L^p-CONJECTURE FOR LOCALLY COMPACT GROUPS. I

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Let G be a locally compact group with a left Haar measure 'μ'. Let 'p' and 'r' be two real numbers >1. By $L^{r,p}$ conjecture we mean the following assertion: whenever $f \in L^r(G)$ and $g \in L^p(G)$ we have that the convolution product $f * g$ of f and g is defined and belongs to $L^p(G)$ again if and only if G is compact. By L^p conjecture we mean the assertion above with $r=p$. Both the conjectures were widely believed to be true though there was no written statement about these conjectures until recently. But in 1960, Kunze and Stein [3] showed that the $L^{r,p}$ conjecture is false for the unimodular group of 2×2 real matrices. This naturally raises the question whether the L^p-conjecture is true in general. The first published result on the L^p-conjecture is by Zelazko [9] and Urbanik [7] in 1961. They proved the conjecture to be true for the abelian case. Then in [5] the author established the truth of the L^p-conjecture for discrete groups when $p \geqq 2$. The author announced in that paper that the conjecture is true for all groups when $p>2$ and presented this result to Amer. Math. Soc. in August of 1963 at the Boulder meeting. At the same time Zelazko [10] established the conjecture for all $p>2$ for all unimodular groups. He claims to have established the conjecture for $p>2$ for all groups in that paper but his crucial Lemma 1 of that paper contains a gap in the proof. In a private communication, Zelazko agreed to this gap. The problem is still open in general when $p>1$.

In this paper we prove the following:

The L^p-conjecture is true for all locally compact groups when $p>2$.

The L^p-conjecture is true for totally disconnected groups when $p=2$.

The methods used in this paper yield the truth of the conjecture for all nilpotent groups, and all unimodular C-groups of Iwasawa when $p>1$. But this result will appear elsewhere.

This is a portion of the author's doctoral dissertation submitted to Yale University in 1963 under the guidance of Professor C. E. Rickart. The author had a grant AFOSR 62–20 from the Air Force when this work was done. The author thanks Professor C. Ionescu-Tulcea for having brought the paper [9] of Zelazko to his attention.

Notations and conventions. All purely topological notions are taken from [2]. All topological spaces occurring in this paper are taken to be Hausdorff. All notions in topological groups and integration on locally compact groups are

Presented to the Society, August 30, 1963; received by the editors February 26, 1964.

taken in general from [8]. By a normal algebra we mean a Banach space which is also a ring where multiplication is bicontinuous. A Banach algebra is a normed algebra where we have further the inequality $\|xy\| \leqq \|x\| \, \|y\|$ for all elements 'x' and 'y' of the algebra. The symbols μ, ν, θ are used for measures. When only one left Haar measure μ is used on a locally compact group G we write sometimes $\int_G f(x)\, dx$ or $\int f(x)\, dx$ instead of $\int_G f(x)\, d\mu(x)$. If $H \subset G$ is a subset of a group G, then $\chi_H(x)$ will denote the characteristic function of H. $L(G)$ will denote the class of complex valued continuous functions on G with compact support.

If G is a locally compact group with a left Haar measure 'μ' and $1 \leqq p < \infty$ then $L^p(G)$ will denote the equivalence classes of Borel measurable functions f on G with complex values such that $\int |f(x)|^p \, dx < \infty$. If $f \in L^p(G)$ then $\|f\|_p$ will denote $(\int_G |f|^p \, dx)^{1/p}$ when $1 \leqq p \leqq \infty$.

1. L^p-conjecture for the case $p > 2$, and some general results.

DEFINITION 1.1. Let G be a locally compact group with a left Haar measure 'μ'. Let f and g be two Borel measurable complex valued functions on G. Then the convolution $f * g$ of 'f' and 'g' is said to exist if the integral $\int_G |f(y)g(y^{-1}x)| \, dy$ exists for almost all $x \in G$. In this case $f * g(x)$ is defined to be $\int_G f(y)g(y^{-1}x)\, dy$. If $1 \leqq p < \infty$, we say that $L^p(G)$ is closed under convolution if whenever f and g belong to $L^p(G)$ we have that $f * g$ exists and again belongs to $L^p(G)$.

