THE FREDHOLM METHOD IN POTENTIAL THEORY[1]

BY
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Introduction. Let G be an open set with a compact boundary B in R^m, the Euclidean m-space. If h is a harmonic function in G such that

$$(1) \qquad \int_P |\text{grad}\, h(x)|\, dx < \infty$$

for every bounded open set $P \subset G$, one may form the distribution Nh over the space D of all infinitely differentiable functions ψ with compact support in R^m defining

$$\langle \psi, Nh \rangle = \int_G \text{grad}\, \psi(x) \cdot \text{grad}\, h(x)\, dx.$$

This distribution will be termed the generalized normal derivative of h (compare [CC], [M], [Y]). It is easily seen that Nh has support in B. In general, Nh need not be a measure in the sense usual in distribution theory [S]. §1 of the present paper deals with generalized normal derivatives of Newtonian potentials. We denote by $C^*(B)$ the Banach space of all finite signed Borel measures with support in B; total variation is taken as a norm in $C^*(B)$. With every $\mu \in C^*(B)$ we associate the corresponding Newtonian potential

$$U\mu(x) = \int_{R^m} p(x-y)\, d\mu(y),$$

where $p(z) = |z|^{2-m}/m-2$ or $p(z) = \log(1/|z|)$ according as $m > 2$ or $m = 2$, and we ask what necessary and sufficient condition is to be imposed on G in order that $NU\mu$ be a measure for every $\mu \in C^*(B)$. For this purpose it is useful to introduce the concept of a hit of a half-line $\{y + t\theta : t > 0\}$ on G (cf. Definition 1.5). If $n(\theta, y)$ denotes the number of such hits, then $n(\theta, y)$ is a Baire function of the variable θ on $\Gamma = R^m \cap \{\theta : |\theta| = 1\}$ and the above mentioned condition reads as follows:

$$(2) \qquad \sup_{y \in B} \int_\Gamma n(\theta, y)\, dH_{m-1}(\theta) < \infty,$$

where H_{m-1} stands for the $(m-1)$-dimensional Hausdorff measure.

Presented to the Society, January 27, 1966 under the title *Normal derivatives of potentials*, received by the editors May 31, 1966.

(1) This work was supported by the National Science Foundation, United States of America.

If G fulfills (2), then the operator

$$NU: \mu \to NU\mu$$

is bounded on $C^*(B)$ and has the form $\frac{1}{2}AI + \overline{W}^*$, where $A = H_{m-1}(\Gamma)$, I is the identity operator and $\overline{W}^*$ is adjoint to an operator $\overline{W}$ acting on the space $C(B)$ of all continuous functions on B. Some properties of $\overline{W}$, which is connected with the classical double-layer potential, are investigated in §§2–3. In particular, in §3 we show that, in case B has no isolated points, the Fredholm radius of $\overline{W}$ is the reciprocal of the quantity

$$V_0 = \lim_{r \downarrow 0} \sup_{y \in B} \left[A|d(y) - \tfrac{1}{2}| + \int_\Gamma n_r(\theta, y)\, dH_{m-1}(\theta) \right],$$

where $d(y)$ denotes the m-density of G at y and $n_r(\theta, y)$ is the number of hits of $\{y + t\theta : 0 < t < r\}$ on G. Relations between V_0 and the geometric structure of B are also investigated in §3. In case V_0 is sufficiently small, these results apply to the Neumann problem where the boundary condition is given by an arbitrary measure $\nu \in C^*(B)$, as treated in §4. By duality based on the Fredholm theory one obtains, as a by-product, representation of solutions of the Dirichlet problem by means of double-layer potentials.

Methods and concepts employed here are those of geometric measure theory; they have their origin in investigations connected with the Gauss-Green theorem, sets with finite perimeter and functions whose partial derivatives are measures [DG], [F], [FL], [FY], [KR], [MA], [P].

1. Normal derivatives of potentials.

1.1. *Terminology and notation.* The symbols R^m, $C^*(B)$, p, $U\mu$, D will have the meaning described in the introduction. For $M \subset R^m$ we shall denote by cl M, int M, fr M and diam M the closure, interior, boundary and diameter of M, respectively. H_k will stand for the k-dimensional Hausdorff measure; H_m coincides with the Lebesgue measure in R^m. We put $\Omega_r(y) = R^m \cap \{z : |z-y| < r\}$, $\Omega = \Omega_1(0)$, $\Gamma_r(y) = \text{fr}\, \Omega_r(y)$, $\Gamma = \Gamma_1(0)$, $A = H_{m-1}(\Gamma)$. Throughout this paragraph $G \subset R^m (m \geqq 2)$ will be a fixed set with a compact boundary B. We shall tacitly assume that G is open. On several places, however, it will be useful to allow G to be a Borel set; this will be always pointed out explicitly.

