

# DISCONJUGACY OF COMPLEX DIFFERENTIAL SYSTEMS

BY  
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1. **Introduction.** In his paper *On an inequality of Lyapunov* [7] Nehari considered a linear differential system

$$(1) \quad y'_i(x) = \sum_{k=1}^n a_{ik}(x)y_k(x), \quad i = 1, \dots, n,$$

where the real (or complex) functions  $a_{ik}(x)$  are continuous on the interval  $a \leq x \leq b$ . From the coefficients  $a_{ik}(x)$  Nehari built a matrix  $A$  with nonnegative constants as elements. Denoting the maximal characteristic value of  $A$  by  $\lambda(A)$ , he showed that  $\lambda(A) \leq 1$  ensures the disconjugacy of (1) in the sense defined below [7, Theorem III]. His proof, based on a variational property of  $\lambda(A)$ , holds also for the complex system

$$(2) \quad w'_i(z) = \sum_{k=1}^n a_{ik}(z)w_k(z), \quad i = 1, \dots, n,$$

where the  $a_{ik}(z)$  are analytic functions in a bounded convex domain (Theorem 1 below). By means of Gronwall's inequality this result can be strengthened (Theorem 2).

The disconjugacy of system (2) can be interpreted in different ways; it is equivalent to a simple property of the determinant of any fundamental system of solutions of (2) (Theorem 3). This interpretation leads to applications which may be of independent interest. One application is a condition which implies that two analytic functions map a given domain onto disjoint domains (Theorem 4); the other one implies that  $f(z)$  satisfies  $\prod_{i=1}^n f(z_i) \neq 1$  for all sets  $(z_1, \dots, z_n)$  of a given domain (Theorem 5).

2. **Preliminaries and Theorem 1.** We shall write the system (2) in matrix notation as

$$(2) \quad w'(z) = A(z)w(z).$$

Here  $A(z)$  is the matrix  $(a_{ik}(z))_1^n$  and  $w(z)$  is the column vector  $[w_1(z), \dots, w_n(z)]$ . We consider only the case where the  $n^2$  analytic functions  $a_{ik}(z)$  are regular in a bounded simply connected domain  $D$ . We now define: *the differential system (2) is called disconjugate in  $D$  if, for every choice of  $n$  (not necessarily distinct) points*

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$z_1, \dots, z_n$  in  $D$ , the only solution of (2) which satisfies  $w_i(z_i) = 0, i = 1, \dots, n$  is the trivial one  $w(z) \equiv 0$  (i.e.,  $w_i(z) = 0, i = 1, \dots, n$  and all  $z \in D$ ). Note that for  $n = 1$  every "system" is disconjugate.

A  $(n \times n)$  matrix with constant elements  $A = (\alpha_{ik})_1^n$  is nonnegative,  $A \geq 0$ , if  $\alpha_{ik} \geq 0, i, k = 1, \dots, n$ . We denote the maximal characteristic value of such a nonnegative matrix  $A$  by  $\lambda(A)$  and we shall use the following variational property of  $\lambda(A)$ :

LEMMA 1. *If  $A$  is a  $(n \times n)$  nonnegative matrix and  $x$  a nonnegative nonvanishing  $n$ -dimensional vector (i.e.,  $A \geq 0, x \geq 0, x \neq 0$ ) then  $Ax \geq \lambda x, \lambda \geq 0$  implies  $\lambda(A) \geq \lambda$ .*

This property of the Perron-Frobenius maximal characteristic value  $\lambda(A)$  was first proved by Collatz [2] and Wielandt [10]. For recent independent proofs see Nehari [7] and Ostrowski [9]. (Lemma 1 is a slight modification of the lemma in [9, p. 82]; the corresponding modification of the elegant proof given there is obvious.)

After these preparations we now state

THEOREM 1. *Let the analytic functions  $a_{ik}(z), i, k = 1, \dots, n, n \geq 2$ , be regular and bounded in the bounded convex domain  $D$ . Set*

$$(3) \quad \alpha_{ik} = \sup_{z \in D} |a_{ik}(z)|, \quad i, k = 1, \dots, n,$$

and let  $\lambda(A)$  be the maximal characteristic value of the matrix  $A = (\alpha_{ik})_1^n$ . Let  $d$  be the diameter of  $D$ . If

$$(4) \quad d\lambda(A) < 1,$$

then the differential system (2) is disconjugate in  $D$ .

**Proof.** (Cf. [7].) Assume, to the contrary, that there exist points  $\alpha_i \in D, i = 1, \dots, n$  and a nontrivial solution  $w(z) = [w_1(z), \dots, w_n(z)]$  of (2) such that  $w_i(\alpha_i) = 0, i = 1, \dots, n$ . The  $n$  points  $\alpha_i$  cannot all coincide, as this would imply  $w(z) \equiv 0$ . Their convex hull  $H = H(\alpha_1, \dots, \alpha_n)$  is thus either a segment or a closed convex polygon belonging to  $D$ . Set now

$$(5) \quad m_i = \max_{z \in H} |w_i(z)|, \quad i = 1, \dots, n.$$

The vector  $m = [m_1, \dots, m_n]$  is nonnegative and nonvanishing. If  $m_i > 0$ , let  $\beta_i$  be a point in  $H$  (necessarily on the boundary of  $H$  if  $H$  is a polygon) satisfying  $|w_i(\beta_i)| = m_i$ . Integrating the  $i$ th component of (2) along the segment from  $\alpha_i$  to  $\beta_i$ , we obtain

$$(6) \quad \begin{aligned} 0 < m_i &= |w_i(\beta_i) - w_i(\alpha_i)| = \left| \int_{\alpha_i}^{\beta_i} w'_i(z) dz \right| \leq \int_{\alpha_i}^{\beta_i} |w'_i(z)| dz \\ &\leq \sum_{k=1}^n \int_{\alpha_i}^{\beta_i} |a_{ik}(z)w_k(z)| dz < d \sum_{k=1}^n \alpha_{ik}m_k. \end{aligned}$$

