NONCOMMUTATIVE MARKOV PROCESSES

BY

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Preview of results. Let \( \mathbb{A} \) be the algebra of linear operators on finite-dimensional Hilbert space \( \mathcal{H} \). Let \( \eta_1 \) be a state on \( \mathbb{A} \), i.e., there exists a positive semidefinite (psd) operator \( P_0 \in \mathbb{A} \) such that \( \text{tr}(P_0) = 1 \) and for all \( A \in \mathbb{A} \), \( \eta_1(A) = \text{trace}(AP_0) \). The linear functional \( \eta_n \) defined on the \( n \)-fold tensor product \( \otimes^n \mathbb{A} \) is defined by setting \( \eta_n(A_1 \otimes A_2 \otimes \cdots \otimes A_n) = \eta_1(A_1T(A_2 \cdots T(A_n))) \) for \( A_i \in \mathbb{A} \), where \( T \) is some linear map sending \( \mathbb{A} \) to itself with the properties that

(i) \( T(A) \) is psd whenever \( A \) is psd, and

(ii) \( T(I) = I \), where \( I \) is the identity operator on \( \mathcal{H} \). \( K_1 \) will denote the set of all linear maps constrained by properties (i) and (ii).

The theory of Markov processes gives rise to the following noncommutative problem (so-called since the algebra \( \mathbb{A} \) is noncommutative): What relations hold between \( \eta_1 \) and \( T \) in order that the induced linear functional \( \eta_n \) is, in fact, a state on \( \otimes^n \mathbb{A} \)? §1 contains a more thorough discussion of this noncommutative analog. Some of the results we obtain in answer to this question are the following: If \( \text{rg}(T) \), the range of \( T \) is commutative, then \( \eta_2 \) is a state on \( \mathbb{A} \otimes \mathbb{A} \) if and only if \( P_0 \) commutes with \( \text{rg}(T) \) (Theorem 2.3). Corollary 2.9, incidentally, says that it is not unusual that \( \text{rg}(T) \) be commutative; in fact, if we suppose that the linear functional \( \eta_n \) is a bona fide state for all \( n \), and if \( \eta_1(\cdot) = \text{trace}(\cdot P_0) \) where \( P_0 \) has distinct eigenvalues, then \( \text{rg}(T) \) is necessarily commutative. We obtain a result which tells us under which circumstances \( \eta_n \) is a state for all \( n \) given only that the functionals \( \eta_1 \) and \( \eta_2 \) are states. In fact, if \( \eta_1(\cdot) = \text{trace}(\cdot P_0) \), where \( P_0 \) is non-singular (positive definite), then the functional \( \eta_n \) is a state on \( \otimes^n \mathbb{A} \) for all \( n \) if and only if \( \eta_2 \) is a state on \( \mathbb{A} \otimes \mathbb{A} \) and \( \text{rg}(T) \) commutes with \( \text{rg}(T^*) \), where \( T^* \) is the Hilbert space adjoint of \( T \) (Theorems 2.6 and 2.7).

It is to be observed that those operators \( T \in K_1 \) which induce a state \( \eta_2 \) on \( \mathbb{A} \otimes \mathbb{A} \) is a convex compact set \( C_{\eta_1} \), i.e., \( C_{\eta_1} = \{ T : T \in K_1 \text{ and } \eta_2 \text{ is a state on } \mathbb{A} \otimes \mathbb{A} \} \). §3 is devoted to characterizing the extreme points of \( C_{\eta_1} \) for the case where \( \mathbb{A} \) is the algebra of \( 2 \times 2 \) matrices over the complex field.

1. Introduction. Let \( \mathcal{X} \) be any abstract set of points \( \xi \), and let \( I \) be an index set. \( \Omega = \mathcal{X}^I \) will represent the product space whose points \( \omega \) are the functions from \( I \)
to \( X \). Suppose we are given a Borel field \( \mathcal{F}_X \) of sets in \( X \). For fixed \( i \in I \) and \( A \in \mathcal{F}_X \) define the set \( S(A; i) \) to be all points \( \omega \in \Omega \) such that \( \omega(i) \in A \). \( \mathcal{F} \) will denote the Borel field generated by the sets \( S(A; i) \) for \( i \in I \) and \( A \in \mathcal{F}_X \).

Suppose \( p_0 \) is a probability measure on \( \mathcal{F}_X \). Suppose \( p \) is a function of \((\xi, A)\) where \( \xi \in X \) and \( A \in \mathcal{F}_X \) such that

(i) For fixed \( \xi \in X \), \( p(\xi; \cdot) \) is a probability measure on \( \mathcal{F}_X \) and

(ii) For fixed \( A \in \mathcal{F}_X \), \( p(\cdot; A) \) is measurable with respect to the Borel field \( \mathcal{F}_X \).

Such a \( p \) is called a Markov transition function. Utilizing \( p_0 \) and \( p \), a probability measure on \( \mathcal{F} \) is defined as follows: Let \( \Lambda \) be a finite intersection of sets \( S(A; i) \), i.e., \( \Lambda = S(A_1; i_1) \cap S(A_2; i_2) \cap \cdots \cap S(A_n; i_n) \). Then

\[
q(\Lambda) = \int_{A_1} p_0(d\xi_1) \int_{A_2} p(\xi_1; d\xi_2) \int_{A_3} p(\xi_2; d\xi_3) \cdots \int_{A_n} p(\xi_{n-1}; d\xi_n)
\]

defines a set function \( q \) which can be extended to a probability measure on sets in \( \mathcal{F} \), where integration is from right to left. This construction is given in a more general setting in Doob [3, p. 613, example 2.6].

In approaching this from a linear functional point of view, suppose \( X \) has a topology and is compact so that the topology for \( \Omega = X' \) is the product topology. Suppose, too, that \( I \) is countable. Let \( \mathcal{C}(X) \) denote the algebra of continuous, complex-valued functions on \( X \).

