POINTWISE ERGODIC THEOREMS

BY

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Introduction. The purpose of this paper is to extend various known results in operator ergodic theory to give a direct approach to pointwise ergodic theorems. The main step in this approach is a maximal ergodic theorem (Theorem 1), which is a generalization of the result given in [1]. A corollary (Theorem 2) of this theorem is Chacon’s ergodic theorem [6] for positive operators which contains both Birkhoff’s ergodic theorem [2] (or, more generally, Dunford-Schwartz’s theorem [9] for positive contractions) and Chacon-Ornstein’s ratio ergodic theorem [4]. Theorem 1 is also used to obtain the identification of the limit in a straightforward way. In the final part of the paper Theorem 2 is generalized to nonpositive operators to give a direct proof of Chacon’s general ergodic theorem [6].

Definitions and basic lemmas. Let \((X, \mathcal{F}, \mu)\) be a \(\sigma\)-finite measure space and \(L_1=L_1(X, \mathcal{F}, \mu)\) the Banach space of equivalence classes of integrable complex valued functions on \(X\). The elements of \(L_1\) will be identified by their representative functions and the relations between them will be considered holding up to sets of zero measure. \(L_1^+\) is the positive cone of \(L_1\), consisting of nonnegative \(L_1\)-functions. All subsets of \(X\) considered in this paper are measurable either by assumption or construction.

Let \(T\) be a contraction on \(L_1\), i.e., a linear operator \(T: L_1 \to L_1\) for which \(\|T\| \leq 1\) with the usual definition of the norm of an operator on a Banach space. We denote by \(\tau\) a positive contraction on \(L_1\), i.e., a contraction such that \(\tau L_1^+ \subseteq L_1^+\).

If \(F=\{f_0, f_1, f_2, \ldots\}\) is a sequence of complex valued functions on \(X\) and \(\alpha\) is a complex number then \(\alpha F=\{\alpha f_0, \alpha f_1, \alpha f_2, \ldots\}\), \(S_n F=\sum_{k=0}^{n} f_k, n \geq 0\) and \(S_\infty F=\sum_{k=0}^{\infty} f_k\) (a.e.), if it exists. For an \(L_1\)-function \(f\), let \(f_\tau=\{f, Tf, T^2 f, \ldots\}\) and \(S_n(f, T)=S_n f_\tau\).

**Definition 1.** A sequence \(P=\{p_0, p_1, p_2, \ldots\}\) of nonnegative, finite and measurable functions on \(X\) is called \(T\)-admissible (or, simply, admissible) if \(f \in L_1\), \(n \geq 0\) and \(|f| \leq p_n\) imply that \(|Tf| \leq p_{n+1}\).

Admissible sequences have been introduced by Chacon [6], to give a mutual generalization (Theorem 6 of the present paper) of the Chacon-Ornstein theorem [4] and the Hopf-Dunford-Schwartz theorem [9]. Note that if \(f \in L_1^+\) then \(f_\tau=\{f, \tau f, \tau^2 f, \ldots\}\) is a \(\tau\)-admissible sequence. For other examples of admissible sequences we refer to [6].

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Throughout this paper $P = \{p_0, p_1, p_2, \ldots\}$ will denote an admissible sequence. We note that, if $\alpha \geq 0$, $\alpha P$ is also an admissible sequence. Also, if $f \in L_1$ and $|f| \leq \sum_{n=n'}^{n} p_k$, ($0 \leq n \leq n'$) then $|T'f| \leq \sum_{n=n+1}^{n+1} p_k$.

The following lemma was first given, in a slightly different form, in [4]. In the following we indicate an outline of its proof and refer the reader for a complete proof to [4], [11], and [12].

**Lemma 1.** For any $f \in L_1$, $\lim_{n \to \infty} (\tau^n f \text{Sn} P) = 0$ a.e. on the set $\{x \mid 0 < S_\infty P(x)\}$.

**Proof.** Assume that $f \in L_1^+$ and let, for a fixed $\epsilon > 0$, $E_n$ be the support of $(\tau^n f - \epsilon S_n P)^+$. Then one can obtain that

$$e \sum_{n=1}^{\infty} \int_{E_n} p_0 < \infty,$$

from which the proof follows.

Let $E \in \mathcal{F}$. Associated with $\tau$, $P$, and $E$ define a possibly finite sequence

$$A = \{a_0, a_1, a_2, \ldots\}$$

of measurable functions as follows:

$$a_0 = \chi p_0,$$

$$a_n = \chi \left[ p_n - \sum_{k=1}^{n} \tau^k a_{n-k} \right], \quad n \geq 1,$$

where $\chi$ is the characteristic function of $E$. An induction argument shows that the functions $a_n$ are nonnegative. The sequence $A$ is infinite if it consists of $L_1$-functions. Otherwise it terminates with the first function which does not belong to $L_1$.

**Definition 2.** We let $Q_E(P) = \sum_{k} a_k$, ($0 \leq \Omega_E(P) \leq \infty$) where $\{a_0, a_1, \ldots\}$ is given by (*) and the summation is taken over the set of indices $k$ for which $a_k$ is defined. If $f \in L_1^+$ we write $\Omega_E(f)$ instead of $\Omega_E(f_k)$.