The generalized normal derivative of a harmonic function h (satisfying (1) for every bounded open $P \subset G$) is defined as in the introduction; we shall write $N^G h$ instead of Nh if it is necessary to specify G. The reason for the terminology is obvious: if G has a smooth boundary with exterior normal n and h is smooth up to B, then

$$\langle \psi, Nh \rangle = \int_B \psi(\partial h/\partial n)\, dH_{m-1}.$$

If spt ψ (=the support of ψ) does not meet B, then there is an open set Q with a smooth boundary such that spt $\psi \cap G \subset Q$, cl $Q \subset G$, so that

$$\langle \psi, N^G h\rangle = \langle \psi, N^Q h\rangle = 0.$$

In particular, if $N^G h$ is a (Borel) measure ν, which means that

$$\langle \psi, N^G h\rangle = \int_{R^m} \psi \, d\nu$$

for every $\psi \in D$, then $\nu \in C^*(B)$.

Variation of a (signed) measure μ on a Borel set M will be denoted by $|\mu|(M)$; for $\mu \in C^*(B)$, $|\mu|(B) = \|\mu\|$ is the norm of μ.

Simple calculation shows that, for $\mu \in C^*(B)$ and $x \in G$,

$$|\text{grad } U\mu(x)| \leqq \int_B |x-y|^{1-m} \, d|\mu|(y),$$

whence we obtain for any bounded Borel $P \subset G$

$$\int_P |\text{grad } U\mu(x)| \, dx \leqq A \text{ diam } (B \cup P)\|\mu\|. \tag{1.1}$$

We see that $NU\mu$ is meaningful for every $\mu \in C^*(B)$. Our main objective in §1 is to answer the following question:

1.2. *What necessary and sufficient restrictions are to be imposed on G in order that $NU\mu$ be a measure for every $\mu \in C^*(B)$?*

1.3. REMARK. Let us agree to denote by δ_y the Dirac measure concentrated at $y \in R^m$. We have for any $\psi \in D$ and any $y \in B$

$$\langle \psi, NU\delta_y\rangle = \int_G \text{grad } \psi(x) \cdot \frac{y-x}{|y-x|^m} \, dx.$$

Direct calculation shows that, in case $Q = R^m - \{y\}$, $N^Q U\delta_y = A\delta_y$.

Let us also observe that, for $\psi \in D$ and $\mu \in C^*(B)$,

$$\langle \psi, NU\mu\rangle = \int_B \langle \psi, NU\delta_y\rangle \, d\mu(y). \tag{1.2}$$

Indeed, if $P = G \cap \text{spt } \psi$ and $K = \sup |\text{grad } \psi|$, then

$$\iint_{G\times B} \left|\text{grad } \psi(x) \cdot \frac{y-x}{|y-x|^m}\right| dx \, d|\mu|(y) \leqq KA \text{ diam } (P \cup B)\|\mu\|,$$

so that Fubini's theorem applies to

$$\iint_{G\times B} \text{grad } \psi(x) \cdot (y-x)|y-x|^{-m} \, dx \, d\mu(y); \tag{1.3}$$

it remains to notice that the two repeated integrals derived from (1.3) occur in (1.2).

Before investigating the problem 1.2 we shall answer the following simpler question:

1.4. *Fix $y \in B$. What must be the shape of G in order that $NU\delta_y$ be a measure?*

Let us first introduct a concept which will be useful later.

1.5. DEFINITION. If $M \subset R^k$ is a Borel set and $S \subset R^k$ is an open segment or half-line then $z \in S$ will be termed a hit of S on M provided both $S \cap M \cap \Omega_r(z)$ and $(S-M) \cap \Omega_r(z)$ have a positive linear measure for every $r>0$.

An answer to 1.4 is included in the following proposition, which will be needed later.

1.6. PROPOSITION. *Suppose that G is a Borel set. Fix $y \in R^m$, $r>0$ and put*

$$E_r(y) = D \cap \{\psi : \operatorname{spt} \psi \subset \Omega_r(y), |\psi| \leqq 1\},$$

$$D_r(y) = E_r(y) \cap \{\psi : y \notin \operatorname{spt} \psi\}.$$

If $n_r(\theta, y)$ denotes the number (possibly 0 or ∞) of all hits of $\{y+\rho\theta : 0<\rho<r\}$ on G, then $n_r(\theta, y)$ is a Baire function of the variable θ on Γ, the integral

$$v_r(y) = \int_\Gamma n_r(\theta, y)\, dH_{m-1}(\theta)$$

is equal to

$$\sup\left\{\int_G \operatorname{grad} \psi(x) \cdot \frac{y-x}{|y-x|^m}\, dx : \psi \in D_r(y)\right\}$$

and

$$\sup\left\{\int_G \operatorname{grad} \psi(x) \cdot \frac{y-x}{|y-x|^m}\, dx : \psi \in E_r(y)\right\} \leqq A + v_r(y).$$

If $y \in B$ and G is open, then $NU\delta_y$ is a measure if and only if $v_\infty(y)<\infty$.

