\[
\tilde{F}(x) = \tilde{F}(x), \quad [x \in JS^{n-2}(x, r)].
\]

Since $\tilde{F}(M) = F(M)$, $\tilde{F}$ is elementary. Hence, using the Schoenflies extension theorem, $\tilde{F}$ admits extension to a homeomorphism of $E^n$ onto itself, which we still denote by $\tilde{F}$. Moreover, using the tubular neighborhood theorem, $\tilde{F}|JS^{n-2}(x, s)$ may be extended over an open neighborhood of $JS^{n-2}(x, s)$ in $E^n$ as a diffeomorphism. Hence, we can assume (cf. [9]) that $\tilde{F}|V$ is a diffeomorphism for some open neighborhood $V$ of $x$ in $E^n$. Therefore $\tilde{F}$ is stable. Since

\[
\tilde{F}|(M - B) = F|(M - B),
\]

we can assume (cf. [9]), moreover, that $\tilde{F} \sim F$. Hence $F$ is stable.

Remark. One verifies that the hypotheses of Lemma 1 can be weakened by allowing $P$ and $P'$ to be diffeomorphs of $E^{n-1}$ in $E^n$.

Proof of Theorem 2. Using induction, the above remark, and the fact that all orientation-preserving homeomorphisms of $E^n$ onto itself are stable for $n = 1, 2, 3$, it suffices to prove the following proposition $(\gamma)$.

$(\gamma)$ If the homeomorphism $G \in H(E^n)$ has property $Q$ at some point, then $G \sim F$, where $F(U) \subset P_1$ for an open set $U$ in $P_2$, and $P_1, P_2$ are diffeomorphs of $E^{n-1}$ in $E^n$.

To verify $(\gamma)$, suppose $G$ has property $Q$ at a point $x_0 \in E^n$. Hence there exist $\sigma, H, P, W$, and $\alpha_2$ in Definition 3 relative to $G$ and $x_0$. To simplify our discussion, we note that for the purposes of verifying $(\gamma)$, we can assume without loss of generality that $L$ is the identity diffeomorphism of $E$. Indeed, $GL$ has property $Q$
at $L^{-1}(x_0)$ using $\sigma$, identity, $H, P, W$, and it is clear that $L$ can be taken as orientation-preserving, which implies that $GL \sim G$.

Now if $V$ is a sufficiently small neighborhood of $HG(x_0)$ in $W$, the straight line segments $\sigma_x([-1, 1])$, as $x$ varies throughout $V$, are mutually disjoint. To see this, note first that Definition 1 and condition (iii) of Definition 3 (with $L = 1$) imply that $\sigma_x([-1, 1]) \cap G^{-1}H^{-1}(W) = \emptyset$ for all $x \in W$. Since the segments $\sigma_x([-1, 1])$ are all translates of one another, we see that if $V \subset W$ is small enough, the $\sigma_x([-1, 1]) \cap \sigma_y([-1, 1]) \neq \emptyset$ for $x, y \in V$ implies that $x \in \sigma_x([-1, 1])$, and hence $x = y$. Actually, it can be proved that the segments $\sigma_x([-1, 1])$ are mutually disjoint for all $x \in W$, but we won’t need this fact.

Let $B = JS^{n-2}(HG(x_0), r)$ be an $(n-1)$-ball in $P$ such that $B \subset V$. Let $P^*$ be the $(n-1)$-hyperplane in $E^n$ going through $x_0 = \sigma(0)$ and such that the segment $\sigma([-1, 1])$ is normal to $P^*$ at $\sigma(0)$. We also suppose $r$ chosen so small that the $(n-1)$-hyperplanes $P', P''$ which are parallel to $P^*$ and go through $\sigma(\frac{1}{2})$, $\sigma(-\frac{1}{2})$, respectively, have the following property: for each $x \in G^{-1}H^{-1}(B)$, the segments $\sigma_x([-1, 1])$ intersect $P'$, $P''$ in (continuously varying) points $\sigma_x(t'_x), \sigma_x(t''_x)$, respectively, where $-1 < t'_x < 0 < t''_x < 1$. Note that $t''_x = -\frac{1}{2}, t'_x = \frac{1}{2}$. Hence the segments $\sigma_x([t''_x, t'_x])$, as $x$ varies throughout $G^{-1}H^{-1}(B)$, “fiber” the neighborhood

$$N = \{\bigcup \sigma_x([t''_x, t'_x]) | x \in G^{-1}H^{-1}(B)\}.$$  

Let $B' = JS^{n-1}(HG(x_0), r/2)$, and let $B* = \{\bigcup \sigma_x([t''_x, t'_x]) | x \in G^{-1}H^{-1}(B')\}$. It is clear that

$$BdB* = \{\bigcup \sigma_x([t''_x, t'_x]) | x \in G^{-1}H^{-1}(S^{n-2}(HG(x_0), r/2)) \cup \{B* \cap (P' \cup P'')\}$$

