$L^p$-CONJECTURE FOR
LOCALLY COMPACT GROUPS. I

BY
M. RAJAGOPALAN

Let $G$ be a locally compact group with a left Haar measure $\mu$. Let $p$ and $r$ be two real numbers $> 1$. By $L^{r,p}$ conjecture we mean the following assertion: whenever $f \in L^r(G)$ and $g \in L^p(G)$ we have that the convolution product $f * g$ of $f$ and $g$ is defined and belongs to $L^p(G)$ again if and only if $G$ is compact. By $L^p$ conjecture we mean the assertion above with $r = p$. Both the conjectures were widely believed to be true though there was no written statement about these conjectures until recently. But in 1960, Kunze and Stein [3] showed that the $L^{r,p}$ conjecture is false for the unimodular group of $2 \times 2$ real matrices. This naturally raises the question whether the $L^p$-conjecture is true in general. The first published result on the $L^p$-conjecture is by Zelazko [9] and Urbanik [7] in 1961. They proved the conjecture to be true for the abelian case. Then in [5] the author established the truth of the $L^p$-conjecture for discrete groups when $p \geq 2$. The author announced in that paper that the conjecture is true for all groups when $p > 2$ and presented this result to Amer. Math. Soc. in August of 1963 at the Boulder meeting. At the same time Zelazko [10] established the conjecture for all $p > 2$ for all unimodular groups. He claims to have established the conjecture for $p > 2$ for all groups in that paper but his crucial Lemma 1 of that paper contains a gap in the proof. In a private communication, Zelazko agreed to this gap. The problem is still open in general when $p > 1$.

In this paper we prove the following:

The $L^p$-conjecture is true for all locally compact groups when $p > 2$.

The $L^p$-conjecture is true for totally disconnected groups when $p = 2$.

The methods used in this paper yield the truth of the conjecture for all nilpotent groups, and all unimodular $C$-groups of Iwasawa when $p > 1$. But this result will appear elsewhere.

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Notations and conventions. All purely topological notions are taken from [2]. All topological spaces occurring in this paper are taken to be Hausdorff. All notions in topological groups and integration on locally compact groups are

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taken in general from [8]. By a normal algebra we mean a Banach space which is also a ring where multiplication is bicontinuous. A Banach algebra is a normed algebra where we have further the inequality $\|xy\| \leq \|x\| \|y\|$ for all elements $x$ and $y$ of the algebra. The symbols $\mu$, $\nu$, $\theta$ are used for measures. When only one left Haar measure $\mu$ is used on a locally compact group $G$ we write sometimes $\int_G f(x) \, dx$ or $\int f(x) \, dx$ instead of $\int_G f(x) \, d\mu(x)$. If $H \subseteq G$ is a subset of a group $G$, then $\chi_H(x)$ will denote the characteristic function of $H$. $L(G)$ will denote the class of complex valued continuous functions on $G$ with compact support.

If $G$ is a locally compact group with a left Haar measure $\mu$ and $1 \leq p < \infty$ then $L^p(G)$ will denote the equivalence classes of Borel measurable functions $f$ on $G$ with complex values such that $\int |f(x)|^p \, dx < \infty$. If $f \in L^p(G)$ then $\|f\|_p$ will denote $(\int_G |f|^p \, dx)^{1/p}$ when $1 \leq p \leq \infty$.

1. $L^p$-conjecture for the case $p > 2$, and some general results.

Definition 1.1. Let $G$ be a locally compact group with a left Haar measure $\mu$. Let $f$ and $g$ be two Borel measurable complex valued functions on $G$. Then the convolution $f \ast g$ of $f$ and $g$ is said to exist if the integral $\int_G |f(y)g(y^{-1}x)| \, dy$ exists for almost all $x \in G$. In this case $f \ast g(x)$ is defined to be $\int_G f(y)g(y^{-1}x) \, dy$.

If $1 \leq p < \infty$, we say that $\mathcal{P}(G)$ is closed under convolution if whenever $f$ and $g$ belong to $L^p(G)$ we have that $f \ast g$ exists and again belongs to $L^p(G)$.