We note that $\Omega_E(\alpha P) = \alpha \Omega_E(P)$ for any $\alpha \geq 0$, $\Omega_E(f) \leq \|f\|$ for any $f \in L_1^+$ and $\Omega_E(P) = 0$ if and only if $S_\infty P = 0$ a.e. on $E$.

A maximal ergodic theorem. The following theorem, which will be fundamental for the rest of this paper, is an extension of an ergodic lemma [1] to admissible sequences. Since its proof is similar to the proof given in [1], here we give only an outline.

**Theorem 1.** Let $E \in \mathcal{F}$ and consider two $\tau$-admissible sequences $P = \{p_0, p_1, \ldots\}$
and \( Q = \{q_0, q_1, \ldots \} \). Then \( \limsup_{n \to \infty} (S_n P - S_n Q) \geq 0 \) a.e. on \( E \) implies that \( \Omega_{\mathcal{E}}(P) \geq \Omega_{\mathcal{E}}(Q) \).

For the proof we need the following

**Lemma 2.** Let \( E \in \mathcal{F} \) and let \( A' = \{a_0', a_1', \ldots \} \) be a sequence of nonnegative measurable functions such that \( a_0' \leq \chi P_0 \) and if \( a_0', a_1', \ldots, a_{n-1}' \in L_1 \) then

\[
a_n' \leq \chi \left( p_n - \sum_{k=1}^{n} \tau^k a_{n-k}' \right),
\]

where \( \chi \) is the characteristic function of \( E \). Then \( \int S_\infty A' \leq \Omega_{\mathcal{E}}(P) \).

**Proof.** Assuming \( \Omega_{\mathcal{E}}(P) < \infty \) we can define

\[
b_n = p_n - \sum_{k=0}^{n} \tau^k a_{n-k}, \quad n \geq 0.
\]

An induction argument shows that a corresponding sequence

\[
b_n' = p_n - \sum_{k=0}^{n} \tau^k a_{n-k}', \quad n \geq 0
\]

is also defined and \( (b_n' - b_n)'s \) are nonnegative \( L_1 \)-functions for all \( n \geq 0 \). Now consider \( G_n = \int [ (S_n A - S_n A') - (b_n' - b_n) ] \). Then one obtains that \( G_0 = 0 \) and \( G_n \leq G_{n+1} \) for all \( n \geq 0 \) which shows that \( 0 \leq G_n \), or \( \int S_n A' \leq \int S_n A \) for all \( n \geq 0 \).

**Proof of Theorem 1.** Let \( \chi \) be the characteristic function of \( E \). Assuming \( \Omega_{\mathcal{E}}(P) < \infty \), we define a sequence \( R = \{r_0, r_1, r_2, \ldots \} \) as follows:

\[
r_0 = \chi P_0 \wedge q_0,
\]

\[
r_n = \chi \left( p_n - \sum_{k=0}^{n-1} \tau^k r_k \right) \wedge (q_0 - S_{n-1} R), \quad n \geq 1.
\]

Lemma 2 shows that these functions are in \( L_1^+ \) and \( \int S_\infty R \leq \Omega_{\mathcal{E}}(P) \). Using the hypotheses on \( P \) and \( Q \) one obtains that \( \int S_\infty R = \chi q_0 \), hence \( \chi q_0 \leq \Omega_{\mathcal{E}}(P) \).

Let \( C = \{c_0, c_1, c_2, \ldots \} \) be the sequence associated with \( \tau, Q, \) and \( E \), as defined by (\*), so that \( \Omega_{\mathcal{E}}(Q) = \sum_k c_k \). In the previous paragraph we obtained that \( \int c_0 \leq \Omega_{\mathcal{E}}(P) \). An induction argument, analogous to that given in [1], shows that, if \( \int S_\infty C \leq \Omega_{\mathcal{E}}(P) \) then \( \int S_{n+1} C \leq \Omega_{\mathcal{E}}(P) \), hence completes the proof of the theorem.

A ratio ergodic theorem for admissible sequences is a direct consequence of Theorem 1.
Theorem 2. Let \( P \) and \( Q \) be two \( \tau \)-admissible sequences and \( E \in \mathcal{F} \). Then \( \Omega_\tau(Q) < \infty \) implies that \( \lim_{n \to \infty} (S_n Q / S_n P) \) exists (and is finite) a.e. on

\[
E' = E \cap \{ x \mid S_\infty P(x) > 0 \}.
\]

Before the proof we note that this theorem will only be used for the special case where \( Q = f \), \( f \in L_1^\tau \). The theorem is stated for a general admissible sequence \( Q \) for reasons of symmetry, since this does not add any difficulty.

The following proof is analogous to that given in [3] for the proof of the Chacon-Ornstein theorem.

Proof of Theorem 2. If \( \lim \sup_{n \to \infty} (S_n Q / S_n P) \) is infinite a.e. on a nonnegligible subset \( G \) of \( E' \), then Theorem 1 gives that \( \Omega_\tau(Q) \geq \alpha \Omega_\tau(P) \) for all \( \alpha \geq 0 \). This is a contradiction, since \( \Omega_\tau(Q) < \infty \) by Lemma 2 and \( \Omega_\tau(P) > 0 \).