We define

\[
f_1 \otimes f_2 \otimes \cdots \otimes f_n \in \mathcal{C}(\Omega)
\]

by

\[
f_1 \otimes f_2 \otimes \cdots \otimes f_n(\omega) = f_1(\omega_1)f_2(\omega_2)\cdots f_n(\omega_n)
\]

where each

\[
f_i \in \mathcal{C}(X), \quad i = 1, 2, \ldots, n,
\]

and

\[
\omega = (\omega_1, \omega_2, \ldots, \omega_n, \omega_{n+1}, \ldots) \in \Omega = X'.
\]

Since these functions \( f_1 \otimes f_2 \otimes \cdots \otimes f_n, n = 1, 2, \ldots \), separate points of \( \Omega \), the selfadjoint algebra they generate is uniformly dense in \( \mathcal{C}(\Omega) \). If \( \mathcal{C}_n(\Omega) \) is the subalgebra of \( \mathcal{C}(\Omega) \) generated by the functions \( f_1 \otimes f_2 \otimes \cdots \otimes f_n \), then the measure \( q \) defined in (1.1) may be identified with the following linear functional \( \eta(n) \) on \( \mathcal{C}_n(\Omega) \).

\[
\eta(n) \to \int_X f_1(\xi_1)p_0(d\xi_1) \int_X f_2(\xi_2)p(\xi_1; d\xi_2) \cdots \int_X f_n(\xi_n)p(\xi_{n-1}; d\xi_n)
\]

where integration proceeds from right to left.
If we define the bounded linear operator $T$ from $\mathcal{C}(X)$ to $\mathcal{C}(X)$ by

$$T(f)(\xi_0) = \int_X f(\xi)p(\xi_0; d\xi) \quad \text{for all } \xi_0 \in X,$$

then it is to be observed that (1.2) assumes the form

$$(1.4) \quad f_1 \otimes f_2 \otimes \cdots \otimes f_n \xrightarrow{\eta(n)} \eta_1(f_1T(f_2T(\cdots T(f_{n-1}T(f_n))))),$$

where $\eta_1$ is the functional on $\mathcal{C}(X)$, $\eta_1 : f \rightarrow \int_X f(\xi)p_0(d\xi)$.

We notice that if $f \in \mathcal{C}(X)$ is positive ($f(\xi) \geq 0$ for all $\xi \in X$), then so is $T(f)$, and that $T(1_x) = 1_x$ where $1_x \in \mathcal{C}(X)$ is the function whose value everywhere on $X$ is one. Moreover, $\eta(n)$ is a state on $\mathcal{C}_n(\Omega)$ for every $n = 1, 2, \ldots$. That is, $\eta(n)(g) \geq 0$ whenever $g \in \mathcal{C}(\Omega)$ is positive and $\eta(n)(1_\Omega) = 1$ where $1_\Omega \in \mathcal{C}(\Omega)$ is the function whose value everywhere on $\Omega$ is one, and $\eta(n)/\mathcal{C}_m(\Omega) = \eta(m)$ whenever $n > m$.

The general problem that we consider is the following: Replace the commutative algebra $\mathcal{C}(X)$ by a noncommutative algebra $\mathfrak{A}$, the algebra of linear operators on finite-dimensional Hilbert space, say. In this case, we will require that $T$ be a linear operator from $\mathfrak{A}$ into itself such that $T(1) = 1$, where $1$ is the identity operator in $\mathfrak{A}$, and $T(A)$ is psd whenever $A$ is psd. Thus, given a state $\eta_1$ on the algebra $\mathfrak{A}$, we define the linear functional $\eta(n)$ on $\otimes^n \mathfrak{A}$ by

$$(1.5) \quad \eta(n)(A_1 \otimes A_2 \otimes \cdots \otimes A_n) = \eta_1(A_1T(A_2 \cdots T(A_n)))$$

for $A_i \in \mathfrak{A}$, which is well defined due to the properties of the tensor product. Our problem concerns the relations between the initial state $\eta_1$ and the operator $T$ which insures that $\eta(n)$ is, in fact, a state on $\otimes^n \mathfrak{A}$.

If (1.5) defines $\eta(n)$ as a state, we shall say that $\eta_1$ is $n$-extendable by $T$. If $\eta(n)$ is a state for all $n = 1, 2, 3, \ldots$, then we say simply that $\eta_1$ is extendable by $T$, or $T$ extends $\eta_1$. In what follows, $\mathfrak{A}$ will represent a factor of finite type (type $I_n$ or $II_1$). In this case, there exists a unique linear functional $\text{tr}$ defined on $\mathfrak{A}$ so that

$$(1.6) \quad \begin{align*}
(1) \quad & \text{tr} (1) = 1. \\
(2) \quad & \text{tr} (\alpha A) = \alpha \text{tr} (A) \quad \text{for all scalars } \alpha, \text{ all } A \in \mathfrak{A}. \\
(3) \quad & \text{tr} (A + B) = \text{tr} (A) + \text{tr} (B) \quad \text{for all } A, B \in \mathfrak{A}. \\
(4) \quad & \text{tr} (P) \geq 0 \quad \text{if } P \text{ is positive definite in } \mathfrak{A}. \\
(5) \quad & \text{tr} (AB) = \text{tr} (BA) \quad \text{for all } A, B \in \mathfrak{A}.
\end{align*}$$

If $P$ is a projection, then $\text{tr} (P) = 0 \Rightarrow P = 0$.

By means of this functional, $\mathfrak{A}$ is given a pre-Hilbert space structure by defining the inner product

$$(1.7) \quad [A, B] = \text{tr} (B^*A) \quad \text{for all } A, B \in \mathfrak{A}.$$
\( \otimes^n \mathcal{A} \) inherits a pre-Hilbert space structure, where for

\[
A_1 \otimes A_2 \otimes \cdots \otimes A_n, B_1 \otimes B_2 \otimes \cdots \otimes B_n \in \otimes^n \mathcal{A},
\]

we define

\[
(1.8) \quad [A_1 \otimes A_2 \otimes \cdots \otimes A_n, B_1 \otimes B_2 \otimes \cdots \otimes B_n] = [A_1, B_1][A_2, B_2] \cdots [A_n, B_n].
\]

Extending linearly defines the inner product on all of \( \otimes^n \mathcal{A} \). Now suppose \( \eta_1(\cdot) \) is a countable additive (c.a.) state on \( \mathcal{A} \). From condition (4) of (1.6), \( \eta_1 \) is completely continuous with respect to \( \text{tr} (\cdot) \). Hence the structure theorem of I. E. Segal [8] (also H. A. Dye [4, Theorem 4, p. 268]) allows us to write

\[
\eta_1(\cdot) = \text{tr} (\cdot P) = [\cdot, P]
\]

where \( P \) is a psd operator in \( \mathcal{A}^c \), the Hilbert space completion of \( \mathcal{A} \). \( P \) is closed and densely defined.

The question arises; given \( P \in \mathcal{A} \), where \( P \) is psd, then which positive-preserving, identity-preserving (bounded) linear operators \( T \) from \( \mathcal{A} \) to \( \mathcal{A} \), extend the state \( \eta_1(\cdot) = [\cdot, P] \) to a state \( \eta_1(\cdot) \) (see (1.5)) satisfying the following:

\[
(1.9) \quad \eta_1(A_1T(A_2 \cdots T(A_n))) = [A_1 \otimes A_2 \otimes \cdots \otimes A_n, Q_n]
\]

for all \( A_i \in \mathcal{A} \) where \( Q_n \) exists in \( \otimes^n \mathcal{A}^c \) and is psd?

2. The extendability of \( T \). It is easy to establish conditions which guarantee that \( \eta_{(n)} \) is a real linear functional for \( n = 2 \). (\( \eta_{(n)}(H) \) is real if and only if \( \eta_{(n)}(H) \) is real whenever \( H = H^* \).)