$L^p$-conjecture 1.2. This is the following statement: Let $G$ be a locally compact group with a left Haar measure $\mu$. Then $L^p(G)$ is closed under convolution for some $p$ such that $1 < p < \infty$ if and only if $G$ is compact.

Remark. The "if" part of the $L^p$-conjecture is trivial to establish. So we consider the "only if" part in this paper.

Theorem 1.3. Let $G$ be a locally compact group with a left Haar measure $\mu$. Let $L^p(G)$ be closed under convolution for some $p > 1$ and $\infty$. Then $L^p(G)$ is a normed algebra with convolution as multiplication. Moreover, in this case we can choose a suitable left Haar measure $\mu_1$ such that $L^p(\mu_1)$ is a Banach algebra.

Proof. Let $1/p + 1/q = 1$ and $\Delta(x)$ the modular function of $G$. Let $f$, $g \in L^p(G)$ and $h \in L^q(G)$. Let $(f, h) = \int_G f(x)h(x) \, dx$ and let $T_f$ be the operator $g \mapsto f \ast g$ in $L^p(G)$. Let $f(x) = f(x^{-1})\Delta(x^{-1})$. Then by a routine calculation it follows that $(T_f(g), h) = (f \ast g, h) = (g, f \ast h)$ for all $f, g \in L^p(G)$ and $h \in L^q(G)$. So by an easy application of the closed graph theorem we get that $T_f$ is continuous in $L^p(G)$.

Similarly we get that the right multiplication is continuous in $L^p(G)$. So by an application of the principle of uniform boundedness we get $L^p(G)$ is a normed algebra. So there is a constant $K$ such that $\|f \ast g\|_p \leq K \|f\|_p \|g\|_p$ for all $f, g \in L^p(G)$. Now choose a left Haar measure $\mu_1$ on $G$ by the relation $d\mu_1(x) = K^p \, d\mu(x)$. Then $L^p(\mu_1)$ will be a Banach algebra under convolution.
Lemma 1.4. Let $G$ be a locally compact group with a left Haar measure $\mu$. Let $H \subseteq G$ be an open subgroup of $G$. Let $L^p(G)$ be closed under convolution for some $p > 1$. Then $L^p(H)$ is also closed under convolution. If $G$ is the direct product $G_1 \times G_2$ of two closed subgroups $G_1$ and $G_2$ with left Haar measures $\mu_1$ and $\mu_2$ respectively and if $L^p(G)$ is closed under convolution then $L^p(G_1)$ and $L^p(G_2)$ are also closed under convolution.

Proof. Obvious.

Lemma 1.5. Let $G$ be a locally compact group with a left Haar measure $\mu$. Let $\varphi$ be a number such that $1 < \varphi < \infty$. Let $H \subseteq G$ be a compact normal subgroup of $G$. Let $L^\varphi(G)$ be closed under convolution. Then $L^\varphi(G/H)$ is also closed under convolution.

Proof. Let $\nu$ be the normalized Haar measure of $H$ and $\varphi: G \to G/H$ be the canonical map from $G$ onto $G/H$. Let $\theta$ be a left Haar measure on $G/H$ such that the relation $\int_G f(x) \, d\nu(x) = [\int_{G/H} (\int_H f(tx) \, d\theta(t)) \, d\nu(x)]$ holds for all $f \in L(G)$ where $\varphi(x) = \bar{x}$. Then the following relations are easily deduced: If $T(f) = \int_H f(tx) \, d\nu(t)$ where $f \in L(G)$ then
1. $\varphi \in L(G/H)$ whenever $f \in L(G)$.
2. $T$ is linear from $L(G)$ onto $L(G/H)$.
3. $T(f * g) = T(f) \ast T(g)$ for all $f, g \in L(G)$.
4. $\|Tf\|_\varphi = \|f\|_\varphi$ for all $f \in L(G)$.

