INTRODUCTION. The purpose of this paper is to extend various known results in operator ergodic theory to give a direct approach to pointwise ergodic theorems. The main step in this approach is a maximal ergodic theorem (Theorem 1), which is a generalization of the result given in [1]. A corollary (Theorem 2) of this theorem is Chacon's ergodic theorem [6] for positive operators which contains both Birkhoff's ergodic theorem [2] (or, more generally, Dunford-Schwartz's theorem [9] for positive contractions) and Chacon-Ornstein's ratio ergodic theorem [4]. Theorem 1 is also used to obtain the identification of the limit in a straightforward way. In the final part of the paper Theorem 2 is generalized to nonpositive operators to give a direct proof of Chacon's general ergodic theorem [6].

DEFINITIONS AND BASIC LEMMAS. Let \((X, \mathcal{F}, \mu)\) be a \(\sigma\)-finite measure space and \(L_1 = L_1(X, \mathcal{F}, \mu)\) the Banach space of equivalence classes of integrable complex valued functions on \(X\). The elements of \(L_1\) will be identified by their representative functions and the relations between them will be considered holding up to sets of zero measure. \(L_1^+\) is the positive cone of \(L_1\), consisting of nonnegative \(L_1\)-functions. All subsets of \(X\) considered in this paper are measurable either by assumption or construction.

Let \(T\) be a contraction on \(L_1\), i.e., a linear operator \(T : L_1 \to L_1\) for which \(\|T\| \leq 1\) with the usual definition of the norm of an operator on a Banach space. We denote by \(\tau\) a positive contraction on \(L_1\), i.e., a contraction such that \(\tau L_1^+ \subset L_1^+\).

If \(F = \{f_0, f_1, f_2, \ldots\}\) is a sequence of complex valued functions on \(X\) and \(\alpha\) is a complex number then \(\alpha F = \{\alpha f_0, \alpha f_1, \alpha f_2, \ldots\}\), \(S_n F = \sum_{k=0}^{\infty} f_k, n \geq 0\) and \(S_n \alpha F = \sum_{k=0}^{\infty} \alpha f_k\) (a.e.), if it exists. For an \(L_1\)-function \(f\), let \(f_\tau = \{f, Tf, T^2f, \ldots\}\) and \(S_n(f, T) = S_n f_\tau\).

DEFINITION 1. A sequence \(P = \{p_0, p_1, p_2, \ldots\}\) of nonnegative, finite and measurable functions on \(X\) is called \(T\)-admissible (or, simply, admissible) if \(f \in L_1, n \geq 0\) and \(|f| \leq p_n\) imply that \(|Tf| \leq p_{n+1}\).

Admissible sequences have been introduced by Chacon [6], to give a mutual generalization (Theorem 6 of the present paper) of the Chacon-Ornstein theorem [4] and the Hopf-Dunford-Schwartz theorem [9]. Note that if \(f \in L_1^+\) then \(f_\tau = \{f, \tau f, \tau^2 f, \ldots\}\) is a \(\tau\)-admissible sequence. For other examples of admissible sequences we refer to [6].

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Throughout this paper $P = \{p_0, p_1, p_2, \ldots\}$ will denote an admissible sequence. We note that, if $\alpha \geq 0$, $\alpha P$ is also an admissible sequence. Also, if $f \in L_1$ and $|f| \leq \sum_{n=0}^{\infty} p_n (0 \leq n \leq n')$ then $|Tf| \leq \sum_{n=0}^{n+1} p_n$.

The following lemma was first given, in a slightly different form, in [4]. In the following we indicate an outline of its proof and refer the reader for a complete proof to [4], [11], and [12].

**Lemma 1.** For any $f \in L_1$, $\lim_{n \to \infty} (\tau^n f|S_n P) = 0$ a.e. on the set $\{x \mid 0 < S_\infty P(x)\}$.

**Proof.** Assume that $f \in L_1^+$ and let, for a fixed $\epsilon > 0$, $E_n$ be the support of $(\tau^n f - \epsilon S_n P)^+$. Then one can obtain that

$$\epsilon \sum_{n=1}^{\infty} \int_{E_n} p_0 < \infty,$$

from which the proof follows.

Let $E \in \mathcal{F}$. Associated with $\tau$, $P$, and $E$ define a possibly finite sequence

$$A = \{a_0, a_1, a_2, \ldots\}$$

of measurable functions as follows:

$$a_0 = \chi p_0,$$

$$a_n = \chi \left[ p_n - \sum_{k=1}^{n} \tau^k a_{n-k} \right], \quad n \geq 1,$$

where $\chi$ is the characteristic function of $E$. An induction argument shows that the functions $a_n$ are nonnegative. The sequence $A$ is infinite if it consists of $L_1$-functions. Otherwise it terminates with the first function which does not belong to $L_1$.

**Definition 2.** We let $Q_E(P) = \int_k \int_{E_k} a_k (0 \leq \Omega_E(P) \leq \infty)$ where $\{a_0, a_1, \ldots\}$ is given by (*) and the summation is taken over the set of indices $k$ for which $a_k$ is defined. If $f \in L_1^+$ we write $\Omega_E(f)$ instead of $Q_E(f)$.