L^p-conjecture 1.2. This is the following statement: Let G be a locally compact group with a left Haar measure μ. Then $L^p(G)$ is closed under convolution for some p such that $1 < p < \infty$ if and only if G is compact.

REMARK. The "if" part of the L^p-conjecture is trivial to establish. So we consider the "only if" part in this paper.

THEOREM 1.3. *Let G be a locally compact group with a left Haar measure 'μ'. Let $L^p(G)$ be closed under convolution for some $p > 1$ and $< \infty$. Then $L^p(G)$ is a normed algebra with convolution as multiplication. Moreover, in this case we can choose a suitable left Haar measure μ_1 such that $L^p(\mu_1)$ is a Banach algebra.*

Proof. Let $1/p + 1/q = 1$ and $\Delta(x)$ the modular function of G. Let $f, g \in L^p(G)$ and $h \in L^q(G)$. Let $(f, h) = \int_G f(x)h(x)\, dx$ and let T_f be the operator $g \to f * g$ in $L^p(G)$. Let $\tilde{f}(x) = f(x^{-1})\, \Delta(x^{-1})$. Then by a routine calculation it follows that $(T_f(g), h) = (f * g, h) = (g, \tilde{f} * h)$ for all $f, g \in L^p(G)$ and $h \in L^q(G)$. So by an easy application of the closed graph theorem we get that T_f is continuous in $L^p(G)$. Similarly we get that the right multiplication is continuous in $L^p(G)$. So by an application of the principle of uniform boundedness we get $L^p(G)$ is a normed algebra. So there is a constant K such that $\|f * g\|_p \leqq K \|f\|_p \|g\|_p$ for all $f, g \in L^p(G)$. Now choose a left Haar measure μ_1 on G by the relation $d\mu_1(x) = K^p \, d\mu(x)$. Then $L^p(\mu_1)$ will be a Banach algebra under convolution.

LEMMA 1.4. *Let G be a locally compact group with a left Haar measure μ. Let $H \subset G$ be an open subgroup of G. Let $L^p(G)$ be closed under convolution for some $p > 1$. Then $L^p(H)$ is also closed under convolution. If G is the direct product $G_1 \times G_2$ of two closed subgroups G_1 and G_2 with left Haar measures μ_1 and μ_2 respectively and if $L^p(G)$ is closed under convolution then $L^p(G_1)$ and $L^p(G_2)$ are also closed under convolution.*

Proof. Obvious.

LEMMA 1.5. *Let G be a locally compact group with a left Haar measure μ. Let 'p' be a number such that $1 < p < \infty$. Let $H \subset G$ be a compact normal subgroup of G. Let $L^p(G)$ be closed under convolution. Then $L^p(G/H)$ is also closed under convolution.*

Proof. Let ν be the normalized Haar measure of H and $\varphi: G \to G/H$ be the canonical map from G onto G/H. Let θ be a left Haar measure on G/H such that the relation $\int_G f(x)\, d\mu(x) = [\int_{G/H} (\int_H f(tx)\, d\nu(t))\, d\theta(\tilde{x})]$ holds for all $f \in L(G)$ where $\varphi(x) = \tilde{x}$. Then the following relations are easily deduced: If $T(f) = \int_H f(tx)\, d\nu(t)$ where $f \in L(G)$ then

1. $Tf \in L(G/H)$ whenever $f \in L(G)$.
2. T is linear from $L(G)$ onto $L(G/H)$.
3. $T(f * g) = T(f) * T(g)$ for all $f, g \in L(G)$.
4. $\|Tf\|_p = \|f\|_p$ for all $f \in L(G)$.

Now $L^p(G)$ is closed under convolution. So there is a constant K such that $\|f * g\|_p < K \|f\|_p \|g\|_p$ from Theorem 1.3. So we have that $\|\tilde{f} * \tilde{g}\|_p \leqq K \|\tilde{f}\|_p \|\tilde{g}\|_p$ for all $\tilde{f}$ and $\tilde{g} \in L(G/H)$ from 1, 2, 3, and 4 above. Since $p < \infty$, we have that $L(G/H)$ is dense in $L^p(G/H)$. Then we get by repeated use of Fatou's lemma, Fubini's theorem and monotone convergence theorem that if $\tilde{f}$ and $\tilde{g}$ belong to $L^p(G/H)$ then $\tilde{f} * \tilde{g}$ is defined and again belongs to $L^p(G/H)$.