**Proposition 2.1.** \( T \) induces a real c.a. linear functional \( \eta_2 \) on \( \mathcal{A} \otimes \mathcal{A} \) if and only if

\[
AT(B) - T(B)A \quad \text{belongs to the null space of} \quad \eta_1, \quad \text{where} \quad T: \mathcal{A} \to \mathcal{A} \quad \text{is a bounded positive semidefinite-preserving, identity-preserving linear operator, and} \quad \eta_1 \quad \text{is a c.a. state on} \quad \mathcal{A}.
\]

**Proof.** \( \eta_2 \) is real if and only if for all \( A, B \in \mathcal{A} \)

\[
(2.1) \quad [\eta_2(A \otimes B)]^- = \eta_2(A^* \otimes B^*),
\]

where \( [\cdot]^\pm \) denotes complex conjugation. That is,

\[
(2.2) \quad [\eta_2(AT(B))]^- = \eta_1(A^*T(B^*)) \quad \text{for all} \quad A, B \in \mathcal{A}.
\]

But \( \eta_1 \) is a state, so that

\[
(2.3) \quad [\eta_1(AT(B))]^- = \eta_1((AT(B))^*) = \eta_1(T(B^*)A^*).
\]

Line (2.3) uses the fact that \( T: \text{psd} \to \text{psd} \) so that in particular \( T(B^*) = T(B^*) \) for all \( B \in \mathcal{A} \). Subtracting (2.2) from (2.3) proves the result. We are led to a fact which will be of some importance later.

**Proposition 2.2.** If the induced linear functional \( \eta_2 \) is positive then \( \text{rg} (T), \text{the range of} \ T, \text{commutes with} \ P, \text{where} \ \eta_1(\cdot) = [\cdot, P].\)
Proof. On one hand,

\[ \eta_1(\mathcal{A}T(B)) = [\mathcal{A}T(B), P] \]

(2.4)

\[ = [A, PT(B^*)] \]

which follows since \([AX, B] = [A, BX^*] \) for all \( A, B, X \in \mathcal{A} \), and since \( T(B)^* = T(B^*) \) for all \( B \in \mathcal{A} \).

On the other hand,

(2.5)

\[ \eta_1(\mathcal{A}T(B)) = \eta_2(A \otimes B). \]

Since \( \eta_2 \) is assumed to be a state, \( \eta_2(A \otimes B) \geq 0 \) whenever \( A, B \) are psd in \( \mathcal{A} \).

From (2.4) and (2.5) we conclude that

(2.6)

\[ [A, PT(B)] \geq 0 \]

if \( A \) and \( B \) are psd. But this implies that for any fixed psd \( B \), \([\cdot, PT(B)] \) is a positive functional, which is possible only if \( PT(B) \) is psd. The product of the psd operators \( P \) and \( T(B) \) is again psd if and only if they commute, and hence, \( P \) and \( T(B) \) commute for all psd \( B \). Extending by linearity, we see that \( P \) commutes with \( \text{rg} \( T \) \).

In what follows, \( \mathcal{A} \) will be the full algebra of linear operators on finite-dimensional Hilbert space (\( \mathcal{A} \) is a factor of type \( I_n \)).

The next theorem characterizes the states which are 2-extendable by an operator \( T \), provided the range of \( T \) is commutative.

Theorem 2.3. Let \( \mathcal{A} \) be a factor of type \( I_n \), and let \( T \in K_{\mathcal{A}} \). Assume, moreover, that the range of \( T \), \( \text{rg} \( T \) \), is commutative. Then the state \( \eta_1(\cdot) = [\cdot, P] \) is 2-extendable by \( T \) if and only if \( P \) commutes with \( \text{rg} \( T \) \), where \( P \) is psd in \( \mathcal{A} \).

Proof. If \( \eta_2(\cdot) \) defined by \( \eta_2(A \otimes B) = \eta_1(\mathcal{A}T(B)) \) is a state on \( \otimes^2 \mathcal{A} \), then \( P \) commutes with \( \text{rg} \( T \) \) as stated in Proposition 2.2.

Conversely, assume \( P \) commutes with \( \text{rg} \( T \) \). For all \( B \in \mathcal{A} \), we note that

\[ T(B) = \sum [B, P_i]P_{x_i} \]

where each \( P_i \) is psd in \( \mathcal{A} \), \( \text{tr} \( P_i \) = 1 \), \( i = 1, 2, \ldots, n \), \( \{P_{x_1}, P_{x_2}, \ldots, P_{x_n}\} \) is a spectral family of orthogonal one-dimensional projections for \( \text{rg} \( T \) \), which is commutative. Now

\[ \eta_1(\mathcal{A}T(B)) = [\mathcal{A}T(B), P] \]

(2.7)

\[ = [A, PT(B^*)] \]

\[ = \sum_{i=1}^{n} [A, P[B^*, P_i]P_{x_i}] \]

\[ = \sum_{i=1}^{n} [A, PP_{x_i}][B, P_i] \]

\[ = \left[ A \otimes B, \sum_{i=1}^{n} PP_{x_i} \otimes P_i \right]. \]
Since $P$ is assumed to commute with $\text{rg} (T)$, $P$ necessarily commutes with the spectral family $\{P_{x_1}, P_{x_2}, \ldots, P_{x_n}\}$. Thus, $P \otimes 1$ commutes with $P_{x_i} \otimes P_i$ for all $i = 1, 2, \ldots, n$, and the product of commuting psd operators, $PP_{x_i} \otimes P_i$ is again psd. We conclude that $\sum_{i=1}^{n} PP_{x_i} \otimes P_i$ is psd in $\otimes^{2} \mathcal{A}$. Since $\text{tr} (\sum_{i=1}^{n} PP_{x_i} \otimes P_i) = 1$,

$$\eta_1(AT(B)) = \left[ A \otimes B, \sum_{i=1}^{n} PP_{x_i} \otimes P_i \right] = \eta_2(A \otimes B)$$

defines the linear functional $\eta_2$ as a state on $\otimes^{2} \mathcal{A}$, and the theorem is proved.

So far we have concerned ourselves with the question of 2-extendability of the c.a. state $\eta_1(\cdot)$ by the operator $T$. Our next theorem demonstrates that if $\text{rg} (T)$ commutes with $\text{rg} (T^*)$, then 2-extendability is sufficient to assure $n$-extendability of a state $\eta_1(\cdot)$ for $n = 1, 2, 3, \ldots$.

After this theorem is proved, we will prove the converse in that if the state $\eta_1(\cdot) = [\cdot, P]$, and the psd operator $P$ has an inverse $P^{-1}$, then every operator $T$ which extends the state $\eta_1(\cdot)$ for all $n = 1, 2, 3, \ldots$ has the property that $\text{rg} (T)$ commutes with $\text{rg} (T^*)$.