We note that $\Omega_E(\alpha P) = \alpha \Omega_E(P)$ for any $\alpha \geq 0$, $\Omega_E(f) \leq \|f\|$ for any $f \in L_1^+$ and $\Omega_E(P) = 0$ if and only if $S_\infty P = 0$ a.e. on $E$.

**A maximal ergodic theorem.** The following theorem, which will be fundamental for the rest of this paper, is an extension of an ergodic lemma [1] to admissible sequences. Since its proof is similar to the proof given in [1], here we give only an outline.

**Theorem 1.** Let $E \in \mathcal{F}$ and consider two $\tau$-admissible sequences $P = \{p_0, p_1, \ldots\}$.
and \( Q = \{ q_0, q_1, \ldots \} \). Then \( \limsup_{n \to \infty} (S_n P - S_n Q) \geq 0 \) a.e. on \( E \) implies that \( \Omega_\xi(P) \geq \Omega_\xi(Q) \).

For the proof we need the following

**Lemma 2.** Let \( E \in \mathcal{F} \) and let \( A' = \{ a'_0, a'_1, \ldots \} \) be a sequence of nonnegative measurable functions such that \( a'_0 \leq \chi P_0 \) and if \( a'_0, a'_1, \ldots, a'_{n-1} \in L_1 \) then

\[
d_n \leq \chi \left( p_n - \sum_{k=1}^{n} \tau^k a_{n-k} \right),
\]

where \( \chi \) is the characteristic function of \( E \). Then \( \int S_\infty A' \leq \Omega_\xi(P) \).

**Proof.** Assuming \( \Omega_\xi(P) < \infty \) we can define

\[
b_n = p_n - \sum_{k=0}^{n} \tau^k a_{n-k}, \quad n \geq 0.
\]

An induction argument shows that a corresponding sequence

\[
b'_n = p_n - \sum_{k=0}^{n} \tau^k a'_{n-k}, \quad n \geq 0
\]

is also defined and \((b'_n - b_n)'s\) are nonnegative \( L_1\)-functions for all \( n \geq 0 \). Now consider \( G_n = \int [ (S_n A - S_n A') - (b_n' - b_n) ] \). Then one obtains that \( G_0 = 0 \) and \( G_n \leq G_{n+1} \) for all \( n \geq 0 \) which shows that \( 0 \leq G_n \), or \( \int S_\infty A' \leq \int S_\infty A \) for all \( n \geq 0 \).

**Proof of Theorem 1.** Let \( \chi \) be the characteristic function of \( E \). Assuming \( \Omega_\xi(P) < \infty \), we define a sequence \( R = \{ r_0, r_1, r_2, \ldots \} \) as follows:

\[
r_0 = \chi E_0 \land q_0, \\
r_n = \chi \left( p_n - \sum_{k=0}^{n-1} \tau^k r_k \right) \land (q_0 - S_{n-1} R), \quad n \geq 1.
\]

Lemma 2 shows that these functions are in \( L_1^+ \) and \( \int S_\infty R \leq \Omega_\xi(P) \). Using the hypotheses on \( P \) and \( Q \) one obtains that \( \int S_\infty R = \chi q_0 \), hence \( \chi q_0 \leq \Omega_\xi(P) \).

Let \( C = \{ c_0, c_1, c_2, \ldots \} \) be the sequence associated with \( \tau, Q, \) and \( E \), as defined by \((*)\), so that \( \Omega_\xi(Q) = \sum c_k \int c_k \). In the previous paragraph we obtained that \( \int c_0 \leq \Omega_\xi(P) \). An induction argument, analogous to that given in [1], shows that, if \( \int S_n C \leq \Omega_\xi(P) \) then \( \int S_{n+1} C \leq \Omega_\xi(P) \), hence completes the proof of the theorem.

A ratio ergodic theorem for admissible sequences is a direct consequence of Theorem 1.
Theorem 2. Let $P$ and $Q$ be two $\tau$-admissible sequences and $E \in \mathcal{F}$. Then $\Omega_\tau(Q) < \infty$ implies that $\lim_{n \to \infty} (S_nQ/S_nP)$ exists (and is finite) a.e. on $E' = E \cap \{x \mid S_\infty P(x) > 0\}$.

Before the proof we note that this theorem will only be used for the special case where $Q = f$, $f \in L^1$. The theorem is stated for a general admissible sequence $Q$ for reasons of symmetry, since this does not add any difficulty.

The following proof is analogous to that given in [3] for the proof of the Chacon-Ornstein theorem.

Proof of Theorem 2. If $\limsup_{n \to \infty} (S_nQ/S_nP)$ is infinite a.e. on a nonnegligible subset $G$ of $E'$, then Theorem 1 gives that $\Omega_{\tau}(Q)^0a(QG)$ for all $a \geq 0$. This is a contradiction, since $QG(\sigma) < \infty$ by Lemma 2 and $QG(P) > 0$.