(We used (3), (5), and  $|\beta_i - \alpha_i| < d$  for the last inequality sign.) Hence,

$$\sum_{k=1}^n \alpha_{ik} m_k \geq \frac{1}{d} m_i,$$

which holds for all  $i$  (i.e., also if  $m_i = 0$ ). We thus obtained the vector inequality

$$Am \geq \frac{1}{d} m, \quad m \geq 0, \quad m \neq 0.$$

By Lemma 1 this implies  $\lambda(A) \geq 1/d$ , which contradicts assumption (4). This completes the proof of the theorem.

We add some remarks.

(i) If we assume that the functions  $a_{ik}(z)$  are regular in the closure  $\bar{D}$  of the bounded convex domain  $D$ , then (4) implies the disconjugacy of (2) in  $\bar{D}$ . The proof is now simpler; (5) has to be replaced by

$$(5') \quad m_i = \max_{z \in \bar{D}} |w_i(z)|, \quad i = 1, \dots, n,$$

and there is no need to consider closed subdomains of  $D$ .

(ii) The convexity of  $D$  was used only to obtain an upper bound,  $d$ , for the lengths of all paths which had to be considered. Let  $D$  be a domain bounded by a, not necessarily convex, rectifiable Jordan curve  $C$  of length  $L$ . It follows from the isoperimetric inequality that any two points in  $\bar{D}$  ( $= D \cup C$ ) can be joined by a path in  $\bar{D}$  of length smaller than  $(2 + \pi)L/2\pi$ . Let  $d$  be the smallest number such that any pair of points in  $\bar{D}$  can be connected by a path in  $\bar{D}$  of length not larger than  $d$ . If the  $a_{ik}(z)$  are regular in  $\bar{D}$  and if (4) holds, then (2) is disconjugate in  $\bar{D}$ .

(iii) Theorem III of [7] is stronger than the restriction of Theorem 1 to an interval. Indeed, the elements of the nonnegative matrix used by Nehari for the system (1) are  $\int_a^b |a_{ik}(x)| dx$  and are thus smaller than  $(b-a) \max |a_{ik}(x)|$ ; the maximal characteristic value of his matrix is therefore smaller than  $d\lambda(A)$  of Theorem 1 ( $d = b - a$ ). A similar idea can be used in the complex case; however the result in this case, which we are now going to state, is not necessarily stronger than Theorem 1. Let  $D$  be bounded by a rectifiable Jordan curve  $C$ . For any analytic function  $f(z)$  regular in  $\bar{D}$  and any pair of points  $\alpha, \beta \in \bar{D}$ , the inequality

$$|f(\beta) - f(\alpha)| \leq \frac{1}{2} \int_C |f'(z)| dz,$$

holds [8, formula (16)]. Assume that the coefficients  $a_{ik}(z)$  of (2) are regular in  $\bar{D}$  and define  $m_i$  by (5'). Instead of (6), we now use

$$m_i = |w_i(\beta_i) - w_i(\alpha_i)| \leq \frac{1}{2} \int_C |w'_i(z)| dz \leq \frac{1}{2} \sum_{k=1}^n m_k \int_C |a_{ik}(z)| dz.$$

Set  $C = (c_{ik})_1^n$ , where  $c_{ik} = \frac{1}{2} \int_C |a_{ik}(z)| dz$ ,  $i, k = 1, \dots, n$ . If  $\lambda(C) < 1$ , then the system (2) is disconjugate in  $\bar{D}$ .

3. **Theorem 2.** To improve the former result we shall use the following

**LEMMA 2.** Let  $u(t)$  be a real continuous function for  $0 \leq t \leq T$  and let  $a \geq 0$ ,  $b > 0$ . Then

$$0 \leq u(t) \leq \int_0^t (a + bu(s)) ds, \quad 0 \leq t \leq T,$$

implies

$$u(t) \leq \frac{a}{b} (e^{bt} - 1), \quad 0 \leq t \leq T.$$

This is a simple case of (well-known generalizations of) Gronwall's inequality and e.g., a special case of an inequality proved in [1, p. 37, problem 1]. For a direct proof, set  $\phi(t) = \int_0^t (a + bu(s)) ds$ , which gives  $\phi'(t) \leq a + b\phi(t)$ . Set now  $\psi(t) = \phi(t)e^{-bt}$ . Then  $\psi'(t) \leq ae^{-bt}$  and therefore  $\psi(t) \leq (a/b)(1 - e^{-bt})$ .

We now state our main result on disconjugacy of differential systems.

**THEOREM 2.** Let the analytic functions  $a_{ik}(z)$ ,  $i, k = 1, \dots, n$ ,  $n \geq 2$ , be regular and bounded in the bounded convex domain  $D$  of diameter  $d$ . Let  $\alpha_{ik}$ ,  $i, k = 1, \dots, n$ , be defined by (3) and set

$$(7) \quad \begin{aligned} b_{ii} &= 0, & i &= 1, \dots, n, \\ b_{ik} &= \frac{\alpha_{ik}}{\alpha_{ii}} (\exp(\alpha_{ii}d) - 1) & \text{if } \alpha_{ii} \neq 0, \\ & & i &\neq k; i, k = 1, \dots, n. \\ b_{ik} &= \alpha_{ik}d & \text{if } \alpha_{ii} = 0, \end{aligned}$$

Let  $\lambda(B)$  be the maximal characteristic value of the matrix  $B = (b_{ik})_1^n$ . If

$$(8) \quad \lambda(B) < 1,$$

then the differential system (2) is disconjugate in  $D$ .

**Proof.** We start as in the proof of Theorem 1; i.e., we assume the existence of  $n$  points  $\alpha_i \in D$  and of a nontrivial solution  $w(z)$  of (2) such that  $w_i(\alpha_i) = 0$ ,  $i = 1, \dots, n$ . We define  $H$ ,  $m_i$  (by (5)) and, if  $m_i > 0$ ,  $\beta_i$  as before. If  $m_i > 0$ , we integrate  $w'_i(z)$  along the segment from  $\alpha_i$  to  $\beta_i$ . For any point  $z$  on this segment we obtain

$$(9) \quad \begin{aligned} |w_i(z)| &= \left| \int_{\alpha_i}^z w'_i(\zeta) d\zeta \right| \leq \int_{\alpha_i}^z |w'_i(\zeta)| d|\zeta| \\ &\leq \int_{\alpha_i}^z \left( \sum_{k \neq i} \alpha_{ik} m_k + \alpha_{ii} |w_i(\zeta)| \right) |d\zeta|. \end{aligned}$$

We now choose the arc length as parameter:

$$\alpha_i + \frac{\beta_i - \alpha_i}{|\beta_i - \alpha_i|} t = z, \quad \alpha_i + \frac{\beta_i - \alpha_i}{|\beta_i - \alpha_i|} s = \zeta,$$

and denote  $u_i(t) = |w_i(z)|$ ,  $u_i(s) = |w_i(\zeta)|$ . (9) yields

$$(10) \quad 0 \leq u_i(t) \leq \int_0^t \left( \sum_{k \neq i} \alpha_{ik} m_k + \alpha_{ii} u_i(s) \right) ds, \quad 0 \leq t \leq |\beta_i - \alpha_i|.$$

If  $\alpha_{ii} \neq 0$ , it follows from Lemma 2 that

$$(11) \quad m_i = |w_i(\beta_i)| = u_i(|\beta_i - \alpha_i|) \leq \sum_{k \neq i} \frac{\alpha_{ik}}{\alpha_{ii}} m_k (\exp(\alpha_{ii} |\beta_i - \alpha_i|) - 1).$$

If  $\alpha_{ii} = 0$ , (10) implies

$$(11') \quad m_i \leq |\beta_i - \alpha_i| \sum_{k \neq i} \alpha_{ik} m_k.$$

(7), (11), (11') and  $|\beta_i - \alpha_i| < d$  give the vector inequality

$$Bm \geq m, \quad m \geq 0, \quad m \neq 0.$$

By Lemma 1 this implies  $\lambda(B) \geq 1$  and we thus obtained the desired contradiction to assumption (8).

We now prove that Theorem 2 is stronger than Theorem 1. To do this we show that  $\lambda(B) \geq 1$  implies  $d\lambda(A) \geq 1$ . Let  $x$  be the nonnegative eigenvector of  $B$  which corresponds to the maximal characteristic value  $\lambda(B)$  [3, p. 66].  $\lambda(B) \geq 1$  implies

$$(12) \quad Bx \geq x, \quad x \geq 0, \quad x \neq 0.$$

If  $\alpha_{ii} \neq 0$ , then the  $i$ th component of (12) is

$$\alpha_{ii}^{-1} (\exp(\alpha_{ii} d) - 1) \sum_{k \neq i} \alpha_{ik} x_k \geq x_i.$$

This gives

$$\sum_{k \neq i} \alpha_{ik} x_k \geq x_i \alpha_{ii} (\exp(\alpha_{ii} d) - 1)^{-1} \geq x_i \left( \frac{1}{d} - \alpha_{ii} \right).$$

Hence,

$$(13) \quad \sum_{k=1}^n \alpha_{ik} x_k \geq \frac{x_i}{d}.$$

If  $\alpha_{ii} = 0$ , then the  $i$ th component of (12) is  $d \sum_{k \neq i} \alpha_{ik} x_k \geq x_i$ . (13) is thus valid for all  $i, i = 1, \dots, n$ . By Lemma 1 this implies  $d\lambda(A) \geq 1$ .

The weaker Theorem 1 has some obvious merits: its elements  $\alpha_{ik}$  are simpler than the  $b_{ik}$  and do not depend explicitly on  $d$ . We mention that remarks similar to (i) and (ii) of the end of the last section hold also for Theorem 2.

Theorem 2 is sharp in the following sense.

Let  $\kappa$  be any given constant larger than 1. Let  $\Delta$  be the diameter of the convex domain  $D$  and set  $d = \Delta/\kappa$ . Let the  $\alpha_{ik}$  be defined by (3) and let the  $b_{ik}$  be defined by (7) (using  $d$ , not  $\Delta$ ). Then (8) does, in general, not imply the disconjugacy of the system (2) in  $D$ .

To prove this, let the matrix  $A(z)$ , defining the system (2), be the constant matrix

$$(14) \quad A(z) = \begin{pmatrix} p & 1 \\ 1 & -p \end{pmatrix}, \quad p \geq 0.$$

The general solution of (2) is in this case given by

$$w_1(z) = c_1(p + (p^2 + 1)^{1/2}) \exp((p^2 + 1)^{1/2}z) - c_2 \exp(-(p^2 + 1)^{1/2}z),$$

$$w_2(z) = c_1 \exp((p^2 + 1)^{1/2}z) + c_2(p + (p^2 + 1)^{1/2}) \exp(-(p^2 + 1)^{1/2}z).$$

$w_1(z_1) = w_2(z_2) = 0$ ,  $w(z) \neq 0$ , imply

$$\exp[2(p^2 + 1)^{1/2}(z_2 - z_1)] = -(p + (p^2 + 1)^{1/2})^2.$$

Hence,

$$(15) \quad |z_2 - z_1| = \frac{1}{2(p^2 + 1)^{1/2}} [\log^2(p + (p^2 + 1)^{1/2})^2 + (2n - 1)^2 \pi^2]^{1/2},$$

here  $n = 1, 2, \dots$ , and  $\log$  denotes the principal branch of the logarithm. For any  $p \geq 0$  we define  $\Delta(p) > 0$  by

$$(16) \quad \Delta(p)^2 = \frac{1}{4(p^2 + 1)} [4 \log^2(p + (p^2 + 1)^{1/2}) + \pi^2].$$

(15) and (16) imply that (2), with  $A(z)$  given by (14), is disconjugate in every convex domain of diameter  $\Delta(p)$ , but that for any  $\varepsilon > 0$  there exist convex domains of diameter  $\Delta(p) + \varepsilon$  for which (2) is not disconjugate.

On the other hand, the matrix  $B$  of Theorem 2 corresponding to our  $A(z)$  becomes, for  $p > 0$ ,

$$(17) \quad B = \begin{pmatrix} 0 & \frac{1}{p}(e^{pd} - 1) \\ \frac{1}{p}(e^{pd} - 1) & 0 \end{pmatrix}.$$

Hence,

$$(18) \quad \lambda(B) = \frac{1}{p}(e^{pd} - 1), \quad p > 0.$$

For given  $p > 0$ , we denote the root of the equation

$$(8') \quad \lambda(B) = 1$$

by  $d(p)$ . Therefore,

$$(19) \quad d(p) = \frac{1}{p} \log(p + 1).$$

Theorem 2 gives that for any  $p > 0$

$$(20) \quad d(p) \leq \Delta(p).$$

(16) and (19) imply

$$(21) \quad \lim_{p \rightarrow \infty} \frac{d(p)}{\Delta(p)} = 1.$$

It follows from (20) and (21) that for any given  $\kappa$ ,  $\kappa > 1$ , we can find  $p$  large enough and  $\varepsilon > 0$  small enough such that

$$(22) \quad \Delta(p) + \varepsilon = \kappa(d(p) - \varepsilon).$$

Consider now the differential system (2), with  $A(z)$  given by (14), such that the constant  $p$  of (14) satisfies (22). By the above, there exist convex domains  $D$  of diameter  $\Delta = \Delta(p) + \varepsilon$  such that this system is not disconjugate in  $D$ . On the other hand, if the constant  $d$ , appearing in the definition (7) of the corresponding  $B$ , is given by  $d = d(p) - \varepsilon = \Delta/\kappa$ , then the inequality (8) holds. This proves the italicized sharpness statement.

We did not prove that the constant 1 on the right-hand side of (8) is the best possible constant. However, it is easily seen that this constant, both in (4) and (8), cannot be replaced by any constant larger than  $\pi/2$ . This follows by considering (14) for  $p=0$ . In this case (17) has to be replaced by

$$(17') \quad B = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix},$$

and it follows that  $d\lambda(A) = \lambda(B) = d$ . Using the former notation, we thus have  $d(0) = 1$ . On the other hand,  $\Delta(0) = \pi/2$ .

To satisfy (22) we had to take  $p$  large, so  $\Delta(p)$  and  $d(p)$  became small. This smallness can be avoided by using the invariance of the theorems under the transformation  $z^* = az$ ,  $a \neq 0$ . Indeed, if we define  $w^*(z^*)$  in  $D^*$  by  $w^*(z^*) = w(z)$ , then (2) becomes

$$(2^*) \quad \frac{dw^*}{dz^*} = A^*(z^*)w^*(z^*), \quad z^* \in D^*,$$

where  $A^*(z^*) = a^{-1}A(z)$ . For the corresponding nonnegative matrices of Theorem 1 it follows that  $A^* = |a|^{-1}A$  (and therefore  $\lambda(A^*) = |a|^{-1}\lambda(A)$ ). As we now have to use  $d^* = |a|d$  in the definition of the matrix  $B^*$ , we obtain  $B^* = B$ . To return to our example, if instead of (14) we use

$$(14^*) \quad A^*(z^*) = \begin{pmatrix} 1 & \frac{1}{p} \\ \frac{1}{p} & -1 \end{pmatrix}, \quad p > 0,$$

then it follows for the corresponding system (2\*) that  $d^*(p) (=pd(p))$  and  $\Delta^*(p) (=p\Delta(p))$  tend, together with  $p$ , to infinity.

**4. Simultaneous disconjugacy.** Until now a given differential system (2) was considered and conditions for its disconjugacy were obtained. We now could consider not only (2), but also the permuted system  $\hat{w}'(z) = \hat{A}(z)\hat{w}(z)$ ; here the permutation  $\hat{A}(z)$  is obtained by a permutation of the rows of the given matrix  $A(z)$  combined with the same permutation of its columns [3, p. 50]. Clearly, (2) and the permuted system are simultaneously disconjugate or not disconjugate. But this elementary remark is worthless for applications. Indeed, in the notation of Theorems 1 and 2, obviously  $\lambda(A) = \lambda(\hat{A})$  and  $\lambda(B) = \lambda(\hat{B})$ . The case which we are going to consider is also elementary, but useful. The nonnegative matrices, built according to the two theorems, for the two simultaneously disconjugate systems (2) and  $(\tilde{2})$  (below) will, in general, have different maximal characteristic values. The following lemma may thus be applied to improve the estimate for the domain of disconjugacy of the given system (2), (see proof of Theorem 5 below).

**LEMMA 3.** *Let the analytic functions  $a_{ik}(z)$  and  $\sigma_i(z)$ ,  $i, k = 1, \dots, n$ , be regular in a bounded simply connected domain  $D$  and assume that  $\sigma_i(z) \neq 0$ , for  $i = 1, \dots, n$  and all  $z \in D$ . Set  $A(z) = (a_{ik}(z))_1^n$ , and  $\tilde{A}(z) = (\tilde{a}_{ik}(z))_1^n$ , where*

$$(23) \quad \tilde{a}_{ik}(z) = a_{ik}(z) \frac{\sigma_k(z)}{\sigma_i(z)} - \delta_{ik} \frac{\sigma'_k(z)}{\sigma_i(z)}, \quad i, k = 1, \dots, n.$$

The systems (2) and

$$(\tilde{2}) \quad \tilde{w}'(z) = \tilde{A}(z)\tilde{w}(z)$$

are together disconjugate or not disconjugate in  $D$ .

To prove this, let  $w(z) = [w_1(z), \dots, w_n(z)]$  be a solution of (2) and define  $\tilde{w}_i(z)$  by

$$(24) \quad w_i(z) = \sigma_i(z)\tilde{w}_i(z), \quad i = 1, \dots, n.$$

(2) and (24) give that  $\tilde{w}(z) = [\tilde{w}_1(z), \dots, \tilde{w}_n(z)]$  satisfies  $(\tilde{2})$ . Conversely, if  $\tilde{w}(z)$  is a solution of  $(\tilde{2})$  and  $w(z)$  is defined by (24), then  $w(z)$  satisfies (2). As  $\sigma_i(z) \neq 0$ ,  $i = 1, \dots, n$ , (24) gives the assertion.

We mention two special cases. (i) For any given matrix  $A(z)$  and any analytic function  $s(z)$  we can always choose  $\tilde{A}(z) = A(z) - s(z)I$ , ( $I = (\delta_{ik})_1^n$ ). This follows from (23) by setting  $\sigma_i(z) = \exp \int_{z_0}^z s(\zeta) d\zeta$ ,  $i = 1, \dots, n$ . (ii) We can always choose  $\tilde{A}(z)$  so that  $\tilde{a}_{ii}(z) = 0$ ,  $i = 1, \dots, n$ . This follows by setting  $\sigma_i(z) = \exp \int_{z_0}^z a_{ii}(\zeta) d\zeta$ ,  $i = 1, \dots, n$ . Theorem 2 reduces in this case to the simpler Theorem 1.

**5. A property of determinants equivalent to disconjugacy.** We now interpret disconjugacy of the linear homogeneous system (2) in terms of the determinant of  $n$  independent solutions, i.e., in terms of the determinant of a fundamental matrix [1, p. 69]. In view of the applications which we shall give in the next sections, it seems preferable to define the system (2) by one of its fundamental matrices (and not vice versa).



**THEOREM 3.** Let the analytic functions  $w_{ik}(z)$ ,  $i, k = 1, \dots, n$ ,  $n \geq 2$ , be regular in the bounded simply connected domain  $D$  and assume that the determinant  $|W(z)|$  of the matrix  $W(z) = (w_{ik}(z))_1^n$  satisfies

$$(25) \quad |W(z)| = |w_{ik}(z)|_1^n = \begin{vmatrix} w_{11}(z) & w_{12}(z) & \cdots & w_{1n}(z) \\ w_{21}(z) & w_{22}(z) & \cdots & w_{2n}(z) \\ \cdot & \cdot & \cdot & \cdot \\ w_{n1}(z) & w_{n2}(z) & \cdots & w_{nn}(z) \end{vmatrix} \neq 0, \quad \text{for all } z \in D.$$

Let the matrix  $A(z) = (a_{ik}(z))_1^n$  be defined by

$$(26) \quad A(z) = W'(z)W^{-1}(z).$$

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$$(2) \quad w'(z) = A(z)w(z)$$

$(w(z) = [w_1(z), \dots, w_n(z)])$  in  $D$  is equivalent to

$$(27) \quad |w_{ik}(z_i)|_1^n = \begin{vmatrix} w_{11}(z_1) & w_{12}(z_1) & \cdots & w_{1n}(z_1) \\ w_{21}(z_2) & w_{22}(z_2) & \cdots & w_{2n}(z_2) \\ \cdot & \cdot & \cdot & \cdot \\ w_{n1}(z_n) & w_{n2}(z_n) & \cdots & w_{nn}(z_n) \end{vmatrix} \neq 0,$$

for every choice of  $n$  (not necessarily distinct) points  $z_1, \dots, z_n$  in  $D$ .

**Proof.** (26) is equivalent to

$$(28) \quad W'(z) = A(z)W(z).$$

The given matrix  $W(z)$  is, by (25), a fundamental solution of the matrix differential equation (28). Keeping thus  $W(z)$  fixed,  $|W(z)| \neq 0$ , the solution vectors  $w(z)$  of (2) are given by

$$(29) \quad w(z) = W(z)c, \quad w(z) = [w_1(z), \dots, w_n(z)], \quad c = [c_1, \dots, c_n].$$

$c = 0$  implies  $w(z) \equiv 0$ , i.e.,  $w(z)$  is the trivial solution of (2). Conversely, if  $w(z) \equiv 0$  then  $c = 0$ . We called (2) disconjugate if  $w_i(z_i) = 0$ ,  $i = 1, \dots, n$ , always implies  $w(z) \equiv 0$ . (2) is thus disconjugate if, for any set  $z_1, \dots, z_n$  in  $D$ ,

$$w_i(z_i) = \sum_{k=1}^n w_{ik}(z_i)c_k = 0, \quad i = 1, \dots, n,$$

implies  $c_1 = \dots = c_n = 0$ . But this holds if, and only if,  $|w_{ik}(z_i)|_1^n \neq 0$ . Disconjugacy of (2) in  $D$  is thus equivalent to the validity of (27) for all sets of  $n$  points in  $D$ .

We mention here another interpretation of disconjugacy, which, however, will not be used in the sequel. Disconjugacy of the homogeneous system (2) in  $D$  is equivalent to the existence and uniqueness of a solution for the nonhomogeneous system

$$(30) \quad w'(z) = A(z)w(z) + b(z), \quad b(z) = [b_1(z), \dots, b_n(z)], \quad \text{all } b_i(z) \text{ regular in } D,$$

under the initial condition  $w_i(z_i) = d_i$ ,  $z_i \in D$ ,  $i = 1, \dots, n$ . This follows by (27) and the well-known relation: general solution of (30) equals particular solution of (30) and general solution of (2) (given by (29)).

**6. Mappings onto disjoint domains.** As first application we obtain

**THEOREM 4.** *Let the analytic functions  $f(z)$  and  $g(z)$  be regular in the bounded convex domain  $D$  of diameter  $d$  and assume that*

$$(31) \quad f(z) \neq g(z), \quad \text{for all } z \in D.$$

Assume also that the following suprema are finite:

$$(32) \quad F = \sup_{z \in D} \left| \frac{f'(z)}{f(z) - g(z)} \right|, \quad G = \sup_{z \in D} \left| \frac{g'(z)}{f(z) - g(z)} \right|.$$

If

$$(33) \quad (e^{Fd} - 1)(e^{Gd} - 1) < 1,$$

then

$$(34) \quad f(z_1) \neq g(z_2), \quad \text{for all pairs } z_1 \in D, z_2 \in D.$$

**Proof.** If one of the functions is constant, then (33) holds trivially; but in this case (31) and (34) are equivalent. We therefore assume that  $F > 0$ ,  $G > 0$ . Set

$$W(z) = \begin{pmatrix} f(z) & 1 \\ g(z) & 1 \end{pmatrix};$$

(31) is thus  $|w_{ik}(z)|_1^2 \neq 0$ . Define

$$(26') \quad A(z) = W'(z)W^{-1}(z) = \begin{pmatrix} \frac{f'}{f-g} & -\frac{f'}{f-g} \\ \frac{g'}{f-g} & -\frac{g'}{f-g} \end{pmatrix},$$

and consider the corresponding differential system

$$(2') \quad w'(z) = A(z)w(z), \quad \text{with } A(z) \text{ given by (26')}.$$

The matrix  $A = (\alpha_{ik})_1^2$  of Theorem 1 is in this case

$$A = \begin{pmatrix} F & F \\ G & G \end{pmatrix},$$

and (as  $F > 0$ ,  $G > 0$ ) the matrix  $B$  of Theorem 2 is

$$B = \begin{pmatrix} 0 & e^{Fd} - 1 \\ e^{Gd} - 1 & 0 \end{pmatrix}.$$

$\lambda(B)^2 = (e^{Fd} - 1)(e^{Gd} - 1)$  and (33) is therefore equivalent to  $\lambda(B) < 1$ . Theorem 2 implies that (2') is disconjugate in  $D$ . This is, by Theorem 3, equivalent to

$$(27') \quad |w_{ik}(z_i)|_1^2 = \begin{vmatrix} f(z_1) & 1 \\ g(z_2) & 1 \end{vmatrix} \neq 0,$$

which is just the conclusion (34) of the theorem.

We add again some remarks.

(i) We do not claim that Theorem 4 is sharp. All we know is that the constant 1 on the right-hand side of (33) cannot be replaced by any constant larger than  $(e-1)^2 = 2.95 \dots$ . This follows by choosing  $f(z) = z$  and  $g(z) = z + 1$ .  $F = G = 1$  and for any  $d > 1$  there exist convex domains of diameter  $d$  for which (34) is invalidated.

(ii) A direct proof gives the following result. Set

$$F^* = \sup_{z \in D} |f'(z)| / \inf_{z \in D} |f(z) - g(z)|, \quad G^* = \sup_{z \in D} |g'(z)| / \inf_{z \in D} |f(z) - g(z)|.$$

Then  $\min(F^*d, G^*d) \leq 1$  implies (34). This follows by supposing that  $f(z_1) = g(z_2)$ . Then

$$f(z_2) - g(z_2) = f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(\zeta) d\zeta,$$

which gives  $F^*d > 1$ ;  $G^*d > 1$  follows similarly. The example in (i) shows that this result is sharp.

(iii) Pairs of univalent functions mapping  $|z| < 1$  onto disjoint domains were considered by Nehari [6]. The necessary conditions obtained by him were generalized by M. Lavie [5] to nonunivalent functions. See also [4, pp. 123, 124].

We finally note that we could modify our theorem according to the remarks at the end of §2. Such a modification, to nonconvex domains, will be convenient for the second application.

**7. Products not taking a fixed value.** As second application we derive the following result.

**THEOREM 5.** *Let  $D$  be the interior of a piecewise smooth Jordan curve  $C$  and let the positive number  $d$  be such that any pair of points in  $\bar{D} = D \cup C$  can be joined by a path in  $\bar{D}$  of length not larger than  $d$ . Let the analytic function  $f(z)$  be regular in  $\bar{D}$  and assume that for a given integer  $n$ ,  $n \geq 2$ , the  $n$ th power of  $f(z)$  satisfies*

$$(35) \quad f^n(z) \neq 1, \quad \text{for all } z \in \bar{D}.$$

Denote

$$(36) \quad F_i = \max_{z \in \bar{D}} \left| \frac{f'(z)f^i(z)}{f^n(z) - 1} \right|, \quad i = 0, \dots, n-2.$$

If

$$(37) \quad d \sum_{i=0}^{n-2} F_i < 1,$$

then

$$(38) \quad \prod_{i=1}^n f(z_i) \neq 1, \quad \text{for all sets } (z_1, \dots, z_n) \subset \bar{D}.$$

**Proof.** We first prove the case  $n=2$ . Set

$$(39) \quad W(z) = \begin{pmatrix} f(z) & 1 \\ 1 & f(z) \end{pmatrix}.$$

The assumption  $f^2(z) \neq 1$  is thus equivalent to  $|w_{ik}(z)|_1^2 \neq 0$ . Define

$$(40) \quad A(z) = W'(z)W^{-1}(z) = \frac{f'(z)}{f^2(z)-1} \begin{pmatrix} f(z) & -1 \\ -1 & f(z) \end{pmatrix},$$

and

$$(41) \quad \tilde{A}(z) = A(z) - \frac{f'(z)f(z)}{f^2(z)-1} I = \frac{f'(z)}{f^2(z)-1} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

It follows from Lemma 3 (see cases (i) or (ii) at the end of §4), that the corresponding differential systems (2) and  $(\tilde{2})$  are together disconjugate (or not disconjugate) in  $\bar{D}$ . The constant matrix  $\tilde{A}$ , whose elements are the maxima, for  $z \in \bar{D}$ , of the absolute values of the elements of  $\tilde{A}(z)$ , is

$$(42) \quad \tilde{A} = \begin{pmatrix} 0 & F_0 \\ F_0 & 0 \end{pmatrix}.$$

Hence,  $\lambda(\tilde{A})=F_0$ . By Theorem 1, modified according to remark (ii) following its proof, our assumption  $(d\lambda(\tilde{A})) dF_0 < 1$  implies that  $(\tilde{2})$  is disconjugate in  $\bar{D}$ . By the above, system (2) is then also disconjugate in  $\bar{D}$ . By Theorem 3, disconjugacy of (2) is equivalent to

$$|w_{ik}(z_i)|_1^2 = \left| \begin{matrix} f(z_1) & 1 \\ 1 & f(z_2) \end{matrix} \right| \neq 0.$$

But this is the desired result for  $n=2$ .

For general  $n$ , the  $(n \times n)$  matrix  $W(z)$  has only  $2n$  elements different from zero:  $f(z)$  in the diagonal, 1 in the first superdiagonal and  $(-1)^n$  in the lower left corner:

$$(39') \quad W(z) = \begin{pmatrix} f & & & & 1 \\ & f & & & \\ & & f & & \\ & & & \ddots & \\ & & & & f \\ (-1)^n & & & & & f \end{pmatrix}.$$

$|W(z)| = |w_{ik}(z)|_1^n \neq 0$  is the assumption (35). Set

$$(40') \quad A(z) = \frac{f'}{f^n-1} \begin{pmatrix} f^{n-1} & -f^{n-2} & f^{n-3} & \dots & (-1)^{n-1} \\ -1 & f^{n-1} & -f^{n-2} & \dots & (-1)^n f \\ f & -1 & f^{n-1} & \dots & (-1)^{n-1} f^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ (-1)^{n-1} f^{n-2} & (-1)^n f^{n-3} & (-1)^{n-1} f^{n-4} & \dots & f^{n-1} \end{pmatrix}.$$

(For even  $n$   $A(z)$  is a circulant.) The validity of  $W'(z) = A(z)W(z)$  is readily checked.  $\tilde{A}(z)$  is again obtained from  $A(z)$  by replacing the elements in the diagonal by zeros:

$$(41') \quad \tilde{A}(z) = A(z) - \frac{f'(z)f^{n-1}(z)}{f^n(z)-1} I.$$

The corresponding nonnegative matrix  $\tilde{A}$  is a circulant (hence generalized stochastic):

$$(42') \quad \tilde{A} = \begin{pmatrix} 0 & F_{n-2} & F_{n-3} & \dots & F_0 \\ F_0 & 0 & F_{n-2} & \dots & F_1 \\ F_1 & F_0 & 0 & \dots & F_2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ F_{n-2} & F_{n-3} & F_{n-4} & \dots & 0 \end{pmatrix}.$$

Clearly,  $\lambda(\tilde{A}) = F_0 + F_1 + \dots + F_{n-2}$ . This and (37) give, by Theorem 1, that the system (2) is disconjugate in  $\bar{D}$ . By Lemma 3, the same holds for (2). Theorem 3 and (39') now yield

$$|w_{ik}(z_i)|_1^n = \prod_{i=1}^n f(z_i) - 1 \neq 0.$$

This completes the proof of Theorem 5.

We now show that for any given even  $n$ , the constant 1 on the right-hand side of (37) cannot be replaced by any constant larger than  $\pi(n-1)/n$ . To prove this, we choose  $f(z) = z$  and consider circular arcs  $C(\epsilon)$ ,  $\epsilon > 0$ , defined by

$$C(\epsilon) = \{z: |z-1| = \epsilon, |z| \leq 1\}.$$

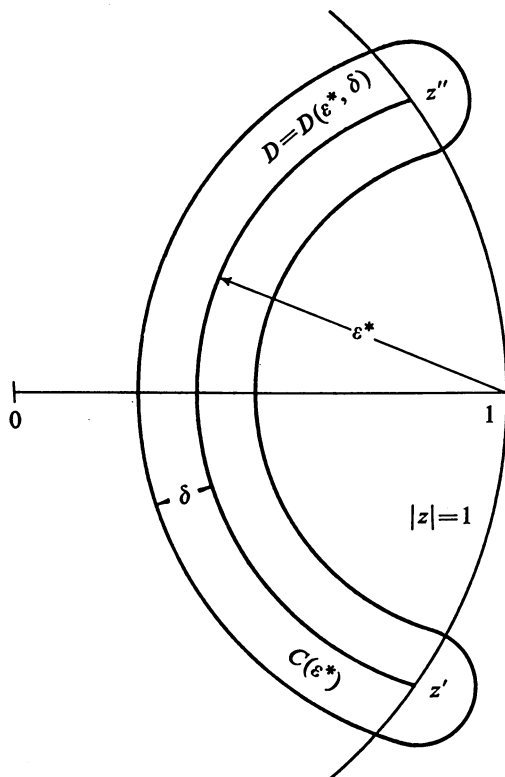
$\epsilon \rightarrow 0$  implies

$$|z^n - 1| = n\epsilon + O(\epsilon^2), \quad z \in C(\epsilon),$$

which holds uniformly in  $z$ ,  $z \in C(\epsilon)$ . It follows, that each of the  $n-1$  fractions, appearing in (36), satisfies

$$(43) \quad \left| \frac{f'(z)f^i(z)}{f^n(z)-1} \right| = \left| \frac{z^i}{z^n-1} \right| \leq \frac{1}{|z^n-1|} = \frac{1}{n\epsilon} + O(1), \quad i = 0, \dots, n-2,$$

which again holds uniformly in  $z$ ,  $z \in C(\epsilon)$ , as  $\epsilon \rightarrow 0$ . Let  $l(\epsilon)$  be the length of



$C(\epsilon)$ . Then  $l(\epsilon) < \pi\epsilon$ . This and (43) imply that, for any given  $\eta > 0$ , we can find a small enough  $\epsilon^*$ ,  $\epsilon^* > 0$ , such that

$$(44) \quad l(\epsilon^*) \sum_{i=0}^{n-2} \max_{z \in C(\epsilon^*)} \left| \frac{z^i}{z^n - 1} \right| < \frac{n-1}{n} \pi + \eta.$$

For any  $\delta$ ,  $0 < \delta < \epsilon^*$ , we now define the  $\delta$ -neighborhood  $D = D(\epsilon^*, \delta)$  of  $C(\epsilon^*)$  by  $D = D(\epsilon^*, \delta) = \{z : |z - w| < \delta, w \in C(\epsilon^*)\}$ . (See figure.) For this domain we set

$$(45) \quad d = d(\epsilon^*, \delta) = l(\epsilon^*) + 2\delta.$$

This  $d$  can serve as the bound for the lengths of all necessary paths in  $\bar{D}$ . By (44), (45), and the continuity of the appearing fractions, it follows that we can choose  $\delta$  so small,  $\delta > 0$ , that

$$(46) \quad d \sum_{i=0}^{n-2} F_i < \frac{n-1}{n} \pi + \eta,$$

where

$$F_i = \max_{z \in D} \left| \frac{z^i}{z^n - 1} \right|, \quad i = 0, \dots, n-2, \quad D = D(\epsilon^*, \delta).$$

If  $\varepsilon^*$  is small enough, then

$$(35') \quad f^n(z) = z^n \neq 1, \quad \text{for all } z \in \bar{D}.$$

On the other hand, denote the endpoints of  $C(\varepsilon^*)$  by  $z'$  and  $z''$  and set

$$z_1 = \cdots = z_{n/2} = z'; \quad z_{n/2+1} = \cdots = z_n = z''.$$

Then

$$\prod_{i=1}^n f(z_i) = \prod_{i=1}^n z_i = 1.$$

This proves that, for even  $n$ , the assumption (37) of the theorem cannot be replaced by (46).

The class of analytic functions  $f(z)$  regular in  $|z| < 1$  and such that  $f(0)=0$ ,  $f(z_1)f(z_2) \neq 1$ ,  $|z_1|, |z_2| < 1$  was studied extensively. These are the Bieberbach-Eilenberg functions [4].

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