First we prove

**Lemma 2.4.** If $T$, a bounded linear operator from $\mathcal{A}$ into $\mathcal{A}$, commutes with the adjoint operation, then $\mathcal{J}(T) = \mathcal{J}(T^*)^{\circ}$ where $\mathcal{J}$ is defined by

$$[\mathcal{J}(T), A \otimes B] = [T(A^*); B]$$

for all $A, B \in \mathcal{A}$ and $X^\circ$ is defined for every $X \in \otimes^{2} \mathcal{A}$ by

$$[X^\circ, A \otimes B] = [X, B \otimes A]$$

for all $A, B \in \mathcal{A}$.

**Proof.**

$$[\mathcal{J}(T), A^* \otimes B] = [T(A^*), B] = [A^*, T^*(B)] = [T^*(B^*), A] = [\mathcal{J}(T^*), B \otimes A] = [\mathcal{J}(T^*)^{\circ}, A \otimes B]$$

for all $A, B \in \mathcal{A}$. Therefore, $\mathcal{J}(T) = \mathcal{J}(T^*)^{\circ}$, and the lemma is proved.

**Lemma 2.5.** For $T$ a bounded linear operator from $\mathcal{A}$ into $\mathcal{A}$ which commutes with the adjoint operation, $\text{rg} (T)$ commutes with $\text{rg} (T^*)$ if and only if $\mathcal{J}(T) \otimes 1$ commutes with $1 \otimes \mathcal{J}(T)$.

**Proof.** Note that $\mathcal{J}(T) \in \otimes^{2} \mathcal{A}$, and that $1$ is the identity operator of $\mathcal{A}$.

There are two identities ((2.10') and (2.11")) we will need. We define an "opposite operator" $\circ$ on $\otimes^{2} \mathcal{A}$, (to be distinguished from the operator $\circ$ on $\otimes^{2} \mathcal{A}$ previously defined) by

$$[X^\circ, A \otimes B \otimes C] = [X, C \otimes B \otimes A]$$

for all $X \in \otimes^3 \mathbb{A}$ and all $A, B, C \in \mathbb{A}$. That is

$$\quad (A \otimes B \otimes C)^{\circ} = C \otimes B \otimes A.$$ 

A relation between the operators $^\circ$ and $^\circ$ is given by

$$\quad ((A_1 \otimes A_2 \otimes 1) \cdot (1 \otimes B_1 \otimes B_2))^\circ = B_2 \otimes A_2 B_1 \otimes A_1$$

$$\quad = (1 \otimes A_2 \otimes A_1)(B_2 \otimes B_1 \otimes 1)$$

$$\quad = 1 \otimes (A_1 \otimes A_2)^{\circ} \cdot (B_1 \otimes B_2)^{\circ} \otimes 1.$$ 

Extension by linearity tells us that for all $X \in \otimes^2 \mathbb{A}, 1 \in \mathbb{A}$,

$$\quad ((X \otimes 1) \cdot (1 \otimes Y))^\circ = (1 \otimes X^\circ)(Y^\circ \otimes 1).$$ 

We consider the other identity. For any state $\eta_1(\cdot) = [\cdot, P]$,

$$\quad \eta_1(AT(B)) = [AT(B), P]$$

$$\quad = [T(B), A^* P]$$

$$\quad = [\mathcal{J}(T), B^* \otimes A^* P]$$

(2.11)

$$\quad = [\mathcal{J}(T)(1 \otimes P), B^* \otimes A^*]$$

$$\quad = [B \otimes A, (1 \otimes P) \cdot \mathcal{J}(T)] \quad \text{since } \mathcal{J}(T) = \mathcal{J}(T)^*$$

$$\quad = [A \otimes B, ((1 \otimes P) \cdot \mathcal{J}(T))^\circ]$$

$$\quad = [A \otimes B, (P \otimes 1) \mathcal{J}(T)^\circ].$$ 

Now for $B$, we substitute $BT(C)$. Hence, (2.11) becomes

(2.11')

$$\quad \eta_1(AT(BT(C))) = [A \otimes BT(C), (P \otimes 1) \mathcal{J}(T)^\circ].$$

Now if $\{E_i\} i = 1, 2, \ldots, n^2$, is a basis for $\mathbb{A}$, then $\{E_i \otimes E_j\}, i, j = 1, 2, \ldots, n^2$ forms a basis for $\otimes^2 \mathbb{A}$. Let $\mathcal{J}(T) = \sum r_{ij}(E_i \otimes E_j)$ $i, j = 1, 2, \ldots, n^2$. Then (2.11') becomes

$$\quad \eta_1(AT(BT(C)))$$

$$\quad = \sum_{i, j = 1}^{n^2} r_{ij}[A \otimes BT(C)][PE_j \otimes E_i]$$

$$\quad = \sum_{i, j = 1}^{n^2} r_{0j}[A, PE_j][BT(C), E_i]$$

(2.11')

$$\quad = \sum_{i, j = 1}^{n^2} r_{ij}[A, PE_j][T(C), B^* E_i]$$

$$\quad = \sum_{i, j = 1}^{n^2} r_{0j}[A, PE_j] \cdot [\mathcal{J}(T), C^* \otimes B^* E_i]$$

$$\quad = \sum_{i, j = 1}^{n^2} r_{0j}[A, PE_j][C \otimes E_i B, \mathcal{J}(T)].$$
\[ r_{ij} = \sum_{i,j} r_{ij}[A, PE_j][C \otimes B, (1 \otimes E_i)\mathcal{J}(T)] \]

\[ = \sum_{i,j} r_{ij}[A, PE_j][B \otimes C, (E_i \otimes 1)\mathcal{J}(T)] \]

\[ (2.11') \]

\[ \text{continued} \]

\[ \left[ A \otimes B \otimes C, \sum_{i,j} r_{ij}PE_i \otimes [(E_i \otimes 1)\mathcal{J}(T)] \right] \]

\[ = \left[ A \otimes B \otimes C, (P \otimes 1 \otimes 1)\left( \sum_{i,j} r_{ij}E_i \otimes E_i \otimes 1 \right) \right] \]

\[ = \left[ A \otimes B \otimes C, (P \otimes 1 \otimes 1)(\mathcal{J}(T)^0 \otimes 1)(1 \otimes \mathcal{J}(T)^0) \right]. \]

Now \( \rho(T) \) commutes with \( \rho(T^*) \) if and only if

\[ (2.12) \quad [B, T^*(A)T(C)] = [B, T(C)T^*(A)] \quad \text{for all } A, B, C \in \mathcal{A}. \]

But the right side of (2.12) can be written

\[ [B, T^*(A)T(C)] = [BT^*(A^*), T(C)] \]

\[ (2.12a) \quad = [T^*(BT^*(A^*)), C] \]

\[ = [C^*T^*(BT^*(A^*)), 1]. \]