If \( \lim_{n \to \infty} (S_n Q / S_n P) \) does not exist on a nonnegligible subset \( H \) of \( E' \) then one can find two numbers \( \alpha < \beta \) and an \( H' \subset H \) with \( \mu(H') > 0 \) such that

\[
\liminf_{n \to \infty} (S_n Q / S_n P) \leq \alpha < \beta \leq \limsup_{n \to \infty} (S_n Q / S_n P)
\]
a.e. on \( H' \). By Theorem 1, this implies that \( \Omega_{H'}(Q) \geq \beta \Omega_{H'}(P) \) and

\[
\alpha \Omega_{H'}(P) \geq \Omega_{H'}(Q)
\]

which is a contradiction.

Identification of the limit. A positive contraction \( \tau \) induces a decomposition of \( X \) into two parts of different characters. We state this result, which is due to Hopf [10] and Chacon [5] in the following form:

Theorem 3. Let \( \tau \) be a positive contraction on \( L_1 = L(X, \mathcal{F}, \mu) \). Then \( X \) can be written as the union of two disjoint sets \( D \) and \( C \), called the dissipative and conservative parts respectively, with the following properties:

(i) For any \( f \in L_1^\tau \), \( S_\infty (f, \tau) < \infty \), a.e. on \( D \).
(ii) For any \( f \in L_1^\tau \), \( S_\infty (f, \tau) = \infty \) or \( 0 \) a.e. on \( C \).
(iii) If \( f \in L_1 \) and \( f = 0 \) a.e. on \( D \) then \( \tau f = 0 \) a.e. on \( D \).
(iv) If \( f \in L_1^\tau \) and \( f = 0 \) a.e. on \( D \) then \( \| \tau f \| = \| f \| \).

Proof. Let \( f \in L_1 \), \( f > 0 \) and

\[
C = \{ x \mid (S_\infty (f, \tau))(x) = \infty \},
\]

\[
D = \{ x \mid (S_\infty (f, \tau))(x) < \infty \},
\]

which are defined up to sets of measure zero. Then Theorem 2 implies that \( D \) and \( C \) have the properties (i) and (ii).
Now assume that (iii) is not true. Then one can find a function \( f \in L^+_1, f = 0 \) a.e. on \( D \) and a nonnegligible subset \( G \) of \( D \) and two numbers \( \alpha, \beta \) such that \( 0 < \alpha < \tau \) a.e. on \( G \) and \( S_n(f, \tau) \leq \beta \) a.e. on \( G \). Since \( S_n(f, \tau) = \infty \) a.e. on the support of \( f \), one can choose \( n \) large enough so that \( S_n(f, \tau) \geq 2\beta /\alpha \) a.e. except on a set \( H \) with \( \int_H f \leq \frac{1}{4} \alpha \mu(G) \). Then it follows that \( S_{n+1}(f, \tau) > \beta \) on a nonnegligible subset of \( G \), which is a contradiction and proves (iii).

Finally assume that (iv) is not true. Then there exists a function \( f \in L^+_1, f = 0 \) a.e. on \( D \) such that \( \|f\| - \|\tau f\| = \lambda > 0 \). Now choose \( n \) large enough so that \( S_n(f, \tau) \geq 2\|f\|\lambda \) a.e. except on a set \( H \) with \( \int_H f \leq \frac{1}{4} \lambda \). Then

\[
\|S_n(f, \tau)\| - \|\tau S_n(f, \tau)\| > \|f\|.
\]

But this is a contradiction, since \( \sum_{k=0}^{\infty} \|\tau^k f\| = \|f\| \).

**Definition 3.** A (measurable) subset \( E \) of \( X \) is invariant (with respect to \( \tau \)) if \( f \in L^1 \) and \( f = 0 \) a.e. on \( X-E \) imply that \( \tau f = 0 \) a.e. on \( X-E \).

Note that the previous theorem gives that \( C \) is invariant.

The following two lemmas formulate the recurrence properties of the conservative part. Using these results we will obtain a simple interpretation of \( \Omega_\mathcal{B}(P) \) (Lemma 5).

**Lemma 3.** Let \( E \) be a (measurable) subset of \( C \), the conservative part, and let \( f \in L^+_1, f = 0 \) a.e. on \( X-E \). Then \( \Omega_\mathcal{B}(\tau f) = \|f\| \) for all \( n \geq 0 \).

**Proof.** For \( n = 0 \) the assertion is trivial. We first show that \( \Omega_\mathcal{B}(\tau f) = \|f\| \).

Let \( \chi \) and \( \chi' \) be the characteristic functions of \( E \) and \( X-E \) respectively and let \( R(\tau f, \tau, E) = \{g_0, g_1, \ldots\} \) and \( R'(\tau f, \tau, E) = \{g'_0, g'_1, \ldots\} \) be the sequences defined as

\[
g_0 = \chi \tau f, \quad g'_0 = \chi' \tau f,
\]

\[
g_n = \chi \tau g'_{n-1}, \quad g'_n = \chi' \tau g_{n-1}, \quad n \geq 1,
\]

and

\[
\tau \mathcal{B} f = S_\mathcal{B}[R(\tau f, \tau, E)].
\]