LEMMA 1.6. *Suppose that the L^p-conjecture is true for a number p $(1 < p < \infty)$ for all totally disconnected locally compact groups and all connected lie groups. Then the conjecture is true for that 'p' for all locally compact groups.*

Proof. By a theorem of Yamabe [4] every locally compact group G contains an open subgroup H and a compact normal subgroup $N \subset H$ such that H/N is a connected lie group (N is normal with respect to H). So if $L^p(G)$ is closed under convolution then $L^p(H)$ is closed under convolution by Lemma 1.4. So $L^p(H/N)$ is closed under convolution by Lemma 1.5. So H/N is compact by hypothesis of the lemma. So the connected component G_0 of the identity 'e' of G is compact. So $L^p(G/G_0)$ is closed under convolution by Lemma 1.5. So G/G_0 is compact by hypothesis of the lemma. So G is compact.

LEMMA 1.7. *Let G be a locally compact group with a left Haar measure μ. Let V be a compact symmetric neighborhood of the identity e of G such that the group generated by V is not compact. Then the following are true:*

(i) *If the set $\{u(V^{n+1})/\mu(V^n) \mid n=1, 2, \ldots\}$ is bounded then $L^p(G)$ is not closed under convolution for any $p>2$.*

(ii) *If the set $\{\mu(V^{2n})/\mu(V^n) \mid n=1, 2, 3, \ldots\}$ is bounded then $L^p(G)$ is not closed under convolution for any $p>1$.*

Proof. Let there be a constant $k>0$ such that $\mu(V^{n+1})/\mu(V^n) \leqq k$ for all $n=1, 2, 3, \ldots$. Let $\chi_{V^n}(x)$ be the characteristic function of V^n for $n=1, 2, 3, \ldots$. Then

$$\chi_{V^n} * \chi_{V^{n+1}}(x) = \int_G \chi_{V^n}(y)\chi_{V^{n+1}}(y^{-1}x)\, d\mu(y)$$

$$\geqq \mu(V^n)\chi_V(x) \quad \text{for all } x \in G \text{ and } n = 1, 2, \ldots.$$

So $\|\chi_{V^n} * \chi_{V^{n+1}}\|_p \geqq \mu(V^n)(\mu(V))^{1/p}$. So

$$\frac{\|\chi_{V^n} * \chi_{V^{n+1}}\|_p}{\|\chi_{V^n}\|_p\|\chi_{V^{n+1}}\|_p} \geqq \frac{\mu(V^n)(\mu(V))^{1/p}}{(\mu(V^n))^{1/p}(\mu(V^{n+1}))^{1/p}}$$

$$= \left(\frac{\mu(V^n)}{\mu(V^{n+1})}\right)^{1/p}(\mu(V))^{1/p}(\mu(V^n))^{1-(2/p)}$$

$$\geqq \left(\frac{\mu(V)}{k}\right)^{1/p}(\mu(V^n))^{1-(2/p)}.$$

Thus $\mathrm{lt}_{n\to\infty} (\|\chi_{V^n} * \chi_{V^{n+1}}\|_p/\|\chi_{V^n}\|_p\|\chi_{V^{n+1}}\|_p)=\infty$ if $p>2$. Thus (i) follows from Theorem 1.3. The statement (ii) can be proved likewise.

LEMMA 1.8. *Let G be a totally disconnected locally compact group with a left Haar measure μ. Let $L^p(G)$ be closed under convolution for a real number p $(1<p<\infty)$. Then there is a maximal compact open subgroup H of G. (That is H is a compact, open subgroup of G and any open compact subgroup of G containing H is H itself.)*