Using (2.11') we obtain

\[ \eta_1(C^*T^*(BT^*(A^*))) = [C^*T^*(BT^*(A^*)), 1], \]

\[ = [C^* \otimes B \otimes A^*, \mathcal{J}(T)^0 \otimes 1)(1 \otimes \mathcal{J}(T)^0)], \]

and by Lemma 2.4, this

\[ = [C^* \otimes B \otimes A^*, (\mathcal{J}(T) \otimes 1)(1 \otimes \mathcal{J}(T))]. \]

On the other hand, the left side of (2.12) becomes

\[ [B, T^*(A)T(C)] = [BT(C^*), T^*(A)] \]

\[ (2.12b) \quad = [T(BT(C^*)), A] \]

\[ = [A^*T(BT(C^*)), 1]. \]

Appealing once again to (2.11'),

\[ = [A^* \otimes B \otimes C^*, (\mathcal{J}(T)^0 \otimes 1)(1 \otimes \mathcal{J}(T)^0)] \]

\[ = [C^* \otimes B \otimes A^*, (\mathcal{J}(T)^0 \otimes 1)(1 \otimes \mathcal{J}(T)^0)]. \]
Using (2.10')

$$[C^* \otimes B \otimes A^*, (1 \otimes \mathcal{J}(T)) \cdot (\mathcal{J}(T) \otimes 1)]$$

for all $A, B, C \in \mathcal{A}$. Comparing (2.12a) with (2.12b) we conclude that $\text{rg}(T)$ commutes with $\text{rg}(T^*)$ if and only if

$$(1 \otimes \mathcal{J}(T))(\mathcal{J}(T) \otimes 1) = (\mathcal{J}(T) \otimes 1)(1 \otimes \mathcal{J}(T));$$

that is, if and only if $1 \otimes \mathcal{J}(T)$ commutes with $\mathcal{J}(T) \otimes 1$ and the lemma is proved.

We come to the theorem which gives a sufficient condition for 2-extendability of a state $\eta_1(\cdot)$ to imply $n$-extendability of $\eta_1(\cdot)$, for all $n$.

**Theorem 2.6.** Suppose $T$ is a linear operator from $\mathcal{A}$ to $\mathcal{A}$ where $T \in K_1$. Suppose $\text{rg}(T)$ commutes with $\text{rg}(T^*)$ and that a state $\eta_1(\cdot)$ on $\mathcal{A}$ is 2-extendable by $T$. Then $\eta_1(\cdot)$ is $n$-extendable by $T$ for all $n=1, 2, 3, \ldots$ whenever $\eta_1(\cdot) = [\cdot, P]$ and $P^{-1}$ exists.

**Proof.** An inductive argument which imitates the demonstration of (2.11") yields

$$\eta_1(A_A, A_{A_2} \cdots T(A_n))$$

$$= [A_1 \otimes A_2 \otimes \cdots \otimes A_n, (P \otimes 1 \otimes \cdots \otimes 1)(\mathcal{J}(T)^0 \otimes 1 \otimes \cdots \otimes 1)$$

$$\cdots (1 \otimes \cdots \otimes \mathcal{J}(T)^0)]$$

where $\eta_1(\cdot) = [\cdot, P]$ defines $P \in \mathcal{A}$.

Now (2.13) defines a state on $\otimes^n \mathcal{A}$ if and only if

$$(P \otimes 1 \otimes \cdots \otimes 1)(\mathcal{J}(T)^0 \otimes I \otimes \cdots \otimes 1)(1 \otimes \mathcal{J}(T)^0 \otimes \cdots \otimes 1)$$

$$\cdots (1 \otimes \cdots \otimes 1 \otimes \mathcal{J}(T)^0)$$

is psd in $\otimes^n \mathcal{A}$. We have assumed that $\eta_1(\cdot)$ is 2-extendable so that we already have, as a special case of (2.14),

$$\eta_1(\cdot) = [\cdot, P] \text{ is psd in } \otimes^2 \mathcal{A}.$$
Theorem 2.7. If the state \( \eta_1(\cdot) = [\cdot, P] \) where \( P^{-1} \) exists, and if \( \eta_1(\cdot) \) is \( n \)-extendable by \( T \) for \( T \) a bounded linear operator from \( \mathcal{A} \) to \( \mathcal{A} \), \( T \in K_1 \), then necessarily, \( \text{rg}(T) \) commutes with \( \text{rg}(T^*) \).

Proof. A proof can be extracted from the techniques of the previous proof using (2.13) and Lemma 2.5, but a brief, direct proof exists, viz.

\[
\eta_3(A \otimes B \otimes C) = \eta_1(AT(BC(C))) = [AT(BC(C)), P] = [T(BC(C)), A^*P] = [BC(C), T^*(A^*P)] = [B, T^*(A^*P)T(C^*)].
\]

Now \( \eta_3 \) is assumed to be a state; also, whenever \( A, B, C \) are psd, then so is \( A \otimes B \otimes C \). Hence, (2.16) is nonnegative under these circumstances. That is, for all \( A, B, C, \) psd,

\[
[B, T^*(A^*P)T(C^*)] \geq 0.
\]

Since \( B \) runs over all psd operators, we conclude

\[
(T^*(A^*P)T(C^*)) \text{ is psd whenever } A, C \text{ are psd.}
\]

Since \( T \in K_1 \) we know that \( T(C^*) \) is psd for \( C^* \) psd. Setting \( C^* = 1 \), (2.18) implies

\[
T^*(A^*P) \text{ is psd for every psd } A.
\]

Thus, (2.18), which is the product of two psd operators, is again psd if and only if the terms \( T^*(A^*P) \) and \( T(C^*) \) commute for all \( A, C \) psd in \( \mathcal{A} \). By linear extension, \( T^*(AP) \) commutes with \( T(C) \) for all \( A, C \in \mathcal{A} \). Replacing \( A \) with \( AP^{-1} \), yields \( T^*(A) \) commutes with \( T(C) \) for all \( A, C \in \mathcal{A} \), which was to be proved.

Having established necessary and sufficient conditions for certain states \( \eta_1 \) to be \( n \)-extendable by certain operators \( T \), we proceed to define the fully extended algebra \( \otimes^\infty \mathcal{A} \) with its fully extended state \( \eta_\infty \).

For all \( n = 1, 2, 3, \ldots \), we shall identify \( \otimes^n \mathcal{A} \) with a subalgebra of \( \otimes^{n+1} \mathcal{A} \) by the following map for all \( A_1, A_2, \ldots, A_n \in \mathcal{A} \);

\[
A_1 \otimes A_2 \otimes \cdots \otimes A_n \rightarrow A_1 \otimes A_2 \otimes \cdots \otimes A_n \otimes 1,
\]

which, when extended by linearity, imbeds \( \otimes^* \mathcal{A} \) into \( \otimes^{n+1} \mathcal{A} \). Thus we have the ordered relation \( \mathcal{A} \subset \otimes^2 \mathcal{A} \subset \otimes^3 \mathcal{A} \subset \cdots \) and so it is meaningful to define \( \otimes^\infty \mathcal{A} = \bigcup_{n=1}^{\infty} \otimes^n \mathcal{A} \). Similarly, \( \eta_{(n)} \), considered as a set of ordered pairs is contained in \( \eta_{(n+1)} \) since the state \( \eta_{(n+1)} \), when restricted to \( \otimes^n \mathcal{A} \), agrees with \( \eta_{(n)} \). Hence \( \eta_\infty = \bigcup_{n=1}^{\infty} \eta_n \) is defined on \( \otimes^\infty \mathcal{A} \). From (2.13) and (2.14), we see that \( \eta_\infty \) is a state on \( \otimes^\infty \mathcal{A} \).
Let \( \mathfrak{A}_n \) be the algebra of linear operators on the \( n \)-dimensional Hilbert space \( \mathcal{H} \). Suppose the state \( \eta_1(\cdot) = [\cdot, P] \) is defined on \( \mathfrak{A}_n \) where \( P \) is psd in \( \mathfrak{A}_n \) and \( \text{tr}(P) = 1 \). The spectral decomposition of \( P \) allows us to represent \( P \) by

\[
P = \sum_{i=1}^{k} \lambda_i \mathcal{M}_i \quad \text{where} \quad \lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0,
\]

and \( \mathcal{M}_i \) is a subspace of \( \mathcal{H}_n \) where \( \dim (\mathcal{M}_i) = n_i \), and \( \mathcal{H}_n = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_k \). Since the \( \lambda_i \)'s are distinct, the only operators \( A \) which commute with \( P \) are of the form

\[
A = A_1 \oplus A_2 \oplus \cdots \oplus A_k
\]

where \( A_i \) is a linear operator from \( \mathcal{H}_i \) into \( \mathcal{M}_i \) (an \( n_i \times n_i \) matrix). Equivalently stated, \( \mathcal{M}_i A \mathcal{M}_j = A_i \mathcal{M}_i \mathcal{M}_j \) for all \( i, j = 1, 2, \ldots, k \) if and only if \( A \) commutes with \( P = \sum_i \lambda_i \mathcal{M}_i \). By Proposition 4.2, and (2.21), we arrive at

**Proposition 2.8.** Suppose \( T \) is linear from \( \mathfrak{A}_n \) into \( \mathfrak{A}_n \) and that \( T \in K_1 \). If \( T \) extends the state \( \eta_1(\cdot) = [\cdot, P] \), then for all \( A \in \mathfrak{A}_n \),

\[
T(A) = T_1(A) \oplus T_2(A) \oplus \cdots \oplus T_k(A)
\]

where

\[
P = \sum_{i=1}^{k} \lambda_i \mathcal{M}_i \quad \text{where} \quad \lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0
\]

and \( T_i(A) \) is an operator from \( \mathcal{M}_i \) into \( \mathcal{M}_i \).

There are a number of observations to be made as to the implications of Proposition 2.8. Since \( T \in K_1 \), each \( T_i(A) \) must be psd whenever \( A \) is psd. Thus, \( T_i \) as defined by (2.22), is an operator from \( \mathfrak{A}_n \) into \( \mathfrak{A}_{n_i} \) where \( T_i(1_n) = 1_{n_i} \) and \( T_i \in K_{1_{n_i}} \), \( 1_{n_i} \) is the identity of \( \mathfrak{A}_{n_i} \).

Statement (2.22) does not say that \( T \) itself is a direct sum of operators, but the range of \( T \) can be thought of as a direct sum. An interesting and easily obtained corollary follows.

**Corollary 2.9.** Suppose \( \eta_1(\cdot) = [\cdot, P] \) is a state on \( \mathfrak{A}_n \). The only operators \( T \) which extend \( \eta_1(\cdot) \), necessarily have commutative range, where \( T \in K_1 \), if it is assumed that the eigenvalues of \( P \) are distinct (of multiplicity one).

**Proof.** Since \( P = \sum_{i=1}^{n} \lambda_i P_{x_i} \), (2.22) implies that \( T(A) = \sum_{i=1}^{n} T_i(A) \cdot P_{x_i} \) where \( T_i \) is a state on \( \mathfrak{A}_n \). I.e., \( T(A) = \sum_{i=1}^{n} [A, P_i] P_{x_i} \) where \( P_i \) is psd of trace one, and \( \sum P_{x_i} = 1 \).

The decomposition of \( \text{rg}(T) \) as given by Proposition 2.8 gives us a means of establishing the structure of \( \mathcal{J}(T) \), where \( [T(A), B] = [\mathcal{J}(T), A^* \otimes B] \) by definition.
Theorem 2.10. Let $\eta_1(\cdot) = [\cdot, P]$ be a state on $\mathcal{B}$, and let $T$ be a linear operator from $\mathcal{B}$ into itself which commutes with the adjoint. Suppose $P = \sum_{i=1}^{k} \lambda_i M_i$ where $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$. If $\eta_1(AT(B)) = [A \otimes B, Q]$ for all $Q \in \mathcal{B}^2$, then

$$Q_2 = J_1(T_1)^0 \otimes J_2(T_2)^0 \otimes \cdots \otimes J_k(T_k)^0$$

where $T_i$ is defined in Proposition 2.8 and $[T_i(A), B] = [J_i(T_i), A^* \otimes B]$ defines $J_i$ for all $A \in \mathcal{B}$, $B \in \mathcal{B}$. Hence, for $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$, $Q_2$ is psd ($T$ 2-extends $\eta$) implies that $J_1(T_1)^0$, $J_2(T_2)^0$, $\ldots$, $J_k(T_k)^0$ are psd.

Proof.

$$\eta_1(AT(B)) = [AT(B), \sum_{i=1}^{k} \lambda_i M_i]$$

$$= \sum_{i=pqrs} [M_pA_{pq}M_q, M_rT(B)M_s, \lambda_i M_i]$$

$$= \sum_{i} \lambda_i [M_iA_{ii}M_i, M_rT(B)M_i, M_i]$$

$$= \sum_{i} \lambda_i [A_{ii}; T_i(B), M_i]$$

where $A_{ii} = M_iA_{ii}M_i$, $T_i(B) = M_iT(B)M_i$ by definition

$$= \sum_{i} \lambda_i [T_i(B), A_{ii}]^0$$

$$= \sum_{i} \lambda_i [J_i(T_i), B^* \otimes A_{ii}]^0$$

$$(2.23)$$

$$= \sum_{i} \lambda_i [A_{ii} \otimes B, J_i(T_i)^0]$$

$$= \sum_{i} \lambda_i [A_{ii} \otimes B, J_i(T_i)^0] = [A \otimes B, Q]$$

by hypothesis. Now if for $A$, $M_pA_{pq}M_q$ is substituted into the second line of the proof, we see that $[M_pA_{pq} \otimes B, Q] = 0$ for all $A, B \in \mathcal{B}$ if $p \neq q$. That is, the only $A \otimes B$ which do not go to zero are of the form

$$(A_{11} \otimes A_{22} \cdots \otimes A_{kk}) \otimes B = \sum E_{ii} \otimes A_{ii} \otimes B; A_{ii} = M_iA_{ii},$$

That is, $[\sum E_{ii} \otimes (A_{ii} \otimes B), Q] \neq 0$ for all $A, B \in \mathcal{B}$ which says that $Q$ is itself a direct sum of the form

$$Q = P_1 \otimes P_2 \otimes \cdots \otimes P_k,$$

where $P_i$ is an $n_i \times n_i$ matrix. Thus from (2.23),

$$[A \otimes B, Q] = \sum_{i=1}^{k} [A_{ii} \otimes B, P_i] = \sum_{i=1}^{k} \lambda_i [A_{ii} \otimes B, J_i(T_i)^0]$$

which says that $P_i = J_i(T_i)^0$ as long as $\lambda_i \neq 0$; that is, for $\lambda_i \in \{\lambda_1, \lambda_2, \ldots, \lambda_k-1, \lambda_k\}$, $Q$ is psd if and only if $P_i$ is psd for each $i$, which obtains if and only if $J_i(T_i)$ is psd and the theorem is proved.
3. Extreme points of $\mathcal{C}_n$. We define the compact, convex set $\mathcal{C}_n$ to be all linear operators $T$ from $\mathfrak{A}_n$ to $\mathfrak{A}_n$ such that $T \in K_1$ and $T$ 2-extends the state $\eta_1(\cdot) = [\cdot, P]$ i.e., $\eta_1(AT(B)) = \eta_2(A \otimes B)$ defines $\eta_2$ as a state on $\otimes^2 \mathfrak{A}$. Similarly, $\mathcal{C}_{tr}$ is the set of 2-extendable operators $T$ which preserve positive semidefiniteness and preserve trace.

If $\mathfrak{A}$ is a commutative, selfadjoint (diagonal) algebra, then any $T \in K_1$ $n$-extends any state $\eta$ on $\mathfrak{A}$, for all $n=1, 2, \ldots$. In this case, we could apply the result of Ionescu-Tulcea [5] (see also R. R. Phelps [7]) which states (in a more general setting) that the extreme points $T$ of the convex set $K_1$ are exactly the multiplicative operators, i.e., $T(AB) = T(A)T(B)$ for all $A, B \in \mathfrak{A}$.

We characterize the extreme points of $\mathcal{C}_n$, denoted $\text{ext} (\mathcal{C}_n)$, for the simplest, noncommutative case $\mathfrak{A}_n = \mathfrak{A}_2$, the $2 \times 2$ matrices. We will use some results of [2], in the following.

Lemma 3.1. Suppose $T \in \mathcal{C}_n$ and $\mathcal{J}(T)$ is positive definite. Then $T \notin \text{ext} (\mathcal{C}_n)$.

Proof. Let $\mathcal{J}(T) = \sum_{i=1}^{n^2} \lambda_i \mathcal{P}(X_i)$, where $\{X_i\}_{i=1}^{n^2}$ is o.n. in $L(H, H)$ and $\lambda_i > 0$ for all $i$. Choose $n^2$ real numbers $\{e_i\}$ so that $\sum e_i X_i = 0$, which is possible, since the $n^2$ psd operators $\{X_i\}_{i=1}^{n^2}$ cannot be linearly independent in $\mathfrak{A} = L(H, H)$. Multiplying by a sufficiently small positive constant, we may assume

\begin{equation}
\max_i |e_i| \leq \min_j \lambda_j.
\end{equation}

Hence $\mathcal{J}(U) = \sum (\lambda_i + e_i) \mathcal{P}(X_i)$; $\mathcal{J}(V) = \sum (\lambda_i - e_i) \mathcal{P}(X_i)$ are psd operators and so $U, V : \text{psd} \to \text{psd}$ [2, Corollary 2.2]. Moreover, $U(1) = T(1) + e_i X_i^* X_i = 1$, $V(1) = T(1) - e_i X_i^* X_i = 1$ [2, Theorem 2.1].

We need only show that if $T$ 2-extends $\eta_1$, then so do $U$ and $V$. But from (2.14) we see that since $\mathcal{J}(U) \circ \mathcal{J}(V)$ are psd, they must commute with $P$ in order that $P \mathcal{J}(U) \circ P \mathcal{J}(V)$ be psd. (That is, in order that $\eta_2$ is a state.) But this follows since $\mathcal{J}(U), \mathcal{J}(V)$ have the same spectral resolution as $\mathcal{J}(T)$, and $\mathcal{J}(T)$ commutes with $P$. The lemma is proved.

At this point, we turn our attention to $\mathfrak{A}_n = \mathfrak{A}_2$ and present

Theorem 3.2. Suppose $\eta_1(\cdot) = [\cdot, P]$ is a state on $\mathfrak{A}_2$. Then $T$ is extreme in $\mathcal{C}_n$ if $T(A) = \omega_1(A)P_{z_1} + \omega_2(A)P_{z_2}$, provided the eigenvalues of $P = \lambda_1 P_{z_1} + \lambda_2 P_{z_2}$ are distinct. Moreover, $P_{z_1} \cdot P_{z_2} = 0$.

Proof. Since the eigenvalues of $P$ are presumed to be distinct, we already have (Corollary 2.9) that $T(A) = \omega_1(A)P_{z_1} + \omega_2(A)P_{z_2}$ for states $\omega_1, \omega_2$ on $\mathfrak{A}_2$. $T$ is extreme if and only if these states are vector states and this concludes the proof.
The other case remaining concerns $\eta(\cdot) = [\cdot, \frac{1}{2} \cdot 1]$ when the eigenvalues of $P$ are equal, i.e., $\eta = \frac{1}{2} \text{tr} (\cdot)$. We may write

$$\mathcal{J}(T) = \begin{pmatrix} T(E_{11}) & T(E_{21}) \\ T(E_{12}) & T(E_{22}) \end{pmatrix}$$

where a basis is chosen so that $T(E_{11})$ (hence $T(E_{22})$) is diagonal. The operators $E_{ij}$ denote the unit matrices. We state a lemma which was proved by C. Davis [1] for operator algebras.

**Lemma 3.3.** If $A_1^{-1}$ exists, then

$$\begin{pmatrix} A_1 & B \\ B^* & A_2 \end{pmatrix}$$

is psd if and only if $A_2 - B^* A_1 B$ is psd.

The matrix representation of $\mathcal{J}(T)$ is given by

$$\begin{pmatrix} P & A \\ A^* & 1 - P \end{pmatrix}$$

and is therefore psd if and only if

$$P + A^* \frac{1}{P} A \leq 1. \tag{3.3}$$

**Lemma 3.4.** Let

$$\mathcal{J}(T) = \begin{pmatrix} P & A \\ A^* & 1 - P \end{pmatrix}$$

if $P^{-1}$ exists. If $T \in \text{ext } (\mathcal{G}_v)$, $A \neq 0$, then $1 - P - A^*(1/P)A = \lambda P_x$ for some $x$, $\lambda \geq 0$.

**Proof.** Suppose not. $-P - A^*(1/P)A \neq \lambda P_x$ means $(1 - P - A^*(1/P)A)^{-1}$ exists, since we excluded the case where the psd $(1 - P - A^*(1/P)A)$ is either of rank 1 (equal to $\lambda P_x$), or of rank 0 (equal to $0 \cdot P_x$). Now it is verified that

$$P + (A \pm \varepsilon A)^* \frac{1}{P} (A \pm \varepsilon A) = P + A^* \frac{1}{P} A \pm |\varepsilon|^2 A^* \frac{1}{P} A \tag{3.4}$$

if $\varepsilon = -\varepsilon$. Now $P + A^*(1/P)A$ is psd if $1 - P - A^*(1/P)A = \lambda P_x$ for some $x$, $\lambda \geq 0$. Thus

$$1 - P - A^* \frac{1}{P} A = Q, \text{ psd} \tag{3.5}$$

which is nonsingular by hypothesis. Thus

$$\min_x [Q, P_x] = \sigma > 0, \tag{3.6}$$

by compactness of the unit sphere in $\mathcal{H}_2$, and

$$\max_y \left[ A^* \frac{1}{P} A, P_y \right] = \Sigma. \tag{3.7}$$
Setting $|e|^2 \leq \sigma$, we have, for all $x$,

\begin{equation}
|e|^2 \begin{bmatrix}
A^* \frac{1}{P} A, P_x
\end{bmatrix} < |e|^2 \Sigma \leq \sigma = \min \{ Q, P_x \} \leq [Q, P_x].
\end{equation}

Thus, $|e|^2 A^* (1/P) A \leq 1 - P - A^* (1/P) A \iff |e|^2 A^* \frac{1}{P} A + P + A^* \frac{1}{P} A \leq 1,$

which says that

\begin{equation}
(\begin{bmatrix}
P \\
A^* - eA^* \\
1 - P
\end{bmatrix}) \text{ and } (\begin{bmatrix}
P \\
A^* + eA^* \\
1 - P
\end{bmatrix}) \text{ are psd.}
\end{equation}

Hence $\mathcal{F}(T)$, being the average of these two matrices, is not extreme so $T \notin \text{ext} (\mathcal{E}_{tr}).$

**Lemma 3.5.** $A = 0$, $P^{-1}$ exists. $T \in \text{ext} (\mathcal{E}_{tr})$ then $T(\cdot) = \omega_x(\cdot) I$ for some vector state $\omega_x$.

**Proof.**

\[ \mathcal{F}(T) = \begin{bmatrix}
P & 0 \\
0 & 1 - P
\end{bmatrix} = \begin{bmatrix}
\lambda & 0 & 0 & 0 \\
0 & \mu & 0 & 0 \\
0 & 0 & 1 - \lambda & 0 \\
0 & 0 & 0 & 1 - \mu
\end{bmatrix}, \]

assuming $P = T(E_{11})$ is diagonal. If $T$ is extreme, Lemma 3.1 says that $\mathcal{F}(T)$ is singular. $P^{-1}$ exists so that

\[ P = \begin{bmatrix}
1 & 0 \\
0 & \mu
\end{bmatrix} \text{ or } \begin{bmatrix}
\lambda & 0 \\
0 & 1
\end{bmatrix}, \]

which is not extremal unless $\lambda$ or $\mu = 1$. Thus $T(E_{11}) = 1$ or $T(\cdot) = \omega_x(\cdot) I$.

**Lemma 3.6.** Suppose $P^{-1}$ and $(1 - P)^{-1}$ do not exist. The extremal $T \in \mathcal{E}_{tr}$ are the antimultiplicative operators $A \mapsto U^* A^t U$, or the multiplicative $A \mapsto U^* A U$ for $U^{-1} = U^*$, where $A^t$ denotes the transpose of $A$.

**Proof.**

\[ \mathcal{F}(T) = \begin{bmatrix}
1 & 0 & 0 & e^{-i\theta} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
e^{i\theta} & 0 & 0 & 1
\end{bmatrix} \]

which is the only possibility for this case, assuming

\[ P = T(E_{11}) = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}. \]
i.e., the corner matrices $T(E_{ij}) = (f_i x f_j)$ and $T$ is antimultiplicative. If

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is assumed, then $T$ is multiplicative.

Lemmas 3.5 and 3.6 have been proved, where nonvoid examples exist. However, no nonvoid example exists for Lemma 3.4. Suppose, in fact, that $P^{-1}$ exists, $A \neq 0$ and $P + A^t (1/P) A = 1 - \lambda P_x$, $0 \leq \lambda \leq 1$. If $P^{-1}$ and $(1 - P)^{-1}$ exist, then the trace of one of the operators $P$ or $1 - P$ exceeds unity; say $\text{tr} (P) > 1$. Thus

$$1 < \text{tr} \left( P + A^t \frac{1}{P} A \right) = \text{tr} (1 - \lambda P_x) = 1 - \lambda \leq 1.$$ 

The first inequality is strict since $A \neq 0$, and a contradiction ensues.

If $P^{-1}$ exists and $1 - P$ has no inverse, then $P$ may be represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}.$$ 

Again $\text{tr} (P) > 1$ and the argument above yields the contradiction. We have proved

**Theorem 3.7.** The extreme points of $\mathcal{C}_n$, where $\eta = \frac{1}{n} \text{tr}$ are the operators

$$T_x : A \rightarrow \omega_x(A) 1, \quad T_U : A \rightarrow U^* A U \text{ or } T_U A \rightarrow U^* A^t U$$

where $U^* = U^{-1}$, and $A^t$ represents the transpose of $A$. The extreme points of $\mathcal{C}_n$ where $\eta(\cdot) = [\cdot, P]$, $P = \lambda_1 P_{z_1} + \lambda_2 P_{z_2}$, $\lambda_1 \neq \lambda_2$ are the operators

$$T_{xy} : A \rightarrow \omega_x(A) P_{z_1} + \omega_y(A) P_{z_2},$$

where $\omega_x, \omega_y$ are vector states and $P_{z_1} \cdot P_{z_2} = 0$.

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