It is easy to check that \( \tau \mathcal{B} \) can be extended linearly to the \( L_1 \) space \( L_1(E) \) of integrable functions with support in \( E \) and be considered as a positive contraction on this space. Note that \( \Omega_\mathcal{B}(\tau f) = \|\tau \mathcal{B} f\| \). An induction argument shows that, for all \( n \geq 0 \), \( S_n(f, \tau) \leq S_n(f, \tau \mathcal{B}) \) a.e. on \( E \). Hence \( \tau \mathcal{B} \) is conservative on \( E \) and

\[
\Omega_\mathcal{B}(\tau f) = \|\tau \mathcal{B} f\| = \|f\|.
\]

Now assume that \( \Omega_\mathcal{B}(\tau^nf) = \|f\| \) for \( n = 0, 1, \ldots, N \). Then

\[
\Omega_\mathcal{B}(\tau^{N+1}f) = \Omega_\mathcal{B}(\tau \chi \tau^Nf) + \Omega_\mathcal{B}(\tau \chi' \tau^Nf) = \|\chi \tau^N f\| + \Omega_\mathcal{B}(\chi' \tau^N f) = \Omega_\mathcal{B}(\tau^N f) = \|f\|.
\]

Here we use the fact that \( \Omega_\mathcal{B}(g) = \Omega_\mathcal{B}(\tau g) \) if \( g = 0 \) a.e. on \( E \).
Definition 4. Let $E$ be a subset of the conservative part $C$ and let $\mathcal{S}_E$ be the class of all invariant subsets of $C$ which contain $E$. Then the measure-theoretic intersection $I(E)$ of the elements of $\mathcal{S}_E$ is the smallest invariant set containing $E$.

Note that $I(E)$ is defined up to a set of measure zero and is an invariant set. If $g$ is an $L^+_1$ function whose support is equal to $E$, then $S_\infty(g, \tau) = \infty$ a.e. on $I(E)$, and $S_\alpha(g, \tau) = 0$ a.e. on $X-\{I(E)\}$.

Lemma 4. Let $E$ be a subset of $C$ and let $f$ be an $L^+_1$ function which has support in $I(E)$. Then $\Omega_E(f) = \|f\|$.

Proof. First observe that if $0 \leq f \leq u$, $u \in L_1$ and $\Omega_E(u) = \|u\|$ then $\Omega_E(f) = \|f\|$. Now let $g \in L^+_1$ and let the support of $g$ be equal to $E$. Then for any $\varepsilon > 0$ there exists an $n$ such that $S_n(g, \tau) \leq \|f\| - \varepsilon$. Since $\Omega_E(S_n(g, \tau)) = \|S_n(g, \tau)\|$, then $\Omega_E(f) \geq \|f\| - \varepsilon$, which proves the lemma.

Lemma 5. If $E$ is a subset of the conservative part $C$, then, for any $\tau$-admissible sequence $P$,

$$\Omega_E(P) = \lim_{n \to \infty} \int_{I(E)} p_n.$$  

Proof. First we have, directly from the definitions, that

$$\Omega_E(P) \leq \lim_{n \to \infty} \int_{I(E)} p_n.$$  

Hence, if $\Omega_E(P) = \infty$ the proof is complete. Otherwise the previous lemma gives that $\Omega_E(P) \leq \Omega_E(p_n')$ where $p_n' = \chi_I p_n \in L^+_1$, $\chi_I$ being the characteristic function of $I(E)$. This implies the conclusion of the lemma.

Lemma 6. The invariant subsets of $C$ form a $\sigma$-field $\mathcal{F}$ with respect to $C$.

Proof. The only nontrivial step of the proof is to show that if $I \subset C$ is invariant then $I' = C - I$ is also invariant. Let $\tau_{I'}$ be defined as in the proof of Lemma 3. If $I'$ is not invariant one can find a function $f \in L^+_1$, $f = 0$ a.e. on $X-I'$ such that $\|\tau_{I'} f\| < \|f\|$. But this contradicts the fact that $\tau_{I'}$ is conservative.

Lemma 7. Let $\Omega_C(Q) < \infty$. Then the restriction of $h = \lim_{n \to \infty} (S_n Q | S_n P)$ to $C$ is an $\mathcal{F}$-measurable function.

Proof. Let $E_\alpha = \{x \mid h(x) \geq \alpha\} \cap C$. If $I(E_\alpha) \cap (C-E_\alpha)$ has nonzero measure, then there exists an $\varepsilon > 0$ such that $I(E_\alpha) \cap (C-E_{\alpha - \varepsilon}) = G$ has also nonzero measure. Let $H = I(G) \cap E_\alpha$. Then it follows that $I(H) = I(G)$, or $\Omega_C(\cdot) = \Omega_H(\cdot)$. Now on $H$, $\lim_{n \to \infty} (S_n Q | S_n P) \geq \alpha$ which implies that $\Omega_H(Q) \geq \alpha \Omega_H(P)$ and from an analogous
consideration for $G$ one obtains that $\Omega_\alpha(Q) \leq (\alpha - \epsilon)\Omega_\alpha(P)$. But this is a contradiction and shows that $E_\alpha \in \mathcal{F}$.

A standard approximation procedure of measurable functions by simple functions leads to the following result.