If $\lim_{n \to \infty} (S_nQ/S_nP)$ does not exist on a nonnegligible subset $H$ of $E'$ then one can find two numbers $\alpha < \beta$ and an $H' \subset H$ with $\mu(H') > 0$ such that

$$\liminf_{n \to \infty} (S_nQ/S_nP) \leq \alpha < \beta \leq \limsup_{n \to \infty} (S_nQ/S_nP)$$

a.e. on $H'$. By Theorem 1, this implies that $\Omega_{\tau}(Q)^0a_{\tau}(H)(P)$ and

$$\alpha \Omega_{\tau}(P) \geq \Omega_{\tau}(Q)$$

which is a contradiction.

Identification of the limit. A positive contraction $\tau$ induces a decomposition of $X$ into two parts of different characters. We state this result, which is due to Hopf [10] and Chacon [5] in the following form:

Theorem 3. Let $\tau$ be a positive contraction on $L_1 = L(X, \mathcal{F}, \mu)$. Then $X$ can be written as the union of two disjoint sets $D$ and $C$, called the dissipative and conservative parts respectively, with the following properties:

(i) For any $f \in L^1_\tau$, $S_\infty(f, \tau) < \infty$, a.e. on $D$.
(ii) For any $f \in L^1_\tau$, $S_\infty(f, \tau) = \infty$ or $0$ a.e. on $C$.
(iii) If $f \in L_1$ and $f = 0$ a.e. on $D$ then $\tau f = 0$ a.e. on $D$.
(iv) If $f \in L_1$ and $f = 0$ a.e. on $D$ then $\|\tau f\| = \|f\|$.

Proof. Let $f \in L_1$, $f > 0$ and

$$C = \{x \mid (S_\infty(f, \tau))(x) = \infty\},$$

$$D = \{x \mid (S_\infty(f, \tau))(x) < \infty\},$$

which are defined up to sets of measure zero. Then Theorem 2 implies that $D$ and $C$ have the properties (i) and (ii).
Now assume that (iii) is not true. Then one can find a function $f \in L^+_1$, $f=0 \text{ a.e.}$ on $D$ and a nonnegligible subset $G$ of $D$ and two numbers $\alpha, \beta$ a.e. on $G$ and $S_\alpha(f, \tau) \leq \beta$ a.e. on $G$. Since $S_\alpha(f, \tau) = \infty$ a.e. on the support of $f$, one can choose $n$ large enough so that $S_n(f, \tau) \geq 2\beta/\alpha$ a.e. except on a set $H$ with $\int_H f \leq 1/\alpha \mu(G)$. Then it follows that $S_{n+1}(f, \tau) > \beta$ on a nonnegligible subset of $G$, which is a contradiction and proves (iii).

Finally assume that (iv) is not true. Then there exists a function $f \in L^+_1$, $f=0 \text{ a.e.}$ on $D$ such that $\|f\| - \|\tau f\| = \lambda > 0$. Now choose $n$ large enough so that $S_n(f, \tau) \geq 2\|f\|\lambda$ a.e. except on a set $H$ with $\int_H f \leq 1/\lambda$. Then

$$\|S_n(f, \tau)\| - \|\tau S_n(f, \tau)\| > \|f\|.$$  

But this is a contradiction, since $\sum_{k=0}^\infty \|\tau^k f\| = \|f\|$.

**Definition 3.** A (measurable) subset $E$ of $X$ is invariant (with respect to $\tau$) if $f \in L^+_1$ and $f=0 \text{ a.e.}$ on $X-E$ imply that $\tau f=0 \text{ a.e.}$ on $X-E$.

Note that the previous theorem gives that $C$ is invariant.

The following two lemmas formulate the recurrence properties of the conservative part. Using these results we will obtain a simple interpretation of $\Omega_b(P)$ (Lemma 5).

**Lemma 3.** Let $E$ be a (measurable) subset of $C$, the conservative part, and let $f \in L^+_1$, $f=0 \text{ a.e.}$ on $X-E$. Then $\|\tau f\| = \|f\|$ for all $n \geq 0$.

**Proof.** For $n=0$ the assertion is trivial. We first show that $\Omega_b(\tau f) = \|f\|$.

Let $\chi$ and $\chi'$ be the characteristic functions of $E$ and $X-E$ respectively and let $R(\tau f, \tau, E) = \{g_0, g_1, \ldots\}$ and $R'(\tau f, \tau, E) = \{g'_0, g'_1, \ldots\}$ be the sequences defined as

$$g_0 = \chi \tau f, \quad g'_0 = \chi' \tau f,$$

$$g_n = \chi \tau g_{n-1}, \quad g'_n = \chi' \tau g'_{n-1}, \quad n \geq 1,$$

and

$$\tau_b f = S_\infty[R(\tau f, \tau, E)].$$

It is easy to check that $\tau_b$ can be extended linearly to the $L_1$ space $L_1(E)$ of integrable functions with support in $E$ and be considered as a positive contraction on this space. Note that $\Omega_b(\tau f) = \|\tau_b f\|$. An induction argument shows that, for all $n \geq 0$, $S_n(f, \tau) \leq S_n(f, \tau_b)$ a.e. on $E$. Hence $\tau_b$ is conservative on $E$ and