Proof. Since G is totally disconnected, there are compact open subgroups in G (see p. 54 of [4]). Suppose there is an ascending sequence $H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots$ of compact open subgroups $H_1, H_2, \ldots, H_n, \ldots$ of G. Let $\chi_{H_n}(x)=1$ if $x \in H_n$ and 0 if $x \in G-H_n$. Put $\varphi_n(x)=(\chi_{H_n}(x)/\mu(H_n))$ for $n=1, 2, 3, \ldots$. Then $\varphi_n * \varphi_n=\varphi_n$ for all $n=1, 2, 3, \ldots$. So

$$\frac{\|\varphi_n * \varphi_n\|_p}{\|\varphi_n\|_p^2} = \frac{\|\varphi_n\|_p}{\|\varphi_n\|_p^2} = \frac{1}{\|\varphi_n\|_p} = \frac{\mu(H_n)}{(\mu(H_n))^{1/p}} = (\mu(H_n))^{1-(1/p)}.$$

By Theorem 1.3 the set $\{\mu(H_n) \mid n=1, 2, 3, \dots\}$ is bounded. Hence there is an n_0 such that $H_{n_0}=H_{n_0+1}=\cdots$. So every ascending sequence of compact, open subgroups of G is finite. Hence the lemma.

LEMMA 1.9. *Let G be a totally disconnected locally compact group with a left Haar measure μ. Let p be a real number $(2<p<\infty)$. Let $L^p(G)$ be closed under convolution. Then G contains an open compact normal subgroup.*

Proof. By Lemma 1.8 there is a maximal, compact, open subgroup H. Now take any element 'a' $\in G-H$ and consider the group generated by $H \cup a^{-1}Ha$. This group should be compact. If not put $V=H \cup a^{-1}Ha$. Then V is a compact symmetric open neighborhood of the identity $e \in G$. Since V^2 is compact there is a finite number of elements $a_1, a_2, \dots, a_k$ of G such that $V^2 \subset (a_1V) \cup (a_2V) \cup \cdots \cup a_kV$. So $V^{n+1} \subset \bigcup_{i=1}^k a_iV^n$ for $n=1, 2, 3, \dots$. So $\mu(V^{n+1}) \leqq k\mu(V^n)$ for all $n=1, 2, 3, \dots$. So $(\mu(V^{n+1})/\mu(V^n)) \leqq k$ for all $n=1, 2, 3, \dots$. Then $L^p(G)$ cannot be closed under convolution by Lemma 1.7 and Lemma 1.4 which contradicts our hypothesis on G. Since H is a maximal open compact subgroup of G we get that $H \cup a^{-1}Ha \subset H$. So $a^{-1}Ha \subset H$ for all 'a' $\in G$. So H is a compact, open, normal subgroup of G.

THEOREM 1.10. *Let G be a locally compact group with a left Haar measure μ. Let $L^p(G)$ be closed under convolution for a real number p $(2<p<\infty)$. Then G must be compact.*

Proof. Let us assume first that G is connected. Let V be a compact symmetric neighborhood of the identity e. Then adopting the proof of Lemma 1.9 we get that the set $\{(\mu(V^{n+1})/\mu(V^n)) \mid n=1, 2, 3, \dots\}$ is bounded. Then by the connectedness of G and by Lemma 1.7 we get that G is compact. Now let us assume that G is totally disconnected. Then by Lemma 1.9 there exists a compact, open, normal subgroup H of G. Then G/H is a discrete group and $L^p(G/H)$ is closed under convolution by Lemma 1.5. So G/H is finite by Theorem 3 of [5]. Then G must be compact. So if G is either connected or totally disconnected the theorem is true. Now the result follows from Lemma 1.6.

2. The case $p=2$ of the L^p-conjecture.

DEFINITION 2.1. An involution $*$ in an algebra A over complex numbers is a one-to-one map from A onto A such that the following hold:

(i) $(x^*)^*=x$ for all $x \in A$.

(ii) $(\lambda x+\mu y)^*=\bar{\lambda}x^*+\bar{\mu}y^*$ for all complex numbers λ and μ and $x, y \in A$.

(iii) $(xy)^*=y^*x^*$ for all $x, y \in A$.

An A^*-algebra is a Banach algebra B with an involution $*$ and an auxiliary norm $| \| \cdot \| |$ such that $| \|xy\| | \leqq | \|x\| | \, | \|y\| |$ and $| \|xx^*\| | = | \|x\| |^2$ for all $x, y \in B$.