is an elementary topological $(n-1)$-sphere in $E^n$. Setting $M = BdB*$, we consider homeomorphism $\Phi: M \to E^n$ which maps the segment $\sigma_x([t''_x, t'_x])$ homeomorphically onto the segment $\sigma_x([t''_x, 0])$, and reduces to the identity $B* \cap P''$. Then $\Phi$ is an elementary homeomorphism such that $\Phi(M) = M'$, where

$$M' = \{\bigcup \sigma_x([t''_x, 0]) | x \in G^{-1}H^{-1}(S^{n-2}(HG(x_0), r/2)) \cup \{B* \cap P''\}.$$

Set $F = G\Phi$. Then $F$ is an elementary homeomorphism of $M$ into $E^n$, and hence may be extended to a homeomorphism of $E^n$ onto itself, which we still denote by $F$. Since $F|\{B* \cap P''\} = G|\{B* \cap P''\}$, we can assume (cf. [9]) that $F \sim G$. Moreover, setting $U = \{\bigcup \sigma_x(t'_x) | x \in G^{-1}H^{-1}(JS^{n-2}(HG(x_0), r/2))\}$, we see that $U$ is an open set in $P'$, and $F(U) = G\Phi(U) = G(G^{-1}H^{-1}(JS^{n-2}(HG(x_0), r/2))) = H^{-1}(JS^{n-2}(HG(x_0), r/2)) \subset H^{-1}(P)$. Then setting $P_1 = H^{-1}(P), P_2 = P'$, we see that Proposition (γ) is verified, which completes the proof of Theorem 2.

We now show that any homeomorphism $F \in H(E^n)$ is equivalent under $\sim$ to a homeomorphism having piercing points.

Since all orientation-preserving homeomorphisms of $E^n$ onto itself are stable
for $n = 1, 2, 3$, it follows that if $F \in H(E^n)$, $n = 1, 2, 3$, then $F \sim H$ where $H$ has an $n$-cell of piercing points. The following theorem extends this latter result to a similar (but weaker) result in the higher dimensions.

**Theorem 3.** Let $F \in H(E^n)$, where $n \geq 4$. Then $F \sim H$ where $H$ has a $k$-cell of piercing points, $k \leq 2n/3 - 1$.

**Proof.** Let $K$ be any $k$-cell in $E^n$. Then $F(K)$ is a flat $k$-cell in $E^n$. Moreover, $F(K)$ is stably flat by a theorem of P. Roy (cf. [10]), and $F^{-1}|F(K) : F(K) \rightarrow K$ admits an extension to a stable homeomorphism $G$ of $E^n$ onto itself. Since $G$ is stable, we can assume that $G|U = 1$, where $U$ is some nonempty open set in $E^n$. Set $H = GF$. Then $H|K = 1$, and hence $K$ consists entirely of piercing points of $H$. Since $H|F^{-1}(U) = F|F^{-1}(U)$, we see that $F \sim H$ and the theorem is proved.

**Remark.** Theorem 3 can be strengthened by requiring that $H|(E^n - V) = F|(E^n - V)$, where $V$ is any nonempty open set in $E^n$. Hence Theorem 3 can be extended to an analogous result about differentiable manifolds.

We conclude by noting a relationship between our notion of piercing point and a notion of "piercing" which has been discussed in the literature. A topological $(n - 1)$-sphere $M$ in $E^n$ is said to be pierced by a straight line segment $yz$ at a point $x_0 \in M$ if $yz \cap M = x_0$, and if $y$ and $z$ lie in opposite components of $E^n - M$. An example was given by Fort (cf. [11]) of a wild sphere which can be pierced at each point by a straight line segment. On the other hand, our example of §8 shows that there exist elementary spheres $M$ in $E^n$ which can not be pierced, even by diffeomorphs of straight line segments, at a dense set of points of $M$. To verify this latter statement, observe first that if $G \in H(E^n)$, and if $M = G(S^{n-1}(c, r))$ can be pierced at a point $x_0$ by a diffeomorph of a straight line segment, then $x_0$ is a piercing point of $G^{-1}$. We now note that the homeomorphism $F \in H(E^n)$ constructed in §8 is such that $F(x)$ is a nonpiercing (spiral) point of $F^{-1}$, for all $x \in X$. Hence if we choose $X$ so that $X \cap S(c, r)$ is dense in $S(c, r)$, then the elementary sphere $M = F(S(c, r))$ can not be pierced at the points of the dense subset $F(X \cap S(c, r))$ of $M$.

**References**

10. P. Roy, *Locally flat k-cells and spheres in $E^n$ are stably flat if $k \leq 2n/3 - 1*}, Abstract 621–66, Notices Amer. Math. Soc. 12 (1965), 323.

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