Now $L^\varphi(G)$ is closed under convolution. So there is a constant $K$ such that $\|f * g\|_\varphi < K \|f\|_\varphi \|g\|_\varphi$ from Theorem 1.3. So we have that $\|\hat{f} \ast \hat{g}\|_\varphi \leq K \|\hat{f}\|_\varphi \|\hat{g}\|_\varphi$ for all $\hat{f}$ and $\hat{g} \in L(G/H)$ from 1, 2, 3, and 4 above. Since $\varphi < \infty$, we have that $L(G/H)$ is dense in $L^\varphi(G/H)$. Then we get by repeated use of Fatou's lemma, Fubini's theorem and monotone convergence theorem that if $\hat{f}$ and $\hat{g}$ belong to $L^\varphi(G/H)$ then $\hat{f} \ast \hat{g}$ is defined and again belongs to $L^\varphi(G/H)$.

Lemma 1.6. Suppose that the $L^p$-conjecture is true for a number $p$ ($1 < p < \infty$) for all totally disconnected locally compact groups and all connected lie groups. Then the conjecture is true for that $\varphi$ for all locally compact groups.

Proof. By a theorem of Yamabe [4] every locally compact group $G$ contains an open subgroup $H$ and a compact normal subgroup $N \subseteq H$ such that $H/N$ is a connected lie group ($N$ is normal with respect to $H$). So if $L^p(G)$ is closed under convolution then $L^p(H)$ is closed under convolution by Lemma 1.4. So $L^p(H/N)$ is closed under convolution by Lemma 1.5. So $H/N$ is compact by hypothesis of the lemma. So the connected component $G_0$ of the identity $e$ of $G$ is compact. So $L^p(G/G_0)$ is closed under convolution by Lemma 1.5. So $G/G_0$ is compact by hypothesis of the lemma. So $G$ is compact.
Lemma 1.7. Let $G$ be a locally compact group with a left Haar measure $\mu$. Let $V$ be a compact symmetric neighborhood of the identity $e$ of $G$ such that the group generated by $V$ is not compact. Then the following are true:

(i) If the set $\{\mu(V^{n+1})/\mu(V^n) | n = 1, 2, \ldots\}$ is bounded then $L^p(G)$ is not closed under convolution for any $p > 2$.

(ii) If the set $\{\mu(V^{2n})/\mu(V^n) | n = 1, 2, 3, \ldots\}$ is bounded then $L^p(G)$ is not closed under convolution for any $p > 1$.

Proof. Let there be a constant $k > 0$ such that $\mu(V^{n+1})/\mu(V^n) \leq k$ for all $n = 1, 2, 3, \ldots$. Let $\chi_{V^n}(x)$ be the characteristic function of $V^n$ for $n = 1, 2, 3, \ldots$. Then

$$\chi_{V^n} \ast \chi_{V^{n+1}}(x) = \int_G \chi_{V^n}(y)\chi_{V^{n+1}}(y^{-1}x) \, d\mu(y)$$

$$\geq \mu(V^n)\chi_{V}(x) \quad \text{for all } x \in G \text{ and } n = 1, 2, \ldots$$

So $\|\chi_{V^n} \ast \chi_{V^{n+1}}\|_p \geq \mu(V^n)(\mu(V))^{1/p}$. So

$$\frac{\|\chi_{V^n} \ast \chi_{V^{n+1}}\|_p}{\|\chi_{V^n}\|_p \|\chi_{V^{n+1}}\|_p} \geq \frac{\mu(V^n)(\mu(V))^{1/p}}{(\mu(V^n))^{1/p}(\mu(V^{n+1}))^{1/p}}$$

$$= \left(\frac{\mu(V^n)}{\mu(V^{n+1})}\right)^{1/p} (\mu(V))^{1/p}(\mu(V^n))^{1-(2/p)}$$

$$\geq \left(\frac{\mu(V)}{k}\right)^{1/p} (\mu(V^n))^{1-(2/p)}.$$

Thus $\lim_{n \to \infty} \left(\|\chi_{V^n} \ast \chi_{V^{n+1}}\|_p/\|\chi_{V^n}\|_p \|\chi_{V^{n+1}}\|_p\right) = \infty$ if $p > 2$. Thus (i) follows from Theorem 1.3. The statement (ii) can be proved likewise.