Let B be a Banach algebra over the complex numbers. An element x is said to be in the radical of B if there is an ideal $I \subset B$ such that $x \in I$ and $\mathrm{lt}_{n\to\infty} (\|y^n\|)^{1/n} \to 0$ as $n \to \infty$ for all $y \in I$. The algebra B is said to be semisimple if 0 is the only element in the radical of B.

LEMMA 2.2. *Let G be a unimodular locally compact group with a left Haar measure μ. Let $L^p(G)$ be closed under convolution for some p $(1 < p < \infty)$. Then $L^p(G)$ is a semisimple Banach algebra assuming that μ was properly chosen to make $L^p(G)$ a Banach algebra.*

Proof. Let $1/p + 1/q = 1$. Let $f^*(x) = \overline{f(x^{-1})}$ for all $f \in L^p(G)$. Then, from the fact that G is unimodular, one can check that $f \to f^*$ is an involution in $L^p(G)$. Moreover, by using standard theorems on integration one can show that $(f * g, h) = (g, f^* * h) = (f, h * g^*)$ for all $f, g \in L^p(G)$ and $h \in L^q(G)$ where $(f, h) = \int_G f(x)\overline{h(x)}\, dx$.

From this it follows easily that if $f \in L^p(G)$ and $g \in L^q(G)$ then $f * g \in L^q(G)$ and $\|f * g\|_q \leqq \|f\|_p \|g\|_q$. From this and the fact that $L^p(G)$ is a Banach algebra and the Riesz convexity theorem it follows that $f * g \in L^2(G)$, and $\|f * g\|_2 \leqq \|f\|_p \|g\|_2$ for all $f \in L^p(G)$ and $g \in L^2(G)$. Now put $| \|f\| | = \sup \{\|f * g\|_2 \mid g \in L^2(G) \text{ and } \|g\|_2 = 1\}$. Then it easily follows that $| \|\cdot\| |$ is a norm in $L^p(G)$ and $| \|f * f^*\| | = | \|f\| |^2$ and $| \|f * g\| | \leqq | \|f\| | \, | \|g\| |$ for all $f, g \in L^p(G)$. So $L^p(G)$ is an A^*-algebra. So it is semisimple by a theorem of Rickart (Theorem 4.1.15 of [6]).

THEOREM 2.3. *Let G be a totally disconnected locally compact group with a left Haar measure μ. Let $L^2(G)$ be closed under convolution. Then G is compact.*

Proof. Assume for the moment that G is unimodular. We may as well assume that 'μ' was properly chosen so as to make $L^2(G)$ a Banach algebra. Let $f^*(x) = \overline{f(x^{-1})}$ for all $f \in L^2(G)$. Then, as was shown in the proof of Lemma 2.2, $*$ is an involution in $L^2(G)$ and $(f * g, h) = (g, f^* * h)$ $(f, h * g^*)$ for all $f, g, h \in L^2(G)$ where (f, g) is the inner product in $L^2(G)$. By Lemma 2.2 we have that $L^2(G)$ is semisimple and hence it is a semisimple H^*-algebra of Ambrose (see [1]). Now let K be a compact, open subgroup of G and let $\varphi(x) = \chi_K(x)/\mu(K)$ where $\chi_K(x)$ is the characteristic function of K.

Then $\varphi = \varphi^*$ and $\varphi * \varphi = \varphi$ and $\varphi \in L^2(G)$. So $\varphi * L^2(G) * \varphi$ is a semisimple H^*-algebra with an identity element and hence is finite dimensional (see [1]). But $\varphi * L^2(G) * \varphi$ consists exactly of those functions in $L^2(G)$ which are constant on double cosets modulo K. So the number of such cosets has to be finite and hence G must be compact.

In the general case let $\Delta(x)$ be the modular function of G and let

$$H = \{x \mid \Delta(x) = 1; x \in G\}.$$

Then H contains all compact, open subgroups of G. So H is an open subgroup of G. So $L^2(H)$ is closed under convolution by Lemma 1.4. Clearly H is unimodular. So H is compact by what was shown above. So $L^2(G/H)$ is closed under convolution, by Lemma 1.5. But G/H is a discrete subgroup of the reals. So it is finite by Theorem 2 of [5]. So G is compact again.

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