**Theorem 4.** Let $I$ be an invariant subset of $C$, $P$ and $Q$ be two admissible sequences, such that $(S_\infty P) > 0$ a.e. on $I$ and $\Omega_\alpha(P) < \infty$, $\Omega_\alpha(Q) < \infty$. Then

$$\lim_{n \to \infty} \int_I q_n = \lim_{n \to \infty} \int_I h p_n$$

with $h = \lim_{n \to \infty} (S_n Q/S_n P)$.

**Corollary.** If $\mu(C) < \infty$, $\Omega_\alpha(Q) < \infty$ and $S_\alpha(P) > 0$ a.e. on $C$, then

$$\lim_{n \to \infty} (S_n Q/S_n P) = \lim_{n \to \infty} \frac{E[q_n]}{E[p_n]}$$

a.e. on $C$.

**Nonpositive operators.** We now consider a general, not necessarily positive contraction $T$. The following result [8], due to Chacon and Krengel, shows that $T$ can always be dominated, in a certain sense, by a positive contraction $\tau$.

**Theorem 5.** For every bounded linear operator $T: L_1 \to L_1$ there is a unique bounded, linear and positive operator $\tau: L_1 \to L_1$ such that

(i) $\|\tau\| \leq \|T\|$.  
(ii) For all $f \in L_1$, $|Tf| \leq \tau |f|$.  
(iii) If $f \in L_1^+$ then $\tau f = \sup_{0 \leq g \leq f} \|Tg\|$.

**Proof.** Let $\mathcal{P}$ be the class of finite (measurable) partitions $\Pi = \{E_1, \ldots, E_n\}$ of $X$, partially ordered in the usual way. For any $\Pi = \{E_1, \ldots, E_n\} \in \mathcal{P}$ and $f \in L_1^+$ let

$$\tau_{\Pi} f = \sum_{k=1}^n |T_{x_k} f|$$

where $x_k$ is the characteristic function of $E_k$, $k = 1, \ldots, n$. Also, let $\Pi_n$ be a sequence of nondecreasing partitions such that

$$\lim_{n \to \infty} \|\tau_{\Pi_n} f\| = \sup_{\Pi \in \mathcal{P}} \|\tau_{\Pi} f\|.$$ 

One then defines $\tau' f = \lim_{n \to \infty} \tau_{\Pi_n} f$ (a.e.) and obtains the transformation $\tau$ of the theorem as the unique linear extension of $\tau'$ to $L_1$. For further details we refer the reader to [8].

**Definition 5.** The operator $\tau$ as given by Theorem 5, is called the linear modulus of $T$.  

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For the rest of this paper, \( \tau \) will denote the linear modulus of \( T \). The quantity \( \Omega_\xi(\cdot) \) is defined, as before, in terms of \( \tau \). We note that a sequence \( P = \{p_0, p_1, \ldots\} \) is \( T \)-admissible if and only if it is \( \tau \)-admissible.

**Definition 6.** Two \( L_1 \) functions \( f \) and \( g \) are called equivalent (with respect to \( T \)), in notation \( f \sim g \), if there exists a strictly positive \( L_1 \) function \( F \), such that

\[
\limsup_{n \to \infty} \frac{|S_n(f-g, T)|}{S_n(F, \tau)} = 0 \quad \text{a.e. on } C
\]

where \( C \) is the conservative part of \( X \) with respect to \( \tau \).

It is clear that this relation is actually an equivalence relation. We also note that, if \( f \sim g \) then \( f + h \sim g + h \) for any \( h \in L_1 \) and also that \( T^n f \sim f \) for all \( n \geq 1 \) by Lemma 1.

**Definition 7.** The nonnegative number

\[
M = M(f) = \inf_{g \sim f} \|g\|
\]

is the minimal norm of \( f \) (with respect to \( T \)).

In terms of these concepts we first note the following

**Lemma 8.** If \( \Omega_\xi(|g|) \geq a \) for all \( g \sim f \) then for any \( \lambda > 0, \eta > 0 \) there exists an \( h \sim f \) such that

\[
\|h\| \leq M + \lambda \quad \text{and} \quad \int_E |h| \geq a - \eta
\]

where \( M \) is the minimal norm of \( f \).

**Proof.** Let \( 0 < \epsilon < \min (\lambda, \frac{1}{2} \eta) \), \( \|g\| \leq M + \epsilon \), \( g \sim f \) and consider

\[
\{\alpha_0, \alpha_1, \ldots\} = R(g, T, E),
\]

\[
\{\beta_0, \beta_1, \ldots\} = R'(g, T, E),
\]

\[
\{\gamma_0, \gamma_1, \ldots\} = R(|g|, \tau, E),
\]

\[
\{\delta_0, \delta_1, \ldots\} = R'(|g|, \tau, E),
\]

where \( R \) and \( R' \) are as defined in the proof of Lemma 3.