$$\Omega_b(\tau f) = \|\tau_b f\| = \|f\|.$$

Now assume that $\Omega_b(\tau^N f) = \|f\|$ for $n=0, 1, \ldots, N$. Then

$$\Omega_b(\tau^{N+1} f) = \Omega_b(\chi \tau^N f) + \Omega_b(\chi' \tau^N f) = \|\chi \tau^N f\| + \Omega_b(\chi' \tau^N f) = \Omega_b(\tau^N f) = \|f\|.$$

Here we use the fact that $\Omega_b(g) = \Omega_b(\tau g)$ if $g=0 \text{ a.e.}$ on $E$. 

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Definition 4. Let $E$ be a subset of the conservative part $C$ and let $\mathcal{F}_E$ be the class of all invariant subsets of $C$ which contain $E$. Then the measure-theoretic intersection $I(E)$ of the elements of $\mathcal{F}_E$ is the smallest invariant set containing $E$.

Note that $I(E)$ is defined up to a set of measure zero and is an invariant set. If $g$ is an $L^+_1$ function whose support is equal to $E$, then $S_\omega(g, \tau) = \infty$ a.e. on $I(E)$, and $S_\omega(g, \tau) = 0$ a.e. on $X - I(E)$.

Lemma 4. Let $E$ be a subset of $C$ and let $f$ be an $L^+_1$ function which has support in $I(E)$. Then $\Omega_E(f) = \|f\|$.

Proof. First observe that if $0 \leq f \leq u$, $u \in L_1$ and $\Omega_E(u) = \|u\|$ then $\Omega_E(f) = \|f\|$. Now let $g \in L^+_1$ and let the support of $g$ be equal to $E$. Then for any $\varepsilon > 0$ there exists an $n$ such that $S_n(g, \tau) \leq \|f\| - \varepsilon$. Since $\Omega_E(S_n(g, \tau)) \geq \|S_n(g, \tau)\|$, we then have $\Omega_E(f) \leq \|f\| - \varepsilon$, which proves the lemma.

Lemma 5. If $E$ is a subset of the conservative part $C$, then, for any $\tau$-admissible sequence $P$,

$$\Omega_E(P) = \lim_{n \to \infty} \int_{I(E)} p_n.$$  

Proof. First we have, directly from the definitions, that

$$\Omega_E(P) \leq \lim_{n \to \infty} \int_{I(E)} p_n.$$  

Hence, if $\Omega_E(P) = \infty$ the proof is complete. Otherwise the previous lemma gives that $\Omega_E(P) \leq \Omega_E(p'_n)$ where $p'_n = \chi_T p_n \in L^+_1$, $\chi_T$ being the characteristic function of $I(E)$. This implies the conclusion of the lemma.

Lemma 6. The invariant subsets of $C$ form a $\sigma$-field $\mathcal{G}$ with respect to $C$.

Proof. The only nontrivial step of the proof is to show that if $I \subseteq C$ is invariant then $I' = C - I$ is also invariant. Let $\tau_{I'}$ be defined as in the proof of Lemma 3. If $I'$ is not invariant one can find a function $f \in L^+_I$, $f = 0$ a.e. on $X - I'$ such that $\|\tau_{I'} f\| < \|f\|$. But this contradicts the fact that $\tau_{I'}$ is conservative.

Lemma 7. Let $\Omega_C(Q) < \infty$. Then the restriction of $h = \lim_{n \to \infty} (S_n Q / S_n P)$ to $C$ is an $\mathcal{G}$-measurable function.

Proof. Let $E_\alpha = \{x \mid h(x) \geq \alpha\} \cap C$. If $I(E_\alpha) \cap (C - E_\alpha)$ has nonzero measure, then there exists an $\varepsilon > 0$ such that $I(E_\alpha) \cap (C - E_\alpha - \varepsilon) = G$ has also nonzero measure. Let $H = I(G) \cap E_\alpha$. Then it follows that $I(H) = I(G)$, or $\Omega_C(\cdot) = \Omega_H(\cdot)$. Now on $H$, $\lim_{n \to \infty} (S_n Q / S_n P) \geq \alpha$ which implies that $\Omega_H(Q) \geq \alpha \Omega_H(P)$ and from an analogous
consideration for $G$ one obtains that $\Omega_o(Q) \leq (\alpha - \epsilon)\Omega_o(P)$. But this is a contradiction and shows that $E_\alpha \in \mathcal{F}$.

A standard approximation procedure of measurable functions by simple functions leads to the following result.