Lemma 1.8. Let $G$ be a totally disconnected locally compact group with a left Haar measure $\mu$. Let $L^p(G)$ be closed under convolution for a real number $p$ ($1 < p < \infty$). Then there is a maximal compact open subgroup $H$ of $G$. (That is $H$ is a compact, open subgroup of $G$ and any open compact subgroup of $G$ containing $H$ is $H$ itself.)

Proof. Since $G$ is totally disconnected, there are compact open subgroups in $G$ (see p. 54 of [4]). Suppose there is an ascending sequence $H_1 \subset H_2 \subset \ldots \subset H_n \subset \ldots$ of compact open subgroups $H_1, H_2, \ldots, H_n, \ldots$ of $G$. Let $\chi_{H_n}(x) = 1$ if $x \in H_n$ and 0 if $x \in G - H_n$. Put $\varphi_n(x) = (\chi_{H_n}(x)/\mu(H_n))$ for $n = 1, 2, 3, \ldots$. Then $\varphi_n \ast \varphi_n = \varphi_n$ for all $n = 1, 2, 3, \ldots$. So

$$\frac{\|\varphi_n \ast \varphi_n\|_p}{\|\varphi_n\|_p} = \frac{\|\varphi_n\|_p^2}{\|\varphi_n\|_p^2} = \frac{1}{\|\varphi_n\|_p^2} = \frac{\mu(H_n)}{(\mu(H_n))^{1/p}} = (\mu(H_n))^{1-(1/p)}.$$
By Theorem 1.3 the set \{\mu(H_n) \mid n = 1, 2, 3, \ldots\} is bounded. Hence there is an \( n_0 \) such that \( H_{n_0} = H_{n_0+1} = \cdots \). So every ascending sequence of compact, open subgroups of \( G \) is finite. Hence the lemma.

**Lemma 1.9.** Let \( G \) be a totally disconnected locally compact group with a left Haar measure \( \mu \). Let \( p \) be a real number \((2 < p < \infty)\). Let \( L^p(G) \) be closed under convolution. Then \( G \) contains an open compact normal subgroup.

**Proof.** By Lemma 1.8 there is a maximal, compact, open subgroup \( H \). Now take any element \( a' \in G - H \) and consider the group generated by \( H \cup a^{-1}Ha \). This group should be compact. If not put \( V = H \cup a^{-1}Ha \). Then \( V \) is a compact symmetric open neighborhood of the identity \( e \in G \). Since \( V^2 \) is compact there is a finite number of elements \( a_1, a_2, \ldots, a_k \) of \( G \) such that \( V^2 \subseteq (a_1V) \cup (a_2V) \cup \cdots \cup a_kV \). So \( V^{n+1} \subseteq \bigcup_{i=1}^k a_iV^n \) for \( n = 1, 2, 3, \ldots \). So \( \mu(V^{n+1}) \leq \mu(V^n) \) for all \( n = 1, 2, 3, \ldots \). Then \( \mu(V^{n+1})/\mu(V^n) \leq k \) for all \( n = 1, 2, 3, \ldots \). Then \( L^2(G) \) cannot be closed under convolution by Lemma 1.7 and Lemma 1.4 which contradicts our hypothesis on \( G \). Since \( H \) is a maximal open compact subgroup of \( G \) we get that \( H \cup a^{-1}Ha \subseteq H \). So \( a^{-1}Ha \subseteq H \) for all \( a' \in G \). So \( H \) is a compact, open, normal subgroup of \( G \).

**Theorem 1.10.** Let \( G \) be a locally compact group with a left Haar measure \( \mu \). Let \( L^p(G) \) be closed under convolution for a real number \( p \) \((2 < p < \infty)\). Then \( G \) must be compact.

**Proof.** Let us assume first that \( G \) is connected. Let \( V \) be a compact symmetric neighborhood of the identity \( e \). Then adopting the proof of Lemma 1.9 we get that the set \{\( (\mu(V^{n+1})/\mu(V^n)) \mid n = 1, 2, 3, \ldots \)\} is bounded. Then by the connectedness of \( G \) and by Lemma 1.7 we get that \( G \) is compact. Now let us assume that \( G \) is totally disconnected. Then by Lemma 1.9 there exists a compact, open, normal subgroup \( H \) of \( G \). Then \( G/H \) is a discrete group and \( L^p(G/H) \) is closed under convolution by Lemma 1.5. So \( G/H \) is finite by Theorem 3 of [5]. Then \( G \) must be compact. So if \( G \) is either connected or totally disconnected the theorem is true. Now the result follows from Lemma 1.6.