Since \( \Omega_\xi(|g|) = \sum_{k=0}^{\infty} \|\gamma_k\| \geq a \), there exists an \( n \) such that \( \sum_{k=0}^{n} \|\gamma_k\| \geq a - \frac{1}{2} \eta \). Let \( h = \sum_{k=0}^{n} \alpha_k + \beta_n \), which is equivalent to \( f \). Then, by simple inductions we obtain that

\[
(M \leq) \|h\| \leq \|g\| \leq M + \epsilon
\]

and

\[
M \leq \sum_{k=0}^{n} \|\alpha_k\| + \|\beta_n\| \leq \sum_{k=0}^{n} \|\gamma_k\| + \|\delta_n\| \leq M + \epsilon
\]

and also \( \|\beta_n\| \leq \|\delta_n\|, \sum_{k=0}^{n} \|\alpha_k\| \leq \sum_{k=0}^{n} \|\gamma_k\| \) which imply that

\[
\int_E |h| = \left\| \sum_{k=0}^{n} \alpha_k \right\| \geq \sum_{k=0}^{n} \|\gamma_k\| - 2 \epsilon \geq a - \eta.
\]
The following lemma shows that if the norm of an $L_1$-function $f$ is very close to its minimal norm, then the action of $T$ on this function can be described in terms of the action of $\tau$ on $|f|$.

**Lemma 9.** Let $f \in L_1$ and let $M$ be its minimal norm. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property. If $g \sim f$ and $\|g\| \leq M + \delta$ then one can find a function $\Delta \in L_1^+$ with $\|\Delta\| < \varepsilon$ and a set $G$ with $\int_G |g| < \varepsilon$ such that, for any $n \geq 0$,

$$\sum_{k=0}^{n} |T^k g - e^{i\theta} \tau^k |g| | \leq \sum_{k=0}^{n} \tau^k \Delta$$

a.e. on $(S \cap C) - G$, where $S$ is the support of $g$ and $\theta : S \to (-\pi, \pi]$ is the phase of $g$.

The proof of this theorem, although quite straightforward, is rather long and will be divided into several sublemmas.

Let $\eta$, $0 < \eta < \pi$, and $\delta > 0$ be fixed and $g \sim f$, $\|g\| \leq M + \delta$. Define two sequences $R = \{r_0, r_1, r_2, \ldots\}$ and $R' = \{r'_0, r'_1, r'_2, \ldots\}$ of $L_1$ functions:

$$r_0 = 0, \quad r'_0 = g,$$

$$r_n = \lambda_n \tau_{n-1}, \quad r'_n = (1 - \lambda_n) \tau_{n-1}, \quad n \geq 1,$$

where $\lambda_n : X \to [0, 1]$ for all $n \geq 1$, such that $\lambda_n = 0$ on the set on which $g = 0$ or $\tau_{n-1} = 0$ or $\theta - \eta < \text{ph} \tau_{n-1} < \theta + \eta$ (mod $2\pi$), where “ph” denotes the phase of its argument. Outside of this set $\lambda_n$ is defined by the following (pointwise) relation

$$\lambda_n = \sup \{t \mid 0 \leq t \leq 1, \theta - \eta/2 \leq \text{ph} [g + S_{n-1} R + t \tau_{n-1}] \leq \theta + \eta/2 \text{ (mod $2\pi$)}\}.$$

In what follows, if

$$A = \{a_0, a_1, a_2, \ldots\}$$

is a sequence of functions, $|A|$ and $\|A\|$ will denote the sequences

$$\{|a_0|, |a_1|, |a_2|, \ldots\} \quad \text{and} \quad \{|a_0|, \|a_1\|, \|a_2\|, \ldots\}$$

respectively.

**Lemma 10.** $(1 - \cos (\eta/2)) S_{\infty} \|R\| \leq 2\varepsilon$.

**Proof.** First, an induction argument shows that $S_n \|R\| + \|r'_n\| \leq M + \delta$. But we also have that $S_n R + r'_n \sim g$, which combined with the first inequality, gives that

$$\|g\| + S_n \|R\| - \|g + S_n R\| \leq 2\delta$$

or

$$\sum_{k=1}^{n} \left[\|g + S_{k-1} R\| + \|r_k\| - \|g + S_k R\|\right] \leq 2\delta.$$
Now, at almost every point on the support of \( r_k \), the difference between the phases of \( g + S_k R \) and \( r_k \) is at least \( \eta/2 \). Hence

\[
|g + S_k R| \leq |g + S_{k-1} R| + |r_k| \cos \frac{\eta}{2}
\]

or

\[
\left( 1 - \cos \frac{\eta}{2} \right) S_n \| R \| \leq 2\delta
\]

for all \( n \geq 0 \).

**Lemma 11.** Let \( G \) be the part of the support of \( g \) on which \( \text{ph} \ (g + S_n R) = \theta - \eta/2 \) or \( \theta + \eta/2 \) for at least one \( n \geq 0 \). Then \( (1 - \cos (\eta/2)) \int_G |g| \leq 2\delta \).

**Proof.** Write \( G \) as the union of disjoint sets \( G_1, G_2, \ldots \) such that on \( G_\alpha \) the phase of \( g + S_k R \) is in the (open) interval \( (\theta - \eta/2, \theta + \eta/2) \) for all \( k = 0, 1, \ldots, n-1 \), but the phase of \( g + S_n R \) is \( \theta - \eta/2 \) or \( \theta + \eta/2 \).

Hence on \( G_\alpha \):

\[
|g + S_n R| \leq |g| \cos \frac{\eta}{2} + S_n \| R \|
\]

or

\[
\left( 1 - \cos \frac{\eta}{2} \right) |g| \leq |g| + S_n \| R \| - |g + S_n R| = c_n
\]

where the last equality defines \( c_n \).

From the proof of the previous lemma we have that \( \int c_n \leq 2\delta \) for all \( n \geq 1 \). Hence,

\[
2\delta \geq \int c_n \geq \int_{\cup_{k=1}^n G_k} c_n \geq \sum_{k=1}^n \int_{G_k} c_k \geq \left( 1 - \cos \frac{\eta}{2} \right) \int_{\cup_{k=1}^n G_k} |g|,
\]

since \( c_n \) is a nondecreasing sequence in \( n \).

**Lemma 12.** Let the sequence \( H = \{h_0, h_1, \ldots\} \) be defined as \( h_0 = 0 \),

\[
h_n = \tau |r_{n-1}'| - |T_{r_{n-1}}|, \quad n \geq 1.
\]

Then

\[
S_\infty \| H \| \leq \delta.
\]

**Proof.** First, an induction argument shows that

\[
\| T_{r_{k-1}} \| \leq \| g \| - S_{k-1} \| R \| - S_k \| H \|, \quad k \geq 1.
\]

Now

\[
S_{n-1} R + r_{n-1}' \sim S_{n-1} R + T_{r_{n-1}} \sim g.
\]
Hence
\[ M \leq S_{n-1}\|R\| + \|Tr_{n-1}\| \]
\[ \leq \|g\| - S_k\|H\| \]
\[ \leq M + \delta - S_k\|H\|, \]
which proves the lemma.

**Lemma 13.** Let
\[ a_n = \tau^n|g| - \sum_{k=1}^{n} \tau^{n-k}|r_k| - |r'_n| \]
and
\[ b_n = e^{i\theta}[\tau^n|g| - \sum_{k=1}^{n} \tau^{n-k}|r_k|] - r'_n. \]

Then
\[ a_n = \sum_{k=1}^{n} \tau^{n-k}r_k \]
and
\[ |b_n| \leq a_n + \eta \tau^n|g| \quad \text{a.e. on } (C \cap S) - G. \]

**Proof.** The first assertion follows directly from the definitions. For the second assertion, we first note that
\[ b_n = e^{i\theta}[\tau^n|g| - \sum_{k=1}^{n} \tau^{n-k}|r_k|] - [r'_n + |r_n|e^{i\theta}]. \]

On \((C \cap S) - G\) we have either \(\lambda_n = 0\) or \(\lambda_n = 1\). If \(\lambda_n = 0\) then \(r_n = 0\) and the phase of \(r'_n = Tr_{n-1}\) is in the interval \((\theta - \eta, \theta + \eta)\). But \(|r'_n| \leq \tau^n|g|\) and \(|b_n| \leq a_n + \eta \tau^n|g|\) follows. If \(\lambda_n = 1\) then \(r'_n = 0\) and \(|b_n| = a_n\).

**Proof of Lemma 9.** From the definitions and the previous lemma we have that
\[ |e^{i\theta}r^n|g| - T^n g| \leq |b_n| + 2 \sum_{k=1}^{n} \tau^{n-k}|r_k| \]
\[ \leq \eta \tau^n|g| + \sum_{k=1}^{n} \tau^{n-k}(h_k + 2|r_k|). \]
Therefore
\[ \sum_{k=0}^{n} |e^{i\theta}r^k|g| - T^k g| \leq \sum_{k=0}^{n} \tau^k[\eta|g| + S_\omega(H + 2|R|)]. \]
Let
\[ \Delta = \eta|g| + S_\omega(H + 2|R|). \]
Then
\[ \|A\| \leq \eta \|g\| + \delta + \frac{4\delta}{1 - \cos \frac{\eta}{2}}. \]

Choose \( \eta \), so that \( M\eta \leq \epsilon/4 \), \( 0 < \eta < \pi \) and choose \( \delta \) satisfying
\[ 0 < \delta < \frac{\epsilon}{16} \left( 1 - \cos \frac{\eta}{2} \right). \]

Then it is easily seen that \( \|A\| \leq \epsilon \) and \( \int_0^\infty |g| < \epsilon \).

**A general ergodic theorem.** We now apply Lemmas 8 and 9 and the results for positive contractions to prove the following general ergodic theorem, due to Chacon [6], [7]. This theorem gives a mutual generalization of the Dunford-Schwartz theorem [9] and the Chacon-Ornstein theorem [4]. We note that Chacon’s original paper [7] does not contain some details of the proof. A complete but rather complicated proof of the theorem appeared in [12].

**Theorem 6.** For any \( f \in L_1 \),

\[ (1) \lim \frac{S_n(f, T)/S_nP}{S_n(f, T)/S_nP} \]

exists (and is finite) a.e. on \( \{x | 0 < S_nP(x)\} \).

*Proof.* We can assume that \( 0 < S_nP \) a.e. Since \( |S_n(f, T)| \leq S_n(|f|, \tau) \), it is also clear that \( \limsup_{n, m \to \infty} (|S_n(f, T)|/S_nP) \) is finite a.e. and that the limit \( (1) \) exists a.e. on \( D \), the dissipative part of \( \tau \). If it fails to exist on a nonnegligible subset of the conservative part \( C \), then there exist a number \( \alpha < 0 \) and a nonnegligible set \( E \subset C \) such that

\[ (2) \limsup_{n, m \to \infty} \frac{|S_n(f, T)/S_nP|}{S_n(f, T)/S_nP} \geq \alpha \quad \text{a.e. on } E. \]

We can also assume the existence of a number \( \beta > 0 \) such that

\[ (3) \limsup_{n \to \infty} \frac{|S_n(f, T)|/S_nP}{S_n(f, T)/S_nP} \leq \beta \quad \text{a.e. on } E. \]

Note that \( (2) \) and \( (3) \) are valid for any \( g \sim f \).

Now \( (2) \) implies that, for any \( g \sim f \), \( \alpha/2 \leq \limsup_{n \to \infty} (S_n(|f|, \tau)/S_nP) \) a.e. on \( E \). Hence, by Theorem 1,

\[ \Omega \left( \frac{\beta}{2} \right) \geq \frac{1}{M + \delta} \Omega \left( \frac{\beta}{2} \right) \]

for any \( g \sim f \), and by Lemma 8, for any \( \delta > 0 \) one can find a \( g \sim f \) such that \( \|g\| \leq M + \delta \) and

\[ (4) \int_E |g| \geq \frac{1}{2} \Omega \left( \frac{\beta}{2} \right). \]
Now let $\varepsilon > 0$ be a fixed number and choose $\delta > 0$, $g, \Delta, G$ as they are given by Lemma 9. We also assume that $g$ satisfies (4).

Let $F = (E - G) \cap S$ where $S$ is the support of $g$. Then, by Lemma 9,

$$\sum_{k=0}^{n} |T^k g - e^{i\theta^*} T^k g| \leq \sum_{k=0}^{n} \tau^k \Delta$$

a.e. on $F$, with $\|\Delta\| \leq \varepsilon$ and

$$\int_F |g| \geq \int_F |g| - \varepsilon \geq \frac{1}{4} \Omega_\varepsilon(P) - \varepsilon$$  \hspace{1cm} (5)

We now have, a.e. on $F$,

$$\beta \geq \limsup_{n \to \infty} \left| \frac{S_n(g, T)}{S_n P} \right|$$

$$\geq \lim_{n \to \infty} \frac{S_n(|g|, \tau)}{S_n P} \left[ 1 - \limsup_{n \to \infty} \left| \frac{S_n(g, T)}{e^{i\theta^*} S_n(|g|, \tau)} - 1 \right| \right]$$

$$\geq \lim_{n \to \infty} \frac{S_n(|g|, \tau)}{S_n P} \left[ 1 - \lim_{n \to \infty} \frac{S_n(\Delta, \tau)}{S_n(|g|, \tau)} \right].$$  \hspace{1cm} (6)

Let

$$l = \lim_{n \to \infty} \frac{S_n(\Delta, \tau)}{S_n(|g|, \tau)}.$$

Then, from Theorem 1 and Lemma 7 it follows that

$$\int_F |g| \leq \|\Delta\| \leq \varepsilon.$$

Since $l \geq 0$, if $R = F \cap \{x \mid l(x) \leq \frac{1}{2}\}$ then

$$\frac{1}{2} \int_{F - R} |g| \leq \varepsilon$$

which, combined with (5) implies that

$$\int_R |g| \geq \frac{1}{4} \Omega_\varepsilon(P) - 3\varepsilon.$$

Now, by (6),

$$\lim_{n \to \infty} \frac{S_n(|g|, \tau)}{S_n P} \leq 2\beta \quad \text{a.e. on } R,$$

hence

$$\Omega_\varepsilon(P) \geq \frac{1}{2\beta} \Omega_\varepsilon(|g|) \geq \frac{1}{2\beta} \left[ \frac{1}{4} \Omega_\varepsilon(P) - 3\varepsilon \right].$$  \hspace{1cm} (7)
On the other hand, (2) implies that, a.e. on $R$,

$$\alpha \leq \limsup_{n,m \to \infty} \left| \frac{S_n(g, T)}{S_nP} - \frac{S_m(g, T)}{S_mP} \right|$$

$$= \lim_{n \to \infty} \frac{S_n(|g|, \tau)}{S_nP} \limsup_{n,m \to \infty} \left| \frac{S_n(g, T)}{e^{i\theta}S_n(|g|, \tau)} - \frac{S_m(g, T)}{e^{i\theta}S_m(|g|, \tau)} \right|$$

$$\leq 2 \lim_{n \to \infty} \frac{S_n(|g|, \tau)}{S_nP} \lim_{n \to \infty} \frac{S_n(\Delta, \tau)}{S_n(|g|, \tau)} = 2 \lim_{n \to \infty} \frac{S_n(\Delta, \tau)}{S_nP}.$$

Hence

$$\Omega_\varepsilon(P) \leq \frac{2}{\alpha} \Omega_\varepsilon(\Delta) \leq \frac{2\varepsilon}{\alpha}.$$  

But, if $\varepsilon$ is sufficiently small, (7) and (8) are incompatible, which means that the assumption (2) cannot be true and completes the proof.

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