**Theorem 4.** Let $I$ be an invariant subset of $C$, $P$ and $Q$ be two admissible sequences, such that $(S_\infty P) > 0 \ a.e. \ on \ I$ and $\Omega_\delta(P) < \infty$, $\Omega_\delta(Q) < \infty$. Then

$$\lim_{n \to \infty} \int_I q_n = \lim_{n \to \infty} \int_I h\rho_n$$

with $h = \lim_{n \to \infty} (S_n Q / S_n P)$.

**Corollary.** If $\mu(C) < \infty$, $\Omega_\delta(Q) < \infty$ and $S_\infty(P) > 0 \ a.e. \ on \ C$, then

$$\lim_{n \to \infty} (S_n Q / S_n P) = \lim_{n \to \infty} E[\rho_n] \ \ a.e. \ on \ C.$$ 

**Nonpositive operators.** We now consider a general, not necessarily positive contraction $T$. The following result [8], due to Chacon and Krengel, shows that $T$ can always be dominated, in a certain sense, by a positive contraction $\tau$.

**Theorem 5.** For every bounded linear operator $T : L_1 \to L_1$ there is a unique bounded, linear and positive operator $\tau : L_1 \to L_1$ such that

(i) $\|\tau\| \leq \|T\|$.  
(ii) For all $f \in L_1$, $|Tf| \leq |f|$.  
(iii) If $f \in L_1^+$ then $rf = \sup_{|\xi| \leq f, \xi \in L_1} |T\xi|$.

**Proof.** Let $\mathcal{P}$ be the class of finite (measurable) partitions $\Pi = [E_1, \ldots, E_n]$ of $X$, partially ordered in the usual way. For any $\Pi = [E_1, \ldots, E_n] \in \mathcal{P}$ and $f \in L_1^+$ let

$$\tau_{nf} = \sum_{k=1}^{n} |T\chi_k f|$$

where $\chi_k$ is the characteristic function of $E_k$, $k = 1, \ldots, n$. Also, let $\Pi_n$ be a sequence of nondecreasing partitions such that

$$\lim_{n \to \infty} \|\tau_{nf}\| = \sup_{n \in \mathcal{P}} \|\tau_{nf}\|.$$ 

One then defines $\tau f = \lim_{n \to \infty} \tau_{nf}$ (a.e.) and obtains the transformation $\tau$ of the theorem as the unique linear extension of $\tau'$ to $L_1$. For further details we refer the reader to [8].

**Definition 5.** The operator $\tau$ as given by Theorem 5, is called the linear modulus of $T$. 

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For the rest of this paper $\tau$ will denote the linear modulus of $T$. The quantity $\Omega_E(\cdot)$ is defined, as before, in terms of $\tau$. We note that a sequence $P = \{p_0, p_1, \ldots\}$ is $T$-admissible if and only if it is $\tau$-admissible.

**Definition 6.** Two $L_1$ functions $f$ and $g$ are called equivalent (with respect to $T$), in notation $f \sim g$, if there exists a strictly positive $L_1$ function $F$, such that

$$\lim_{n \to \infty} \sup \frac{|S_n(f-g, T)|}{S_n(F, \tau)} = 0 \text{ a.e. on } C$$

where $C$ is the conservative part of $X$ with respect to $\tau$.

It is clear that this relation is actually an equivalence relation. We also note that, if $f \sim g$ then $f + h \sim g + h$ for any $h \in L_1$ and also that $T^nf \sim f$ for all $n \geq 1$ by Lemma 1.

**Definition 7.** The nonnegative number

$$M = M(f) = \inf_{g \sim f} \|g\|$$

is the minimal norm of $f$ (with respect to $T$).

In terms of these concepts we first note the following

**Lemma 8.** If $\Omega_E(|g|) \leq a$ for all $g \sim f$ then for any $\lambda > 0$, $\eta > 0$ there exists an $h \sim f$ such that

$$\|h\| \leq M + \lambda \quad \text{and} \quad \int_E |h| \geq a - \eta$$

where $M$ is the minimal norm of $f$.

**Proof.** Let $0 < \epsilon < \min \left(\lambda, \frac{1}{4\eta}\right)$, $\|g\| \leq M + \epsilon$, $g \sim f$ and consider

$$\{\alpha_0, \alpha_1, \ldots\} = R(g, T, E),$$
$$\{\beta_0, \beta_1, \ldots\} = R'(g, T, E),$$
$$\{\gamma_0, \gamma_1, \ldots\} = R(|g|, \tau, E),$$
$$\{\delta_0, \delta_1, \ldots\} = R'(|g|, \tau, E),$$

where $R$ and $R'$ are as defined in the proof of Lemma 3.