2. The case \( p = 2 \) of the \( L^p \)-conjecture.

**Definition 2.1.** An involution \( * \) in an algebra \( A \) over complex numbers is a one-to-one map from \( A \) onto \( A \) such that the following hold:

(i) \( (x^*)^* = x \) for all \( x \in A \).
(ii) \( (\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^* \) for all complex numbers \( \lambda \) and \( \mu \) and \( x, y \in A \).
(iii) \( (xy)^* = y^* x^* \) for all \( x, y \in A \).

An \( A^* \)-algebra is a Banach algebra \( B \) with an involution \( * \) and an auxiliary norm \( \| \cdot \| \) such that \( \| xy \| \leq \| x \| \| y \| \) and \( \| xx^* \| = \| x \|^2 \) for all \( x, y \in B \).
Let $B$ be a Banach algebra over the complex numbers. An element $x$ is said to be in the radical of $B$ if there is an ideal $I \subseteq B$ such that $x \in I$ and $\lim_{n \to \infty} (\|y^n\|)^{1/n} \to 0$ as $n \to \infty$ for all $y \in I$. The algebra $B$ is said to be semisimple if 0 is the only element in the radical of $B$.

**Lemma 2.2.** Let $G$ be a unimodular locally compact group with a left Haar measure $\mu$. Let $L^p(G)$ be closed under convolution for some $p \ (1 < p < \infty)$. Then $L^p(G)$ is a semisimple Banach algebra assuming that $\mu$ was properly chosen to make $L^p(G)$ a Banach algebra.

**Proof.** Let $1/p + 1/q = 1$. Let $f^*(x) = \overline{f(x^{-1})}$ for all $f \in L^p(G)$. Then, from the fact that $G$ is unimodular, one can check that $f \to f^*$ is an involution in $L^p(G)$. Moreover, by using standard theorems on integration one can show that $(f*g,h) = (f,h*g^*)$ for all $f, g \in L^2(G)$ and $h \in L^q(G)$ where $(f,h) = \int_G f(x)h(x)\,dx$.

From this it follows easily that if $f \in L^p(G)$ and $g \in L^q(G)$ then $f*g \in L^1(G)$ and $\|f*g\|_q \leq \|f\|_p \|g\|_q$. From this and the fact that $L^1(G)$ is a Banach algebra and the Riesz convexity theorem it follows that $L^p(G)$ is semisimple and hence it is a semisimple $\mathcal{A}^*$-algebra of Ambrose (see [1]). Now let $A$ be a compact, open subgroup of $G$ and let $\varphi(x) = x_{\kappa}(x)/\mu(K)$ where $x_{\kappa}(x)$ is the characteristic function of $K$.

Then $\varphi = \varphi^*$ and $\varphi * \varphi = \varphi$ and $\varphi \in L^2(G)$. So $\varphi * L^2(G) * \varphi$ is a semisimple $\mathcal{H}^*$-algebra with an identity element and hence is finite dimensional (see [1]). But $\varphi * L^2(G) * \varphi$ consists exactly of those functions in $L^2(G)$ which are constant on double cosets modulo $K$. So the number of such cosets has to be finite and hence $G$ must be compact.

In the general case let $\Delta(x)$ be the modular function of $G$ and let

$$H = \{x \mid \Delta(x) = 1; x \in G\}.$$
Then $H$ contains all compact, open subgroups of $G$. So $H$ is an open subgroup of $G$. So $L^2(H)$ is closed under convolution by Lemma 1.4. Clearly $H$ is unimodular. So $H$ is compact by what was shown above. So $L^2(G/H)$ is closed under convolution, by Lemma 1.5. But $G/H$ is a discrete subgroup of the reals. So it is finite by Theorem 2 of [5]. So $G$ is compact again.

References


BANARAS HINDU UNIVERSITY
VARANASI, INDIA

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS