POINTWISE ERGODIC THEOREMS

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Introduction. The purpose of this paper is to extend various known results in operator ergodic theory to give a direct approach to pointwise ergodic theorems. The main step in this approach is a maximal ergodic theorem (Theorem 1), which is a generalization of the result given in [1]. A corollary (Theorem 2) of this theorem is Chacon's ergodic theorem [6] for positive operators which contains both Birkhoff's ergodic theorem [2] (or, more generally, Dunford-Schwartz's theorem [9] for positive contractions) and Chacon-Ornstein's ratio ergodic theorem [4]. Theorem 1 is also used to obtain the identification of the limit in a straightforward way. In the final part of the paper Theorem 2 is generalized to nonpositive operators to give a direct proof of Chacon's general ergodic theorem [6].

Definitions and basic lemmas. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and $L_1 = L_1(X, \mathscr{F}, \mu)$ the Banach space of equivalence classes of integrable complex valued functions on X. The elements of L_1 will be identified by their representative functions and the relations between them will be considered holding up to sets of zero measure. L_1^+ is the positive cone of L_1, consisting of nonnegative L_1-functions. All subsets of X considered in this paper are measurable either by assumption or construction.

Let T be a contraction on L_1, i.e., a linear operator $T: L_1 \to L_1$ for which $\|T\| \leqq 1$ with the usual definition of the norm of an operator on a Banach space. We denote by τ a positive contraction on L_1, i.e., a contraction such that $\tau L_1^+ \subset L_1^+$.

If $F = \{f_0, f_1, f_2, \ldots\}$ is a sequence of complex valued functions on X and α is a complex number then $\alpha F = \{\alpha f_0, \alpha f_1, \alpha f_2, \ldots\}$, $S_n F = \sum_{k=0}^{n} f_k$, $n \geqq 0$ and $S_\infty F = \sum_{k=0}^{\infty} f_k$ (a.e.), if it exists. For an L_1-function f, let $f_T = \{f, Tf, T^2 f, \ldots\}$ and $S_n(f, T) = S_n f_T$.

Definition 1. A sequence $P = \{p_0, p_1, p_2, \ldots\}$ of nonnegative, finite and measurable functions on X is called T-admissible (or, simply, admissible) if $f \in L_1$, $n \geqq 0$ and $|f| \leqq p_n$ imply that $|Tf| \leqq p_{n+1}$.

Admissible sequences have been introduced by Chacon [6], to give a mutual generalization (Theorem 6 of the present paper) of the Chacon-Ornstein theorem [4] and the Hopf-Dunford-Schwartz theorem [9]. Note that if $f \in L_1^+$ then $f_\tau = \{f, \tau f, \tau^2 f, \ldots\}$ is a τ-admissible sequence. For other examples of admissible sequences we refer to [6].

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Throughout this paper $P=\{p_0, p_1, p_2, \ldots\}$ will denote an admissible sequence. We note that, if $\alpha \geqq 0$, αP is also an admissible sequence. Also, if $f \in L_1$ and $|f| \leqq \sum_{k=n}^{n'} p_k$, $(0 \leqq n \leqq n')$ then $|Tf| \leqq \sum_{k=n+1}^{n'+1} p_k$.

The following lemma was first given, in a slightly different form, in [4]. In the following we indicate an outline of its proof and refer the reader for a complete proof to [4], [11], and [12].

LEMMA 1. *For any* $f \in L_1$, $\lim_{n\to\infty} (\tau^n f / S_n P) = 0$ *a.e. on the set* $\{x \mid 0 < S_\infty P(x)\}$.

Proof. Assume that $f \in L_1^+$ and let, for a fixed $\varepsilon > 0$, E_n be the support of

$$(\tau^n f - \varepsilon S_n P)^+.$$

Then one can obtain that

$$\varepsilon \sum_{n=1}^{\infty} \int_{E_n} p_0 < \infty,$$

from which the proof follows.

Let $E \in \mathscr{F}$. Associated with τ, P, and E define a possibly finite sequence

$$A = \{a_0, a_1, a_2, \ldots\}$$

of measurable functions as follows:

$$(*) \qquad \begin{aligned} a_0 &= \chi p_0, \\ a_n &= \chi\left[p_n - \sum_{k=1}^{n} \tau^k a_{n-k}\right], \qquad n \geqq 1, \end{aligned}$$

where χ is the characteristic function of E. An induction argument shows that the functions a_n are nonnegative. The sequence A is infinite if it consists of L_1-functions. Otherwise it terminates with the first function which does not belong to L_1.

DEFINITION 2. We let $\Omega_E(P) = \sum_k \int a_k$, $(0 \leqq \Omega_E(P) \leqq \infty)$ where $\{a_0, a_1, \ldots\}$ is given by (*) and the summation is taken over the set of indices k for which a_k is defined. If $f \in L_1^+$ we write $\Omega_E(f)$ instead of $\Omega_E(f_\tau)$.

We note that $\Omega_E(\alpha P) = \alpha \Omega_E(P)$ for any $\alpha \geqq 0$, $\Omega_E(f) \leqq \|f\|$ for any $f \in L_1^+$ and $\Omega_E(P) = 0$ if and only if $S_\infty P = 0$ a.e. on E.

A maximal ergodic theorem. The following theorem, which will be fundamental for the rest of this paper, is an extension of an ergodic lemma [1] to admissible sequences. Since its proof is similar to the proof given in [1], here we give only an outline.

THEOREM 1. *Let* $E \in \mathscr{F}$ *and consider two* τ*-admissible sequences* $P = \{p_0, p_1, \ldots\}$

and $Q=\{q_0, q_1, \ldots\}$. *Then* $\limsup_{n\to\infty} (S_nP-S_nQ)\geqq 0$ *a.e. on* E *implies that* $\Omega_E(P)\geqq\Omega_E(Q)$.

For the proof we need the following

LEMMA 2. *Let* $E\in\mathscr{F}$ *and let* $A'=\{a'_0, a'_1, \ldots\}$ *be a sequence of nonnegative measurable functions such that* $a'_0\leqq\chi p_0$ *and if* $a'_0, a'_1, \ldots, a'_{n-1}\in L_1$ *then*

$$a'_n \leqq \chi\left(p_n-\sum_{k=1}^{n}\tau^k a'_{n-k}\right),$$

where χ *is the characteristic function of* E. *Then* $\int S_\infty A'\leqq\Omega_E(P)$.

Proof. Assuming $\Omega_E(P)<\infty$ we can define

$$b_n = p_n-\sum_{k=0}^{n}\tau^k a_{n-k}, \qquad n\geqq 0.$$

An induction argument shows that a corresponding sequence

$$b'_n = p_n-\sum_{k=0}^{n}\tau^k a'_{n-k}, \qquad n\geqq 0$$

is also defined and (b'_n-b_n)'s are nonnegative L_1-functions for all $n\geqq 0$. Now consider $G_n=\int[(S_nA-S_nA')-(b'_n-b_n)]$. Then one obtains that $G_0=0$ and $G_n\leqq G_{n+1}$ for all $n\geqq 0$ which shows that $0\leqq G_n$, or $\int S_nA'\leqq\int S_nA$ for all $n\geqq 0$.

Proof of Theorem 1. Let χ be the characteristic function of E. Assuming $\Omega_E(P)<\infty$, we define a sequence $R=\{r_0, r_1, r_2, \ldots\}$ as follows:

$$r_0 = \chi p_0 \wedge q_0,$$

$$r_n = \chi\left(p_n-\sum_{k=0}^{n-1}\tau^{n-k}r_k\right)\wedge(q_0-S_{n-1}R), \qquad n\geqq 1.$$

Lemma 2 shows that these functions are in L_1^+ and $\int S_\infty R\leqq\Omega_E(P)$. Using the hypotheses on P and Q one obtains that $\int S_\infty R=\chi q_0$, hence $\chi q_0\leqq\Omega_E(P)$.

Let $C=\{c_0, c_1, c_2, \ldots\}$ be the sequence associated with τ, Q, and E, as defined by (*), so that $\Omega_E(Q)=\sum_k\int c_k$. In the previous paragraph we obtained that $\int c_0\leqq\Omega_E(P)$. An induction argument, analogous to that given in [1], shows that, if $\int S_nC\leqq\Omega_E(P)$ then $\int S_{n+1}C\leqq\Omega_E(P)$, hence completes the proof of the theorem.

A ratio ergodic theorem for admissible sequences is a direct consequence of Theorem 1.

THEOREM 2. *Let P and Q be two τ-admissible sequences and $E \in \mathscr{F}$. Then $\Omega_E(Q) < \infty$ implies that* $\lim_{n\to\infty} (S_nQ/S_nP)$ *exists (and is finite) a.e. on*

$$E' = E \cap \{x \mid S_\infty P(x) > 0\}.$$

Before the proof we note that this theorem will only be used for the special case where $Q = f_\tau$, $f \in L_1^+$. The theorem is stated for a general admissible sequence Q for reasons of symmetry, since this does not add any difficulty.

The following proof is analogous to that given in [3] for the proof of the Chacon-Ornstein theorem.

Proof of Theorem 2. If $\limsup_{n\to\infty} (S_nQ/S_nP)$ is infinite a.e. on a nonnegligible subset G of E', then Theorem 1 gives that $\Omega_G(Q) \geqq \alpha\Omega_G(P)$ for all $\alpha \geqq 0$. This is a contradiction, since $\Omega_G(Q) < \infty$ by Lemma 2 and $\Omega_G(P) > 0$.

If $\lim_{n\to\infty} (S_nQ/S_nP)$ does not exist on a nonnegligible subset H of E' then one can find two numbers $\alpha < \beta$ and an $H' \subset H$ with $\mu(H') > 0$ such that

$$\liminf_{n\to\infty} (S_nQ/S_nP) \leqq \alpha < \beta \leqq \limsup_{n\to\infty} (S_nQ/S_nP)$$

a.e. on H'. By Theorem 1, this implies that $\Omega_{H'}(Q) \geqq \beta\Omega_{H'}(P)$ and

$$\alpha\Omega_{H'}(P) \geqq \Omega_{H'}(Q)$$

which is a contradiction.

Identification of the limit. A positive contraction τ induces a decomposition of X into two parts of different characters. We state this result, which is due to Hopf [10] and Chacon [5] in the following form:

THEOREM 3. *Let τ be a positive contraction on $L_1 = L(X, \mathscr{F}, \mu)$. Then X can be written as the union of two disjoint sets D and C, called the dissipative and conservative parts respectively, with the following properties:*

(i) *For any $f \in L_1^+$, $S_\infty(f, \tau) < \infty$, a.e. on D.*
(ii) *For any $f \in L_1^+$, $S_\infty(f, \tau) = \infty$ or 0 a.e. on C.*
(iii) *If $f \in L_1$ and $f = 0$ a.e. on D then $\tau f = 0$ a.e. on D.*
(iv) *If $f \in L_1^+$ and $f = 0$ a.e. on D then $\|\tau f\| = \|f\|$.*

Proof. Let $f \in L_1$, $f > 0$ and

$$C = \{x \mid (S_\infty(f, \tau))(x) = \infty\},$$

$$D = \{x \mid (S_\infty(f, \tau))(x) < \infty\},$$

which are defined up to sets of measure zero. Then Theorem 2 implies that D and C have the properties (i) and (ii).

Now assume that (iii) is not true. Then one can find a function $f \in L_1^+, f=0$ a.e. on D and a nonnegligible subset G of D and two numbers α, β, such that $0<\alpha<\tau f$ a.e. on G and $S_\infty(f, \tau) \leqq \beta$ a.e. on G. Since $S_\infty(f, \tau)=\infty$ a.e. on the support of f, one can choose n large enough so that $S_n(f, \tau) \geqq 2\beta f/\alpha$ a.e. except on a set H with $\int_H f \leqq \frac{1}{4}\alpha\mu(G)$. Then it follows that $S_{n+1}(f, \tau)>\beta$ on a nonnegligible subset of G, which is a contradiction and proves (iii).

Finally assume that (iv) is not true. Then there exists a function $f \in L_1^+, f=0$ a.e. on D such that $\|f\|-\|\tau f\|=\lambda>0$. Now choose n large enough so that $S_n(f, \tau) \geqq 2\|f\| f/\lambda$ a.e. except on a set H with $\int_H f \leqq \frac{1}{4}\lambda$. Then

$$\|S_n(f, \tau)\|-\|\tau S_n(f, \tau)\| \ > \ \|f\|.$$

But this is a contradiction, since $\sum_{k=0}^\infty[\|\tau^k f\|-\|\tau^{k+1}f\|] \leqq \|f\|$.

Definition 3. A (measurable) subset E of X is invariant (with respect to τ) if $f \in L_1$ and $f=0$ a.e. on $X-E$ imply that $\tau f=0$ a.e. on $X-E$.

Note that the previous theorem gives that C is invariant.

The following two lemmas formulate the recurrence properties of the conservative part. Using these results we will obtain a simple interpretation of $\Omega_E(P)$ (Lemma 5).

Lemma 3. *Let E be a (measurable) subset of C, the conservative part, and let $f \in L_1^+, f=0$ a.e. on $X-E$. Then $\Omega_E(\tau^n f)=\|f\|$ for all $n \geqq 0$.*

Proof. For $n=0$ the assertion is trivial. We first show that $\Omega_E(\tau f)=\|f\|$.

Let χ and χ' be the characteristic functions of E and $X-E$ respectively and let $R(\tau f, \tau, E)=\{g_0, g_1, \ldots\}$ and $R'(\tau f, \tau, E)=\{g_0', g_1', \ldots\}$ be the sequences defined as

$$g_0 = \chi\tau f, \qquad g_0' = \chi'\tau f,$$
$$g_n = \chi\tau g_{n-1}', \qquad g_n' = \chi'\tau g_{n-1}', \qquad n \geqq 1,$$

and

$$\tau_E f = S_\infty[R(\tau f, \tau, E)].$$

It is easy to check that τ_E can be extended linearly to the L_1 space $L_1(E)$ of integrable functions with support in E and be considered as a positive contraction on this space. Note that $\Omega_E(\tau f)=\|\tau_E f\|$. An induction argument shows that, for all $n \geqq 0$, $S_n(f, \tau) \leqq S_n(f, \tau_E)$ a.e. on E. Hence τ_E is conservative on E and

$$\Omega_E(\tau f) = \|\tau_E f\| = \|f\|.$$

Now assume that $\Omega_E(\tau^n f)=\|f\|$ for $n=0, 1, \ldots, N$. Then

$$\begin{aligned}\Omega_E(\tau^{N+1}f) &= \Omega_E(\tau\chi\tau^N f)+\Omega_E(\tau\chi'\tau^N f) = \|\chi\tau^N f\|+\Omega_E(\chi'\tau^N f)\\ &= \Omega_E(\tau^N f) = \|f\|.\end{aligned}$$

Here we use the fact that $\Omega_E(g)=\Omega_E(\tau g)$ if $g=0$ a.e. on E.

DEFINITION 4. Let E be a subset of the conservative part C and let $\mathscr{I}_E$ be the class of all invariant subsets of C which contain E. Then the measure-theoretic intersection $I(E)$ of the elements of $\mathscr{I}_E$ is the smallest invariant set containing E.

Note that $I(E)$ is defined up to a set of measure zero and is an invariant set. If g is an L_1^+ function whose support is equal to E, then $S_\infty(g, \tau)=\infty$ a.e. on $I(E)$, and $S_\infty(g, \tau)=0$ a.e. on $X-I(E)$.

LEMMA 4. *Let E be a subset of C and let f be an L_1^+ function which has support in $I(E)$. Then $\Omega_E(f)=\|f\|$.*

Proof. First observe that if $0\leqq f\leqq u$, $u\in L_1$ and $\Omega_E(u)=\|u\|$ then $\Omega_E(f)=\|f\|$. Now let $g\in L_1^+$ and let the support of g be equal to E. Then for any $\varepsilon>0$ there exists an n such that $S_n(g, \tau)\geqq f$ a.e. except on a set G with $\int_G f\geqq \|f\|-\varepsilon$. Since $\Omega_E(S_n(g, \tau))=\|S_n(g, \tau)\|$, we then have $\Omega_E(f)\geqq\|f\|-\varepsilon$, which proves the lemma.

LEMMA 5. *If E is a subset of the conservative part C, then, for any τ-admissible sequence P,*

$$\Omega_E(P) = \lim_{n\to\infty}\int_{I(E)} p_n.$$

Proof. First we have, directly from the definitions, that

$$\Omega_E(P) \leqq \lim_{n\to\infty}\int_{I(E)} p_n.$$

Hence, if $\Omega_E(P)=\infty$ the proof is complete. Otherwise the previous lemma gives that $\Omega_E(P)\geqq\Omega_E(p_n')$ where $p_n'=\chi_I p_n\in L_1^+$, χ_I being the characteristic function of $I(E)$. This implies the conclusion of the lemma.

LEMMA 6. *The invariant subsets of C form a σ-field $\mathscr{I}$ with respect to C.*

Proof. The only nontrivial step of the proof is to show that if $I\subset C$ is invariant then $I'=C-I$ is also invariant. Let $\tau_{I'}$ be defined as in the proof of Lemma 3. If I' is not invariant one can find a function $f\in L_1^+$, $f=0$ a.e. on $X-I'$ such that $\|\tau_{I'}f\|<\|f\|$. But this contradicts the fact that $\tau_{I'}$ is conservative.

LEMMA 7. *Let $\Omega_C(Q)<\infty$. Then the restriction of $h=\lim_{n\to\infty}(S_nQ/S_nP)$ to C is an $\mathscr{I}$-measurable function.*

Proof. Let $E_\alpha=\{x \mid h(x)\geqq\alpha\}\cap C$. If $I(E_\alpha)\cap(C-E_\alpha)$ has nonzero measure, then there exists an $\varepsilon>0$ such that $I(E_\alpha)\cap(C-E_{\alpha-\varepsilon})=G$ has also nonzero measure. Let $H=I(G)\cap E_\alpha$. Then it follows that $I(H)=I(G)$, or $\Omega_G(\cdot)=\Omega_H(\cdot)$. Now on H, $\lim_{n\to\infty}(S_nQ/S_nP)\geqq\alpha$ which implies that $\Omega_H(Q)\geqq\alpha\Omega_H(P)$ and from an analogous consideration for G one obtains that $\Omega_G(Q) \leqq (\alpha-\varepsilon)\Omega_G(P)$. But this is a contradiction and shows that $E_\alpha \in \mathscr{I}$.

A standard approximation procedure of measurable functions by simple functions leads to the following result.

THEOREM 4. *Let I be an invariant subset of C, P and Q be two admissible sequences, such that $(S_\infty P) > 0$ a.e. on I and $\Omega_I(P) < \infty$, $\Omega_I(Q) < \infty$. Then*

$$\lim_{n\to\infty} \int_I q_n = \lim_{n\to\infty} \int_I hp_n$$

with $h = \lim_{n\to\infty} (S_n Q/S_n P)$.

COROLLARY. *If $\mu(C) < \infty$, $\Omega_c(Q) < \infty$ and $S_\infty(P) > 0$ a.e. on C, then*

$$\lim_{n\to\infty} (S_n Q/S_n P) = \lim_{n\to\infty} \frac{E[q_n|\mathscr{I}]}{E[p_n|\mathscr{I}]} \quad \text{a.e. on } C.$$

Nonpositive operators. We now consider a general, not necessarily positive contraction T. The following result [8], due to Chacon and Krengel, shows that T can always be dominated, in a certain sense, by a positive contraction τ.

THEOREM 5. *For every bounded linear operator $T: L_1 \to L_1$ there is a unique bounded, linear and positive operator $\tau: L_1 \to L_1$ such that*

(i) $\|\tau\| \leqq \|T\|$.

(ii) *For all $f \in L_1$, $|Tf| \leqq \tau|f|$.*

(iii) *If $f \in L_1^+$ then $\tau f = \sup_{|g|\leqq f; g\in L_1} |Tg|$.*

Proof. Let $\mathscr{P}$ be the class of finite (measurable) partitions $\Pi = [E_1, \ldots, E_n]$ of X, partially ordered in the usual way. For any $\Pi = [E_1, \ldots, E_n] \in \mathscr{P}$ and $f \in L_1^+$ let

$$\tau_\Pi f = \sum_{k=1}^{n} |T\chi_k f|$$

where χ_k is the characteristic function of E_k, $k = 1, \ldots, n$. Also, let Π_n be a sequence of nondecreasing partitions such that

$$\lim_{n\to\infty} \|\tau_{\Pi_n} f\| = \sup_{\Pi\in\mathscr{P}} \|\tau_\Pi f\|.$$

One then defines $\tau' f = \lim_{n\to\infty} \tau_{\Pi_n} f$ (a.e.) and obtains the transformation τ of the theorem as the unique linear extension of τ' to L_1. For further details we refer the reader to [8].

DEFINITION 5. The operator τ as given by Theorem 5, is called the linear modulus of T.

For the rest of this paper τ will denote the linear modulus of T. The quantity $\Omega_E(\cdot)$ is defined, as before, in terms of τ. We note that a sequence $P=\{p_0, p_1, \ldots\}$ is T-admissible if and only if it is τ-admissible.

DEFINITION 6. Two L_1 functions f and g are called equivalent (with respect to T), in notation $f\sim g$, if there exists a strictly positive L_1 function F, such that

$$\limsup_{n\to\infty} \frac{|S_n(f-g, T)|}{S_n(F, \tau)} = 0 \text{ a.e. on } C$$

where C is the conservative part of X with respect to τ.

It is clear that this relation is actually an equivalence relation. We also note that, if $f\sim g$ then $f+h\sim g+h$ for any $h\in L_1$ and also that $T^n f\sim f$ for all $n\geqq 1$ by Lemma 1.

DEFINITION 7. The nonnegative number

$$M = M(f) = \inf_{g\sim f} \|g\|$$

is the minimal norm of f (with respect to T).

In terms of these concepts we first note the following

LEMMA 8. *If $\Omega_E(|g|)\geqq a$ for all $g\sim f$ then for any $\lambda>0$, $\eta>0$ there exists an $h\sim f$ such that*

$$\|h\| \leqq M+\lambda \quad \text{and} \quad \int_E |h| \geqq a-\eta$$

where M is the minimal norm of f.

Proof. Let $0<\varepsilon<\min(\lambda, \frac{1}{2}\eta)$, $\|g\|\leqq M+\varepsilon$, $g\sim f$ and consider

$$\{\alpha_0, \alpha_1, \ldots\} = R(g, T, E),$$
$$\{\beta_0, \beta_1, \ldots\} = R'(g, T, E),$$
$$\{\gamma_0, \gamma_1, \ldots\} = R(|g|, \tau, E),$$
$$\{\delta_0, \delta_1, \ldots\} = R'(|g|, \tau, E),$$

where R and R' are as defined in the proof of Lemma 3.

Since $\Omega_E(|g|)=\sum_{k=0}^{\infty}\|\gamma_k\|\geqq a$, there exists an n such that $\sum_{k=0}^{n}\|\gamma_k\|\geqq a-\frac{1}{2}\eta$. Let $h=\sum_{k=0}^{n}\alpha_k+\beta_n$, which is equivalent to f. Then, by simple inductions we obtain that

$$(M\leqq)\|h\| \leqq \|g\| \leqq M+\varepsilon$$

and

$$M \leqq \sum_{k=0}^{n}\|\alpha_k\|+\|\beta_n\| \leqq \sum_{k=0}^{n}\|\gamma_k\|+\|\delta_n\| \leqq M+\varepsilon$$

and also $\|\beta_n\|\leqq\|\delta_n\|$, $\sum_{k=0}^{n}\|\alpha_k\|\leqq\sum_{k=0}^{n}\|\gamma_k\|$ which imply that

$$\int_E |h| = \left\|\sum_{k=0}^{n}\alpha_k\right\| \geqq \sum_{k=0}^{n}\|\gamma_k\|-2\varepsilon \geqq a-\eta.$$

The following lemma shows that if the norm of an L_1-function f is very close to its minimal norm, then the action of T on this function can be described in terms of the action of τ on $|f|$.

LEMMA 9. *Let $f \in L_1$ and let M be its minimal norm. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property. If $g \sim f$ and $\|g\| \leqq M + \delta$ then one can find a function $\Delta \in L_1^+$ with $\|\Delta\| < \varepsilon$ and a set G with $\int_G |g| < \varepsilon$ such that, for any $n \geqq 0$,*

$$\sum_{k=0}^{n} |T^k g - e^{i\theta} \tau^k |g| \, | \leqq \sum_{k=0}^{n} \tau^k \Delta$$

a.e. on $(S \cap C) - G$, where S is the support of g and $\theta \colon S \to (-\pi, \pi]$ is the phase of g.

The proof of this theorem, although quite straightforward, is rather long and will be divided into several sublemmas.

Let η, $0 < \eta < \pi$, and $\delta > 0$ be fixed and $g \sim f$, $\|g\| \leqq M + \delta$. Define two sequences $R = \{r_0, r_1, r_2, \ldots\}$ and $R' = \{r_0', r_1', r_2', \ldots\}$ of L_1 functions:

$$r_0 = 0, \qquad r_0' = g,$$

$$r_n = \lambda_n \mathrm{Tr}_{n-1}', \qquad r_n' = (1 - \lambda_n) \mathrm{Tr}_{n-1}', \qquad n \geqq 1,$$

where $\lambda_n \colon X \to [0, 1]$ for all $n \geqq 1$, such that $\lambda_n = 0$ on the set on which $g = 0$ or $\mathrm{Tr}_{n-1}' = 0$ or $\theta - \eta < \mathrm{ph}\, \mathrm{Tr}_{n-1}' < \theta + \eta \pmod{2\pi}$, where "ph" denotes the phase of its argument. Outside of this set λ_n is defined by the following (pointwise) relation $\lambda_n = \sup \{t \mid 0 \leqq t \leqq 1, \theta - \eta/2 \leqq \mathrm{ph}\, [g + S_{n-1} R + t\, \mathrm{Tr}_{n-1}'] \leqq \theta + \eta/2 \pmod{2\pi}\}$.

In what follows, if

$$A = \{a_0, a_1, a_2, \ldots\}$$

is a sequence of functions, $|A|$ and $\|A\|$ will denote the sequences

$$\{|a_0|, |a_1|, |a_2|, \ldots\} \quad \text{and} \quad \{\|a_0\|, \|a_1\|, \|a_2\|, \ldots\}$$

respectively.

LEMMA 10. $(1 - \cos(\eta/2)) S_\infty \|R\| \leqq 2\delta$.

Proof. First, an induction argument shows that $S_n \|R\| + \|r_n'\| \leqq M + \delta$. But we also have that $S_n R + r_n' \sim g$, which combined with the first inequality, gives that

$$\|g\| + S_n \|R\| - \|g + S_n R\| \leqq 2\delta$$

or

$$\sum_{k=1}^{n} [\|g + S_{k-1} R\| + \|r_k\| - \|g + S_k R\|] \leqq 2\delta.$$

Now, at almost every point on the support of r_k, the difference between the phases of $g+S_kR$ and r_k is at least $\eta/2$. Hence

$$|g+S_kR| \leqq |g+S_{k-1}R|+|r_k| \cos\frac{\eta}{2}$$

or

$$\left(1-\cos\frac{\eta}{2}\right)S_n\|R\| \leqq 2\delta$$

for all $n \geqq 0$.

LEMMA 11. *Let G be the part of the support of g on which* ph $(g+S_nR)=\theta-\eta/2$ *or $\theta+\eta/2$ for at least one $n \geqq 0$. Then* $(1-\cos(\eta/2)) \int_G |g| \leqq 2\delta$.

Proof. Write G as the union of disjoint sets $G_1, G_2, \ldots$ such that on G_n the phase of $g+S_kR$ is in the (open) interval $(\theta-\eta/2, \theta+\eta/2)$ for all $k=0, 1, \ldots, n-1$, but the phase of $g+S_nR$ is $\theta-\eta/2$ or $\theta+\eta/2$.

Hence on G_n:

$$|g+S_nR| \leqq |g| \cos\frac{\eta}{2}+S_n|R|$$

or

$$\left(1-\cos\frac{\eta}{2}\right)|g| \leqq |g|+S_n|R|-|g+S_nR| = c_n$$

where the last equality defines c_n.

From the proof of the previous lemma we have that $\int c_n \leqq 2\delta$ for all $n \geqq 1$. Hence,

$$2\delta \geqq \int c_n \geqq \int_{\bigcup_{k=1}^n G_k} c_n \geqq \sum_{k=1}^n \int_{G_k} c_k \geqq \left(1-\cos\frac{\eta}{2}\right) \int_{\bigcup_{k=1}^n G_k} |g|,$$

since c_n is a nondecreasing sequence in n.

LEMMA 12. *Let the sequence $H=\{h_0, h_1, \ldots\}$ be defined as $h_0=0$,*

$$h_n = \tau|r'_{n-1}|-|\mathrm{T}r'_{n-1}|, \qquad n \geqq 1.$$

Then

$$S_\infty\|H\| \leqq \delta.$$

Proof. First, an induction argument shows that

$$\|\mathrm{T}r'_{k-1}\| \leqq \|g\|-S_{k-1}\|R\|-S_k\|H\|, \qquad k \geqq 1.$$

Now

$$S_{n-1}R+r'_{n-1} \sim S_{n-1}R+\mathrm{T}r'_{n-1} \sim g.$$

Hence

$$\begin{aligned} M &\leqq S_{n-1}\|R\| + \|\mathrm{T}r'_{n-1}\| \\ &\leqq \|g\| - S_k\|H\| \\ &\leqq M + \delta - S_k\|H\|, \end{aligned}$$

which proves the lemma.

LEMMA 13. *Let*

$$a_n = \tau^n|g| - \sum_{k=1}^{n} \tau^{n-k}|r_k| - |r'_n|$$

and

$$b_n = e^{i\theta}\left[\tau^n|g| - \sum_{k=1}^{n} \tau^{n-k}|r_k|\right] - r'_n.$$

Then

$$a_n = \sum_{k=1}^{n} \tau^{n-k} h_k$$

and

$$|b_n| \leqq a_n + \eta\tau^n|g| \quad \text{a.e. on } (C \cap S) - G.$$

Proof. The first assertion follows directly from the definitions. For the second assertion, we first note that

$$b_n = e^{i\theta}\left[\tau^n|g| - \sum_{k=1}^{n-1} \tau^{n-k}|r_k|\right] - [r'_n + |r_n|e^{i\theta}].$$

On $(C \cap S) - G$ we have either $\lambda_n = 0$ or $\lambda_n = 1$. If $\lambda_n = 0$ then $r_n = 0$ and the phase of $r'_n = \mathrm{T}r'_{n-1}$ is in the interval $(\theta - \eta,\ \theta + \eta)$. But $|r'_n| \leqq \tau^n|g|$ and $|b_n| \leqq a_n + \eta\tau^n|g|$ follows. If $\lambda_n = 1$ then $r'_n = 0$ and $|b_n| = a_n$.

Proof of Lemma 9. From the definitions and the previous lemma we have that

$$\begin{aligned} |e^{i\theta}\tau^n|g| - T^n g| &\leqq |b_n| + 2\sum_{k=1}^{n} \tau^{n-k}|r_k| \\ &\leqq \eta\tau^n|g| + \sum_{k=1}^{n} \tau^{n-k}(h_k + 2|r_k|). \end{aligned}$$

Therefore

$$\sum_{k=0}^{n} |e^{i\theta}\tau^k|g| - T^k g| \leqq \sum_{k=0}^{n} \tau^k[\eta|g| + S_\infty(H + 2|R|)].$$

Let

$$\Delta = \eta|g| + S_\infty(H + 2|R|).$$

Then

$$\|\Delta\| \leqq \eta\|g\|+\delta+\frac{4\delta}{1-\cos\frac{\eta}{2}}.$$

Choose η, so that $M\eta \leqq \varepsilon/4$, $0<\eta<\pi$ and choose δ satisfying

$$0 < \delta < \frac{\varepsilon}{16}\left(1-\cos\frac{\eta}{2}\right).$$

Then it is easily seen that $\|\Delta\| \leqq \varepsilon$ and $\int_G |g| < \varepsilon$.

A general ergodic theorem. We now apply Lemmas 8 and 9 and the results for positive contractions to prove the following general ergodic theorem, due to Chacon [6], [7]. This theorem gives a mutual generalization of the Dunford-Schwartz theorem [9] and the Chacon-Ornstein theorem [4]. We note that Chacon's original paper [7] does not contain some details of the proof. A complete but rather complicated proof of the theorem appeared in [12].

THEOREM 6. *For any* $f \in L_1$,

$$\lim_{n\to\infty} (S_n(f, T)/S_nP) \tag{1}$$

exists (and is finite) a.e. on $\{x \mid 0<S_\infty P(x)\}$.

Proof. We can assume that $0<S_\infty P$ a.e. Since $|S_n(f, T)| \leqq S_n(|f|, \tau)$, it is also clear that $\limsup_{n\to\infty} (|S_n(f, T)|/S_nP)$ is finite a.e. and that the limit (1) exists a.e. on D, the dissipative part of τ. If it fails to exist on a nonnegligible subset of the conservative part C, then there exist a number $\alpha<0$ and a nonnegligible set $E\subset C$ such that

$$\limsup_{n,m\to\infty} |(S_n(f, T)/S_nP)-(S_m(f, T)/S_mP)| \geqq \alpha \qquad \text{a.e. on } E. \tag{2}$$

We can also assume the existence of a number $\beta>0$ such that

$$\limsup_{n\to\infty} (|S_n(f, T)|/S_nP) \leqq \beta \qquad \text{a.e. on } E. \tag{3}$$

Note that (2) and (3) are valid for any $g\sim f$.

Now (2) implies that, for any $g\sim f$, $\alpha/2 \leqq \limsup_{n\to\infty} (S_n(|f|, \tau)/S_nP)$ a.e. on E. Hence, by Theorem 1,

$$\Omega_E(|g|) \geqq \tfrac{1}{2}\alpha\Omega_E(P) \qquad (>0)$$

for any $g\sim f$, and by Lemma 8, for any $\delta>0$ one can find a $g\sim f$ such that $\|g\| \leqq M+\delta$ and

$$\int_E |g| \geqq \tfrac{1}{4}\alpha\Omega_E(P). \tag{4}$$

Now let $\varepsilon > 0$ be a fixed number and choose $\delta > 0, g, \Delta, G$ as they are given by Lemma 9. We also assume that g satisfies (4).

Let $F=(E-G) \cap S$ where S is the support of g. Then, by Lemma 9,

$$\sum_{k=0}^{n} |T^k g - e^{i\theta}\tau^k |g| \,| \leqq \sum_{k=0}^{n} \tau^k \Delta$$

a.e. on F, with $\|\Delta\| \leqq \varepsilon$ and

$$\int_F |g| \geqq \int_E |g| - \varepsilon \geqq \tfrac{1}{4}\alpha\Omega_E(P) - \varepsilon \tag{5}$$

We now have, a.e. on F,

$$\begin{aligned} \beta &\geqq \limsup_{n\to\infty} \left| \frac{S_n(g, T)}{S_n P} \right| \\ &\geqq \lim_{n\to\infty} \frac{S_n(|g|, \tau)}{S_n P} \left[1 - \limsup_{n\to\infty} \left| \frac{S_n(g, T)}{e^{i\theta} S_n(|g|, \tau)} - 1 \right| \right] \\ &\geqq \lim_{n\to\infty} \frac{S_n(|g|, \tau)}{S_n P} \left[1 - \lim_{n\to\infty} \frac{S_n(\Delta, \tau)}{S_n(|g|, \tau)} \right]. \end{aligned} \tag{6}$$

Let

$$l = \lim_{n\to\infty} \frac{S_n(\Delta, \tau)}{S_n(|g|, \tau)}.$$

Then, from Theorem 1 and Lemma 7 it follows that

$$\int_F l|g| \leqq \|\Delta\| \leqq \varepsilon.$$

Since $l \geqq 0$, if $R = F \cap \{x \mid l(x) \leqq \tfrac{1}{2}\}$ then

$$\tfrac{1}{2} \int_{F-R} |g| \leqq \varepsilon$$

which, combined with (5) implies that

$$\int_R |g| \geqq \tfrac{1}{4}\alpha\Omega_E(P) - 3\varepsilon.$$

Now, by (6),

$$\lim_{n\to\infty} \frac{S_n(|g|, \tau)}{S_n P} \leqq 2\beta \qquad \text{a.e. on } R,$$

hence

$$\Omega_R(P) \geqq \frac{1}{2\beta} \Omega_R(|g|) \geqq \frac{1}{2\beta} [\tfrac{1}{4}\alpha\Omega_E(P) - 3\varepsilon]. \tag{7}$$

On the other hand, (2) implies that, a.e. on R,

$$\begin{aligned}\alpha &\leqq \limsup_{n,m\to\infty} \left| \frac{S_n(g,T)}{S_nP} - \frac{S_m(g,T)}{S_mP} \right| \\ &= \lim_{n\to\infty} \frac{S_n(|g|,\tau)}{S_nP} \limsup_{n,m\to\infty} \left| \frac{S_n(g,T)}{e^{i\theta}S_n(|g|,\tau)} - \frac{S_m(g,T)}{e^{i\theta}S_m(|g|,\tau)} \right| \\ &\leqq 2 \lim_{n\to\infty} \frac{S_n(|g|,\tau)}{S_nP} \lim_{n\to\infty} \frac{S_n(\Delta,\tau)}{S_n(|g|,\tau)} = 2 \lim_{n\to\infty} \frac{S_n(\Delta,\tau)}{S_nP}.\end{aligned}$$

Hence

$$\Omega_R(P) \leqq \frac{2}{\alpha}\Omega_R(\Delta) \leqq \frac{2\varepsilon}{\alpha}. \tag{8}$$

But, if ε is sufficiently small, (7) and (8) are incompatible, which means that the assumption (2) cannot be true and completes the proof.

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