Since $\Omega_E(|g|) = \sum_{k=0}^\infty \|\gamma_k\| \geq a$, there exists an $n$ such that $\sum_{k=0}^n \|\gamma_k\| \geq a - \frac{1}{4}\eta$. Let $h = \sum_{k=0}^n \alpha_k + \beta_n$, which is equivalent to $f$. Then, by simple inductions we obtain that

$$(M \leq) \|h\| \leq \|g\| \leq M + \epsilon$$

and

$$M \leq \sum_{k=0}^n \|\alpha_k\| + \|\beta_n\| \leq \sum_{k=0}^n \|\gamma_k\| + \|\delta_n\| \leq M + \epsilon$$

and also $\|\beta_n\| \leq \|\delta_n\|$, $\sum_{k=0}^n \|\alpha_k\| \leq \sum_{k=0}^n \|\gamma_k\|$ which imply that

$$\int_E |h| = \left\| \sum_{k=0}^n \alpha_k \right\| \geq \sum_{k=0}^n \|\gamma_k\| - 2\epsilon \geq a - \eta.$$
The following lemma shows that if the norm of an $L_1$-function $f$ is very close to its minimal norm, then the action of $T$ on this function can be described in terms of the action of $\tau$ on $|f|$.

**Lemma 9.** Let $f \in L_1$ and let $M$ be its minimal norm. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property. If $g \sim f$ and $\|g\| \leq M + \delta$ then one can find a function $\Delta \in L_1^*$ with $\|\Delta\| < \varepsilon$ and a set $G$ with $\int_G |g| < \varepsilon$ such that, for any $n \geq 0$,

$$\sum_{k=0}^{n} |T^k g - e^{i\theta} r^k| g| \leq \sum_{k=0}^{n} r^k \Delta$$

a.e. on $(S \cap C) - G$, where $S$ is the support of $g$ and $\theta : S \to (-\pi, \pi]$ is the phase of $g$.

The proof of this theorem, although quite straightforward, is rather long and will be divided into several sublemmas.

Let $\eta, 0 < \eta < \pi$, and $\delta > 0$ be fixed and $g \sim f$, $\|g\| \leq M + \delta$. Define two sequences $R = \{r_0, r_1, r_2, \ldots\}$ and $R' = \{r'_0, r'_1, r'_2, \ldots\}$ of $L_1$ functions:

$$\begin{align*}
r_0 &= 0, & r'_0 &= g, \\
r_n &= \lambda_n \tau_{n-1}, & r'_n &= (1 - \lambda_n) \tau_{n-1}, & n \geq 1,
\end{align*}$$

where $\lambda_n : X \to [0,1]$ for all $n \geq 1$, such that $\lambda_n = 0$ on the set on which $g = 0$ or $\tau_{n-1} = 0$ or $\theta - \eta < \text{ph} \tau_{n-1} < \theta + \eta$ (mod $2\pi$), where "ph" denotes the phase of its argument. Outside of this set $\lambda_n$ is defined by the following (pointwise) relation

$$\lambda_n = \sup \{t \mid 0 \leq t \leq 1, \theta - \eta/2 \leq \text{ph} [g + \tau_{n-1} R + t \tau_{n-1}] \leq \theta + \eta/2 \text{ (mod } 2\pi)\}.$$

In what follows, if

$$A = \{a_0, a_1, a_2, \ldots\}$$

is a sequence of functions, $|A|$ and $\|A\|$ will denote the sequences

$$\{|a_0|, |a_1|, |a_2|, \ldots\} \quad \text{and} \quad \{|a_0|, \|a_1\|, \|a_2\|, \ldots\}$$

respectively.

**Lemma 10.** $(1 - \cos (\eta/2))S_n \|R\| \leq 2\delta$.

**Proof.** First, an induction argument shows that $S_n \|R\| + \|r'_n\| \leq M + \delta$. But we also have that $S_n R + r'_n \sim g$, which combined with the first inequality, gives that

$$\|g\| + S_n \|R\| - \|g + S_n R\| \leq 2\delta$$

or

$$\sum_{k=1}^{n} \left[\|g + S_{k-1} R\| + \|r_k\| - \|g + S_k R\|\right] \leq 2\delta.$$
Now, at almost every point on the support of \( r_k \), the difference between the phases of \( g + S_k R \) and \( r_k \) is at least \( \eta/2 \). Hence

\[
|g + S_k R| \leq |g + S_{k-1} R| + |r_k| \cos \frac{\eta}{2}
\]

or

\[
\left( 1 - \cos \frac{\eta}{2} \right) S_n \| R \| \leq 2\delta
\]

for all \( n \geq 0 \).

**Lemma 11.** Let \( G \) be the part of the support of \( g \) on which \( \text{ph} (g + S_n R) = \theta - \eta/2 \) or \( \theta + \eta/2 \) for at least one \( n \geq 0 \). Then

\[
(1 - \cos (\eta/2)) \int_G |g| \leq 2\delta.
\]

**Proof.** Write \( G \) as the union of disjoint sets \( G_1, G_2, \ldots \) such that on \( G_n \) the phase of \( g + S_k R \) is in the (open) interval \((\theta - \eta/2, \theta + \eta/2)\) for all \( k = 0, 1, \ldots, n - 1 \), but the phase of \( g + S_n R \) is \( \theta - \eta/2 \) or \( \theta + \eta/2 \).

Hence on \( G_n \):

\[
|g + S_n R| \leq |g| \cos \frac{\eta}{2} + S_n |R|
\]

or

\[
\left( 1 - \cos \frac{\eta}{2} \right) |g| \leq |g| + S_n |R| - |g + S_n R| = c_n
\]

where the last equality defines \( c_n \).

From the proof of the previous lemma we have that \( \int c_n \leq 2\delta \) for all \( n \geq 1 \). Hence,

\[
2\delta \geq \int_{U_{k+1}} c_n \geq \sum_{k=1}^{n} \int_{\partial_k} c_k \geq \left( 1 - \cos \frac{\eta}{2} \right) \int_{U_n} |g|,
\]

since \( c_n \) is a nondecreasing sequence in \( n \).

**Lemma 12.** Let the sequence \( H = \{h_0, h_1, \ldots\} \) be defined as \( h_0 = 0 \),

\[
h_n = \tau |r'_{n-1}| - |T_{r'_{n-1}}|, \quad n \geq 1.
\]

Then

\[
S_{\infty} \| H \| \leq \delta.
\]

**Proof.** First, an induction argument shows that

\[
\| T_{r'_{k-1}} \| \leq \| g \| - S_{k-1} \| R \| - S_k \| H \|, \quad k \geq 1.
\]

Now

\[
S_{n-1} R + r'_{n-1} \sim S_{n-1} R + T_{r'_{n-1}} \sim g.
\]
Hence
\[ M \leq S_{n-1}\|R\| + \|Tr_{n-1}\| \]
\[ \leq \|g\| - S_k\|H\| \]
\[ \leq M + \delta - S_k\|H\|, \]
which proves the lemma.

**Lemma 13.** Let
\[ a_n = \tau^n|g| - \sum_{k=1}^{n} \tau^{n-k}|r_k| - |r'_n| \]
and
\[ b_n = e^{i\theta} \left( \tau^n|g| - \sum_{k=1}^{n} \tau^{n-k}|r_k| \right) - r'_n. \]
Then
\[ a_n = \sum_{k=1}^{n} \tau^{n-k}h_k \]
and
\[ |b_n| \leq a_n + \eta \tau^n|g| \text{ a.e. on } (C \cap S) - G. \]

**Proof.** The first assertion follows directly from the definitions. For the second assertion, we first note that
\[ b_n = e^{i\theta} \left( \tau^n|g| - \sum_{k=1}^{n-1} \tau^{n-k}|r_k| \right) - [r'_n + |r_n|e^{i\theta}]. \]
On \((C \cap S) - G\) we have either \(\lambda_n = 0\) or \(\lambda_n = 1\). If \(\lambda_n = 0\) then \(r_n = 0\) and the phase of \(r'_n = Tr_{n-1}\) is in the interval \((\theta - \eta, \theta + \eta)\). But \(|r'_n| \leq \tau^n|g|\) and \(|b_n| \leq a_n + \eta \tau^n|g|\) follows. If \(\lambda_n = 1\) then \(r'_n = 0\) and \(|b_n| = a_n\).

**Proof of Lemma 9.** From the definitions and the previous lemma we have that
\[ |e^{i\theta} \tau^n|g| - T^n g| \leq |b_n| + 2 \sum_{k=1}^{n} \tau^{n-k}|r_k| \]
\[ \leq \eta \tau^n|g| + \sum_{k=1}^{n} \tau^{n-k}(h_k + 2|r_k|). \]
Therefore
\[ \sum_{k=0}^{n} |e^{i\theta} \tau^k|g| - T^k g| \leq \sum_{k=0}^{n} \tau^k[|g| + S_\alpha(H + 2|R|)]. \]
Let
\[ \Delta = \eta |g| + S_\alpha(H + 2|R|). \]
Then
\[ \|\Delta\| \leq \eta \|g\| + \delta + \frac{4\delta}{1 - \cos \frac{\eta}{2}}. \]

Choose \( \eta \), so that \( M\eta \leq \varepsilon/4 \), \( 0 < \eta < \pi \) and choose \( \delta \) satisfying
\[ 0 < \delta < \frac{\varepsilon}{16} \left(1 - \cos \frac{\eta}{2}\right). \]

Then it is easily seen that \( \|\Delta\| \leq \varepsilon \) and \( \int_E |g| < \varepsilon. \)

A general ergodic theorem. We now apply Lemmas 8 and 9 and the results for positive contractions to prove the following general ergodic theorem, due to Chacon [6], [7]. This theorem gives a mutual generalization of the Dunford-Schwartz theorem [9] and the Chacon-Ornstein theorem [4]. We note that Chacon's original paper [7] does not contain some details of the proof. A complete but rather complicated proof of the theorem appeared in [12].

**Theorem 6.** For any \( f \in L_1 \),
\[ (1) \quad \lim_{n \to \infty} (S_n(f, T)/S_nP) \]
exists (and is finite) a.e. on \( \{x \mid 0 < S_nP(x)\} \).

**Proof.** We can assume that \( 0 < S_nP \) a.e. Since \( |S_n(f, T)| \leq S_n(|f|, \tau) \), it is also clear that \( \limsup_{n \to \infty} (|S_n(f, T)|/S_nP) \) is finite a.e. and that the limit (1) exists a.e. on \( D \), the dissipative part of \( \tau \). If it fails to exist on a nonnegligible subset of the conservative part \( C \), then there exist a number \( \alpha < 0 \) and a nonnegligible set \( E \subset C \) such that
\[ (2) \quad \limsup_{n, m \to \infty} \left| (S_n(f, T)/S_nP) - (S_m(f, T)/S_mP) \right| \geq \alpha \quad \text{a.e. on } E. \]

We can also assume the existence of a number \( \beta > 0 \) such that
\[ (3) \quad \limsup_{n \to \infty} (|S_n(f, T)|/S_nP) \leq \beta \quad \text{a.e. on } E. \]

Note that (2) and (3) are valid for any \( g \sim f \).

Now (2) implies that, for any \( g \sim f \), \( \alpha/2 \leq \limsup_{n \to \infty} (S_n(|f|, \tau)/S_nP) \) a.e. on \( E \). Hence, by Theorem 1,
\[ \Omega_g(|g|) \geq \frac{1}{2} \alpha \Omega_g(P) \quad (>0) \]
for any \( g \sim f \), and by Lemma 8, for any \( \delta > 0 \) one can find a \( g \sim f \) such that \( \|g\| \leq M + \delta \) and
\[ (4) \quad \int_E |g| \geq \frac{1}{2} \alpha \Omega_g(P). \]
Now let \( \varepsilon > 0 \) be a fixed number and choose \( \delta > 0, g, \Delta, G \) as they are given by Lemma 9. We also assume that \( g \) satisfies (4).

Let \( F = (E - G) \cap S \) where \( S \) is the support of \( g \). Then, by Lemma 9,

\[
\sum_{k=0}^{n} |T^k g - e^{i\theta} T^k g| \leq \sum_{k=0}^{n} \tau^k \Delta
\]
a.e. on \( F \), with \( \|\Delta\| \leq \varepsilon \) and

(5) \[
\int_F |g| \geq \int_F |g| - \varepsilon \geq \frac{1}{4} \Omega_{\varepsilon}(P) - \varepsilon
\]

We now have, a.e. on \( F \),

(6) \[
\lim_{n \to \infty} \frac{S_n(|g|, \tau)}{S_n P} \left[ 1 - \lim_{n \to \infty} \frac{S_n(g, T)}{e^{i\theta} S_n(|g|, \tau)} - 1 \right] \geq \lim_{n \to \infty} \frac{S_n(|g|, \tau)}{S_n P} \left[ 1 - \lim_{n \to \infty} \frac{S_n(\Delta, \tau)}{S_n(|g|, \tau)} \right].
\]

Let

\[
I = \lim_{n \to \infty} \frac{S_n(\Delta, \tau)}{S_n(|g|, \tau)}.
\]

Then, from Theorem 1 and Lemma 7 it follows that

\[
\int_F |g| \leq \|\Delta\| \leq \varepsilon.
\]

Since \( I \geq 0 \), if \( R = F \cap \{x \mid l(x) \leq \frac{1}{2}\} \) then

\[
\frac{1}{2} \int_{F - R} |g| \leq \varepsilon
\]

which, combined with (5) implies that

\[
\int_R |g| \geq \frac{1}{4} \Omega_{\varepsilon}(P) - 3\varepsilon.
\]

Now, by (6),

\[
\lim_{n \to \infty} \frac{S_n(|g|, \tau)}{S_n P} \leq 2\beta \quad \text{a.e. on } R,
\]

hence

(7) \[
\Omega_{\varepsilon}(P) \geq \frac{1}{2\beta} \Omega_{\varepsilon}(|g|) \geq \frac{1}{2\beta} (\frac{1}{4} \Omega_{\varepsilon}(P) - 3\varepsilon).
\]
On the other hand, (2) implies that, a.e. on \( R \),

\[
\alpha \leq \limsup_{n,m \to \infty} \left| \frac{S_n(g, T) - S_m(g, T)}{S_nP - S_mP} \right|
\]

\[
= \lim_{n \to \infty} \frac{S_n(|g|, \tau)}{S_nP} \limsup_{n,m \to \infty} \left| \frac{S_n(g, T) - S_m(g, T)}{e^{i \theta S_n(|g|, \tau)} - e^{i \theta S_m(|g|, \tau)}} \right|
\]

\[
\leq 2 \lim_{n \to \infty} \frac{S_n(|g|, \tau)}{S_nP} \lim_{n \to \infty} \frac{S_n(\Delta, \tau)}{S_n(|g|, \tau)} = 2 \lim_{n \to \infty} \frac{S_n(\Delta, \tau)}{S_nP}.
\]

Hence

\[
(8) \quad \Omega_\tau(P) \leq \frac{2}{\alpha} \Omega_\tau(\Delta) \leq \frac{2\varepsilon}{\alpha}.
\]

But, if \( \varepsilon \) is sufficiently small, (7) and (8) are incompatible, which means that the assumption (2) cannot be true and completes the proof.

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