1.7. REMARK. If it is necessary to specify the set G, we write $n_r^G(\theta, y)$ and $v_r^G(y)$ instead of $n_r(\theta, y)$ and $v_r(y)$.

We postpone the proof of Proposition 1.6 to 1.11. First we establish two lemmas.

1.8. *Notation.* If f is a function in R^1 we denote by var $[f; (a, b)]$ its variation on $(a, b) = R^1 \cap \{t : a<t<b\}$. If f is known to be summable over every compact subset in (a, b), we shall use var ess $[f; (a, b)]$ to denote $\sup_\psi \int_a^b \psi'(t) f(t)\, dt$, ψ ranging over all infinitely differentiable functions with compact spt $\psi \subset (a, b)$ such that $|\psi| \leqq 1$.

REMARK. It follows easily from the Riesz representation theorem and elementary distribution theory that var ess $[f; (a, b)]<\infty$ implies the existence of a function g in (a, b) such that $g=f$ a.e. in (a, b) and var $[g; (a, b)]=$ var ess $[f; (a, b)]$.

Clearly, var $[f; (a, b)]=$ var ess $[f; (a, b)]$ whenever f is continuous in (a, b).

1.9. LEMMA. *If c_M is the characteristic function of a Borel set $M \subset R^1$, then* var ess $[c_M; (a, b)]$ *equals the number of hits of (a, b) on M.*

Proof. Let q stand for the number of all hits of (a, b) on M. If $q<\infty$ and $a_1<\cdots<a_q$ are all the hits, then no (a_j, a_{j+1}) can meet both M and $(a, b)-M$ in a set of positive linear measure. It follows that either M or $(a, b)-M$ is equivalent with $\bigcup_k (a_{2k-1}, a_{2k})$, where $1\leqq k$, $2k\leqq q$. Consequently, var ess $[c_M; (a, b)]=q$. Conversely, if var ess $[c_M; (a, b)]<\infty$, then there is a g with var $[g; (a, b)]<\infty$ such that $g=c_M$ a.e. in (a, b).

Clearly, this implies $q<\infty$.

1.10. Lemma. *Let f be a bounded Baire function in R^m, $y\in R^m$, $0\leqq a<b\leqq\infty$. For $\theta\in\Gamma$ put*

$$f_\theta(t) = f(y+t\theta), \qquad t\in R^1. \tag{1.4}$$

Then var ess $[f_\theta; (a, b)]$ *is a Baire function of the variable θ on Γ and the integral*

$$\int_\Gamma \text{var ess } [f_\theta; (a, b)]\, dH_{m-1}(\theta)$$

equals

$$v(a, b, f) = \sup_\psi \int_{R^m} f(x) \operatorname{grad} \psi(x)\cdot\frac{y-x}{|y-x|^m}\, dx,$$

ψ ranging over all functions in D with

$$\operatorname{spt}\psi \subset R^m \cap \{x : a < |x-y| < b\}, \qquad |\psi| \leqq 1. \tag{1.5}$$

Proof. We may assume $y=0$, $b<\infty$. Using the notation from (1.4) we obtain for any $\psi\in D$ satisfying (1.5)

$$\int_{R^m} f(x) \operatorname{grad}\psi(x)\cdot\frac{x}{|x|^m}\, dx = \int_\Gamma \left(\int_a^b f_\theta(t)\psi_\theta'(t)\, dt\right) dH_{m-1}(\theta),$$

$$\int_a^b f_\theta(t)\psi_\theta'(t)\, dt \leqq \text{var ess } [f_\theta; (a, b)].$$

Assuming that we know already that var ess $[f_\theta; (a, b)]$ is measurable (H_{m-1}) on Γ we get

$$v(a, b, f) \leqq \int_\Gamma \text{var ess } [f_\theta; (a, b)]\, dH_{m-1}(\theta).$$

It remains to prove that var ess $[f_\theta; (a, b)]$ is a Baire function of θ and

$$\int_\Gamma \text{var ess } [f_\theta; (a, b)]\, dH_{m-1}(\theta) \leqq v(a, b, f). \tag{1.6}$$

To show this we first assume, in addition, that

(I). f_θ has a continuous derivative on (a, b) for every $\theta\in\Gamma$ and

$$\sup\{|f_\theta'(t)| : \theta\in\Gamma, c < t < d\} = K(c, d) < \infty$$

whenever $a<c<d<b$.

For every positive integer N we subdivide (a, b) by means of points

$$a_k = a_k^N = a+k2^{-N}(b-a), \qquad 1 \leqq k < 2^N.$$

Consider $k<2^N-2$. Since sign $[f_\theta(a_{k+1})-f_\theta(a_k)]$ is a Baire function of θ, there are functions $\phi_{ks} \in D$ such that $|\phi_{ks}| \leqq 1$ and

$$\lim_{s\to\infty} \phi_{ks}(\theta) = \text{sign}\,[f_\theta(a_{k+1})-f_\theta(a_k)] \qquad \text{a.e. } (H_{m-1})$$

on Γ. Further express the characteristic function of (a_k, a_{k+1}) as $\lim_{s\to\infty} \rho_{ks}$, where ρ_{ks} are infinitely differentiable functions in R^1 with

$$\text{spt}\, \rho_{ks} \subset (a_k, a_{k+1}),\ |\rho_{ks}| \leqq 1,$$

and define

$$\psi_s(t\theta) = -\sum_{k=1}^{2^N-2} \phi_{ks}(\theta)\rho_{ks}(t), \qquad t \geqq 0,\ \theta \in \Gamma.$$

Then

$$\psi_s \in D,\ |\psi_s| \leqq 1,\ \text{spt}\,\psi_s \subset R^m \cap \{x : a < |x| < b\}.$$

Consequently,

$$v(a, b, f) \geqq \int_\Gamma \left[\int_a^b f_\theta(t)\psi'_{s\theta}(t)\, dt\right] dH_{m-1}(\theta).$$

The sequence of integrals

$$\int_a^b f_\theta(t)\psi'_{s\theta}(t)\, dt = \sum_{k=1}^{2^N-2} \phi_{ks}(\theta) \int_{a_k}^{a_{k+1}} \rho_{ks}(t) f'_\theta(t)\, dt$$

is dominated by $(b-a)K(a_1, a_{2^N-1})$ and converges, as $s \to \infty$, to

$$\sigma_N(\theta) = \sum_{k=1}^{2^N-2} |f_\theta(a_{k+1})-f_\theta(a_k)|$$

a.e. (H_{m-1}) on Γ. Hence we conclude

$$v(a, b, f) \geqq \int_\Gamma \sigma_N(\theta)\, dH_{m-1}(\theta).$$

Noting that $\sigma_N(\theta) \uparrow \text{var}\,[f_\theta; (a, b)]$ as $N \to \infty$ we see that var $[f_\theta; (a, b)]$ is a Baire function of θ and (1.6) holds in this special case.

Let us now drop the additional assumptions (I) on f. For every positive integer N we fix a symmetric infinitely differentiable function ω_N in R^1 with

$$\text{spt}\, \omega_N \subset (-1/N, 1/N),\ \int_{R^1} \omega_N(t)\, dt = 1$$

and define f_N so that $f_{N\theta}=f_\theta * \omega_N$ (=the convolution of f_θ and ω_N) on the positive real axis, $f_N(0)=0$. Let $a^N=a+1/N$, $b^N=b-1/N$, $2/N<b-a$. It follows from the first part of the proof that

$$\int_\Gamma \operatorname{var}\,[f_{N\theta};(a^N,b^N)]\,dH_{m-1}(\theta) = v(a^N,b^N,f_N). \tag{1.7}$$

If ψ_N is obtained from ψ in the same way as f_N from f, then

$$\psi\in D,\ |\psi|\leqq 1,\ \operatorname{spt}\psi\subset R^m\cap\{x: a^N<|x|<b^N\}$$

imply

$$\psi_N\in D,\ |\psi_N|\leqq 1,\ \operatorname{spt}\psi_N\subset R^m\cap\{x: a<|x|<b\}$$

and

$$\int_{a^N}^{b^N}\psi_\theta'(t)f_{N\theta}(t)\,dt = \int_a^b \psi_{N\theta}'(t)f_\theta(t)\,dt.$$

Consequently,

$$v(a^N,b^N,f_N)\leqq v(a,b,f). \tag{1.8}$$

The same argument shows that

$$\operatorname{var\,ess}\,[f_{N\theta};(a^N,b^N)]\leqq \operatorname{var\,ess}\,[f_\theta;(a,b)]. \tag{1.9}$$

It is easy to see that

$$\liminf_{N\to\infty}\operatorname{var\,ess}\,[f_{N\theta};(a^N,b^N)]\geqq \operatorname{var\,ess}\,[f_\theta;(a,b)],$$

which together with (1.9) yields

$$\lim_{N\to\infty}\operatorname{var\,ess}\,[f_{N\theta};(a^N,b^N)] = \operatorname{var\,ess}\,[f_\theta;(a,b)]. \tag{1.10}$$

In particular, var ess $[f_\theta;(a,b)]$ is a Baire function of θ. (1.7), (1.8), and (1.10) imply (1.6).

REMARK. The above lemma could also be derived from general theorems on functions, whose partial derivatives are measures; cf. [FL], [KR], [P] on the subject.

Now it is easy to present the following.

1.11. **Proof of Proposition 1.6.** Let f be the characteristic function of G. By 1.9 and 1.10

$$\operatorname{var\,ess}\,[f_\theta;(0,r)] = n_r(\theta,y),$$

$$v_r(y) = v(0,r,f).$$

If $n_r(\theta,y)<\infty$, then $G\cap\{y+t\theta: 0<t<r\}$ is equivalent (H_1) with a finite union

of disjoint segments, whose end points are hits of $\{y+t\theta : 0<t<r\}$ on G and, possibly, y and $y+r\theta$. Hence we conclude for $\psi \in E_r(y)$

$$\left|\int_0^\infty f_\theta(t)\psi'_\theta(t)\,dt\right| \leqq 1+n_r(\theta, y), \qquad \theta \in \Gamma,$$

$$\int_G \operatorname{grad}\psi(x)\cdot(y-x)|y-x|^{-m}\,dx \leqq \int_\Gamma [1+n_r(\theta, y)]\,dH_{m-1}(\theta) = A+v_r(y).$$

It remains to note that, in case $y \in B$ and G is open, $NU\delta_y$ is a measure if and only if

$$\sup\{\langle\psi, NU\delta_y\rangle : \psi \in D, |\psi| \leqq 1\} < \infty.$$

1.12. REMARK. Let us observe that, in case $y \in B$ and $NU\delta_y \in C^*(B)$,

$$v_\infty(y) \leqq \|NU\delta_y\| \leqq A+v_\infty(y). \tag{1.11}$$

Now we are in position to answer the question raised in 1.2.

1.13. THEOREM. *$NU\mu$ is a measure for every $\mu \in C^*(B)$ if and only if*

$$V = \sup_{y\in B} v_\infty(y) < \infty. \tag{1.12}$$

If this is the case, then

$$NU : \mu \to NU\mu$$

is a bounded linear operator on $C^*(B)$,

$$\|NU\| \leqq A+V$$

and (1.2) *holds for every bounded Baire function ψ on B. In particular,*

$$NU\mu(M) = \int_B NU\delta_y(M)\,d\mu(y)$$

for $\mu \in C^*(B)$ and every Borel set $M \subset B$.

Proof. With every $\psi \in D$ we associate a linear functional L_ψ over $C^*(B)$ defined by

$$\langle\mu, L_\psi\rangle = \langle\psi, NU\mu\rangle, \qquad \mu \in C^*(B).$$

Denoting

$$P_\psi = G \cap \operatorname{spt}\psi, \qquad s_\psi = \sup|\operatorname{grad}\psi|,$$

we obtain from (1.1)

$$|\langle\mu, L_\psi\rangle| \leqq s_\psi A \operatorname{diam}(B \cup P_\psi)\|\mu\|$$

which shows that every L_ψ is bounded on $C^*(B)$. Let $E = D \cap \{\psi : |\psi| \leqq 1\}$. Then $NU\mu$ is a measure if and only if

$$\sup_{\psi\in E}\langle\psi, NU\mu\rangle < \infty.$$

In particular, if $NU\mu$ is a measure for every $\mu \in C^*(B)$, then the class of functionals $\{L_\psi\}_{\psi \in E}$ must be pointwise bounded on $C^*(B)$ and, by the uniform boundedness principle,

$$\sup_{\psi \in E} \|L_\psi\| = K < \infty.$$

Employing (1.11) we get for every $y \in B$

$$v_\infty(y) \leqq \sup_{\psi \in E} \langle \psi, NU\delta_y \rangle \leqq K.$$

Conversely, if (1.12) holds, then (1.2) together with (1.11) imply

$$\sup_{\psi \in E} |\langle \psi, NU\mu \rangle| \leqq (A+V)\|\mu\|$$

for every $\mu \in C^*(B)$. It is also easily seen that in this case (1.2) extends to any bounded Baire function ψ.

2. **Double layer potentials.**

2.1. *Notation.* Throughout this paragraph $C \subset R^m$ will denote a Borel set with a compact boundary B. Given $z \in R^m$ we put

$$D(z) = D \cap \{\psi : z \notin \operatorname{spt} \psi\}$$

and define

$$W_\psi(z) = \int_C \operatorname{grad} \psi(x) \cdot \frac{x-z}{|x-z|^m}\, dx, \qquad \psi \in D(z). \tag{2.1}$$

If it is necessary to specify C we write W_ψ^C instead of W_ψ. In case C has a smooth boundary with exterior normal n the integral (2.1) reduces to

$$\int_B \psi(y) \frac{(y-z)\cdot n(y)}{|y-z|^m}\, dH_{m-1}(y),$$

which is the classical double-layer potential. If ψ vanishes in some neighborhood of B then there is a $Q \subset R^m$ with a smooth boundary such that

$$\operatorname{spt} \psi \cap C \subset \operatorname{int} Q, \qquad \operatorname{cl} Q \subset \operatorname{int} C,$$

whence

$$W_\psi(z) = W_\psi^Q(z) = 0.$$

If $z \notin B$, we use this observation to extend $W_\psi(z)$ from $D(z)$ to D defining

$$W_\psi(z) = W_{\tilde\psi}(z),$$

where $\tilde\psi$ is an arbitrary function in $D(z)$ coinciding with given $\psi \in D$ in some neighborhood of B. $W_\psi(z)$ may thus be considered as a distribution over D with support in B (compare [D, Chapter III, p. 157]).

For fixed $\psi \in D$, $W_\psi(z)$ is a harmonic function of z in $R^m - B$. Indeed, if O is an open set with $B \cap \text{cl } O = \varnothing$, then there is a $\bar{\psi} \in D$ coinciding with ψ in some neighborhood of B and vanishing on O; clearly,

$$W_\psi(z) = W_{\bar{\psi}}(z) = \int_{C-O} \text{grad } \bar{\psi}(x) \cdot \frac{x-z}{|x-z|^m} \, dx$$

is a harmonic function of z in O.

Our main objective in this paragraph is to find necessary and sufficient geometric conditions on C securing natural extendability of W_ψ from D to broader class of continuous functions and also "nice behaviour" (e.g., boundedness) of W_ψ near B for each continuous ψ.

2.2. LEMMA. *Fix $z \in R^m$. Then*

$$v_\infty^C(z) < \infty \tag{2.2}$$

is a necessary and sufficient condition to secure

$$\lim_{k \to \infty} W_{\psi_k}(z) = W_\psi(z)$$

for every sequence of $\psi_k \in D(z)$ converging uniformly (as $k \to \infty$) to $\psi \in D(z)$. If (2.2) holds then there is a $\nu_z \in C^*(B)$ such that

$$W_\psi(z) = \int_B \psi(y) \, d\nu_z(y), \qquad \psi \in D(z), \tag{2.3}$$

$$\nu_z(\{z\}) = 0, \tag{2.4}$$

$$\|\nu_z\| = v_\infty^C(z). \tag{2.5}$$

(2.3) *together with any of the two conditions* (2.4), (2.5) *determine ν_z uniquely.*

Proof. This follows at once from the equality

$$v_\infty^C(z) = \sup \{W_\psi(z) : \psi \in D(z), |\psi| \leqq 1\} \tag{2.6}$$

established in 1.6.

2.3. REMARK. If (2.2) holds we extend $W \cdots (z)$ defining

$$Wf(z) = \int_B f(y) \, d\nu_z(y)$$

for any bounded Baire function f on B.

In order to present another integral representation for $Wf(z)$ we introduce the following.

2.4. *Notation.* Fix $z \in R^m$ and $\theta \in \Gamma$. We put for $t > 0$

$$s(t; z, \theta) = \sigma \, (= \pm 1)$$

if there is a $\delta > 0$ such that

$$z+(t+\sigma\tau)\theta \in R^m - C, \qquad z+(t-\sigma\tau)\theta \in C$$

for a.e. $\tau \in (0, \delta)$; otherwise we set $s(t; z, \theta)=0$.

Clearly, $s(t; z, \theta) \neq 0$ only if $z+t\theta$ is a hit of $\{z+\tau\theta : \tau > 0\}$ on C.

2.5. LEMMA. If $v_\infty^C(z) < \infty$ *then*

$$Wf(z) = \int_\Gamma \Big\{\sum_{t>0} f(z+t\theta)s(t; z, \theta)\Big\}\, dH_{m-1}(\theta) \tag{2.7}$$

for any bounded Baire function f on B.

Proof. Let $v_\infty^C(z) < \infty$. If $f \in D(z)$ then

$$Wf(z) = \int_C \operatorname{grad} f(x) \cdot \frac{x-z}{|x-z|^m}\, dx$$
$$= \int_\Gamma \Big\{\int_{C_\theta} \partial_\theta f(z+t\theta)\, dt\Big\}\, dH_{m-1}(\theta),$$

where

$$C_\theta = \{t : t > 0, z+t\theta \in C\}, \partial_\theta f = \theta \cdot \operatorname{grad} f. \tag{2.8}$$

Noting that $n_\infty^C(\theta, z) < \infty$ implies

$$\int_{C_\theta} \partial_\theta f(z+t\theta)\, dt = \sum_{t>0} f(z+t\theta)s(t; z, \theta)$$

we obtain (2.7).

If $\{f_k\}$ is a pointwise convergent sequence of functions on B such that, for all k, $|f_k| \leqq K$ and (2.7) holds with f replaced by f_k, then

$$\Big|\sum_{t>0} f_k(z+t\theta)s(t; z, \theta)\Big| \leqq K n_\infty^C(\theta, z)$$

a.e. $\{H_{m-1}\}$ on Γ and, by the Lebesgue convergence theorem, (2.7) holds for $f = \lim_k f_k$ as well.

We conclude that (2.7) is valid for every bounded Baire function f vanishing at z; in view of (2.4), vanishing at z is irrelevant.

2.6. PROPOSITION. *Let $v_\infty^C(z) < \infty$. Denote by K_z and L_z the set of all $\theta \in \Gamma$ for which there is an $\varepsilon = \varepsilon(\theta) > 0$ such that*

$$H_1(\{z+t\theta : 0 < t < \varepsilon\} \cap C) = 0$$

and

$$H_1(\{z+t\theta : 0 < t < \varepsilon\} - C) = 0,$$

respectively. Then K_z, L_z *are measurable* (H_{m-1}),

$$H_{m-1}(\Gamma-(K_z \cup L_z)) = 0 \tag{2.9}$$

and $\nu_z(B)=H_{m-1}(L_z)$ *or* $\nu_z(B)=-H_{m-1}(K_z)$ *according as* C *is bounded or not. If* $\psi \in D$, *then*

$$\int_C \operatorname{grad} \psi(x)\cdot \frac{x-z}{|x-z|^m}\,dx = W_\psi(z)-H_{m-1}(L_z)\psi(z). \tag{2.10}$$

If Q *is a convex Borel set, then*

$$|\nu_z(B\cap Q)| \leqq A. \tag{2.11}$$

Proof. It is easily seen that

$$\Gamma \cap \{\theta : n_\infty^C(\theta, z) < \infty\} \subset L_z \cup K_z$$

whence (2.9) follows at once.

Fix now a $\theta \in \Gamma$ with $n_\infty^C(\theta, z)<\infty$. Let

$$t_1 < \cdots < t_q \tag{2.12}$$

be all the points $t \in (0, \infty)$ with $s(t; z, \theta)\neq 0$ (cf. 2.4). Clearly,

$$s(t_{j+1};\cdots) = -s(t_j;\cdots), \qquad 1 \leqq j < q \tag{2.13}$$

and $s(t_1;\cdots)=1$ or $s(t_1;\cdots)=-1$ according as $\theta \in L_z$ or $\theta \in K_z$. If C is bounded, then $s(t_q;\cdots)=1$, while $s(t_q;\cdots)=-1$ in the opposite case. We conclude that

$$\sum(\theta) = \sum_{t>0} s(t; z, \theta)$$

almost (H_{m-1}) equals the characteristic function of L_z if C is bounded, while $-\sum(\theta)$ almost equals the characteristic function of K_z in the opposite case. Employing (2.7) with $f\equiv 1$ we get the first part of our proposition.

Let now f be the characteristic function of a convex Borel set Q. Consider again a fixed $\theta \in \Gamma$, $n_\infty^C(\theta, z)<\infty$, and the corresponding sequence (2.12). If t_i and t_k are the first and the last members of (2.12) with $z+t_j\theta \in Q$, respectively, then (2.13) implies

$$\left|\sum_{j=1}^{q} f(z+t_j\theta)s(t_j; z, \theta)\right| = \left|\sum_{j=i}^{k} s(t_j; z, \theta)\right| \leqq 1,$$

whence (2.11) follows by 2.5. If $\psi \in D$ then we have with the notation from (2.8)

$$\int_{C_\theta} \partial_\theta\psi(z+t\theta)\,dt = \sum_{t>0} \psi(z+t\theta)s(t; z, \theta)$$

for $\theta \in K_z \cap \{\theta : n_\infty^C(\theta, z)<\infty\}$, while

$$\int_{C_\theta} \partial_\theta\psi(z+t\theta)\,dt = \sum_{t>0} \psi(z+t\theta)s(t; z, \theta)-\psi(z)$$

for $\theta \in L_z \cap \{\theta : n_\infty^C(\theta, z) < \infty\}$. Hence

$$\int_C \operatorname{grad} \psi(x) \cdot \frac{x-z}{|x-z|^m} \, dx = \int_\Gamma \left(\int_{C_\theta} \partial_\theta \psi(z+t\theta) \, dt \right) dH_{m-1}(\theta)$$

$$= W\psi(z) - H_{m-1}(L_z)\psi(z).$$

2.7. Lemma. *Let $v_\infty^C(z) < \infty$ and define L_z as in 2.6. If $M \subset \Gamma$ is measurable (H_{m-1}), $H_{m-1}(M) > 0$ and*

$$\Lambda_M = \{z + t\theta : \theta \in M, t > 0\},$$

then

$$\lim_{r \to 0+} \frac{H_m(\Omega_r(z) \cap C \cap \Lambda_M)}{H_m(\Omega_r(z) \cap \Lambda_M)} = \frac{H_{m-1}(L_z \cap M)}{H_{m-1}(M)}. \tag{2.14}$$

In particular, C has an m-dimensional density

$$d_C(z) = H_{m-1}(L_z)/A$$

at z.

Proof. Let $\varepsilon(\theta)$ have the meaning described in the definition of K_z, L_z in 2.6 and put

$$K^r = M \cap \{\theta : \theta \in K_z, \varepsilon(\theta) > r\},$$

$$L^r = M \cap \{\theta : \theta \in L_z, \varepsilon(\theta) > r\}.$$

We have

$$H_m(\Omega_r(z) \cap C \cap \Lambda_M) \geqq m^{-1} r^m \operatorname{inn} H_{m-1}(L^r),$$

$$H_m((\Omega_r(z) - C) \cap \Lambda_M) \geqq m^{-1} r^m \operatorname{inn} H_{m-1}(K^r),$$

where $\operatorname{inn} H_{m-1}$ stands for the inner $(m-1)$-dimensional Hausdorff measure. Denoting

$$d_r = \frac{H_m(\Omega_r(z) \cap C \cap \Lambda_M)}{H_m(\Omega_r(z) \cap \Lambda_M)}$$

and noting that

$$K^r \uparrow (K_z \cap M), \qquad L^r \uparrow (L_z \cap M)$$

as $r \downarrow 0$, we obtain

$$\liminf_{r \to 0+} d_r \geqq H_{m-1}(L_z \cap M)/H_{m-1}(M),$$

$$\liminf_{r \to 0+} (1 - d_r) \geqq H_{m-1}(K_z \cap M)/H_{m-1}(M),$$

whence (2.14) follows by (2.9).

2.8. *Notation.* $P(C)$ will denote the perimeter of C defined by

$$P(C) = \sup_w \int_C \operatorname{div} w(x)\,dx,$$

where $w=[w_1,\ldots,w_m]$ ranges over all vector-valued functions with m components $w_j \in D$ satisfying

$$\left(\sum_{j=1}^m w_j^2\right)^{1/2} = |w| \leqq 1.$$

(Further information on sets with finite perimeter may be found in [DG], [F3], [FL], [MA].)

For $M \subset R^m$ and $z \in R^m$ we let

$$\operatorname{dist}(z, M) = \inf\{|z-y| : y \in M\}.$$

2.9. LEMMA. *$v_\infty^C(z)$ is a lower semicontinuous function of z on R^m satisfying the inequality*

$$v_\infty^C(z) \leqq P(C)(\operatorname{dist}(z, B))^{1-m}, \qquad z \notin B.$$

Proof. If $K < v_\infty^C(z)$, then there is a $\psi \in D(z)$ such that $|\psi| \leqq 1$ and $W_\psi(z) > K$ (see (2.6)). Hence

$$\liminf_{y\to z} v_\infty^C(y) \geqq \lim_{y\to z} W\psi(y) = W\psi(z) > K.$$

Suppose now that $z \notin B$, fix an arbitrary $\psi \in D(z)$ with $|\psi| \leqq 1$ and a positive $\rho < \operatorname{dist}(z, B)$. Then there is a $\tilde\psi \in D$, $|\tilde\psi| \leqq 1$, which coincides with ψ in some neighborhood of B and vanishes on $\Omega_\rho(z)$. Let us define $w(z) = O$ ($\in R^m$),

$$w(x) = \tilde\psi(x)\frac{x-z}{|x-z|^m}, \qquad x \neq z,$$

and observe that $|w| \leqq \rho^{1-m}$,

$$\operatorname{grad} \tilde\psi(x)\cdot\frac{x-z}{|x-z|^m} = \operatorname{div} w(x).$$

Consequently,

$$W\psi(z) = W\tilde\psi(z) = \int_C \operatorname{div} w(x)\,dx \leqq \rho^{1-m}P(C).$$

REMARK. We see that $v_\infty^C(z)$ is finite on $R^m - B$ provided $P(C) < \infty$. The converse is also true as it follows from the following

2.10. PROPOSITION. *If*

$$\sum_{j=1}^{m+1} v_\infty^C(z_j) < \infty$$

for an $(m+1)$-tuple of points $z_1, \ldots, z_{m+1}$ in general position (i.e., not situated on a single hyperplane), then

$$P(C) < \infty. \tag{2.15}$$

Proof. To prove (2.15) it is sufficient to show that

$$\sup\left\{\int_C \partial_\theta \psi(x)\,dx : \psi \in D, |\psi| \leqq 1\right\} < \infty$$

for every $\theta \in \Gamma$. Fix $\theta \in \Gamma$. Let Π_j denote the hyperplane determined by $\{z_k : k \neq j\}$. Since

$$\bigcup_{j=1}^{m+1} (R^m - \Pi_j) = R^m,$$

there are $\alpha_j \in D$ such that

$$\Pi_j \cap \operatorname{spt} \alpha_j = \varnothing$$

and

$$\alpha = \sum_{j=1}^{m+1} \alpha_j = 1$$

in some neighborhood of B.

Noting that

$$\int_C \alpha(x)\partial_\theta \psi(x)\,dx = \int_C \partial_\theta \psi(x)\,dx$$

we see that it is sufficient to prove that

$$\sup\left\{\int_C \alpha_j(x)\partial_\theta \psi(x)\,dx : \psi \in D, |\psi| \leqq 1\right\} < \infty$$

for $j=1, \ldots, m+1$. Consider, for instance, $j=1$. If $x \in \operatorname{spt} \alpha_1$, then $x-z_2, \ldots, x-z_{m+1}$ are linearly independent. Consequently,

$$\theta = \sum_{k=2}^{m+1} a_k(x) \frac{x-z_k}{|x-z_k|^m},$$

where a_k are infinitely differentiable in some neighborhood of spt α_1. Extending a_k arbitrarily to R^m we get

$$\int_C \alpha_1(x)\partial_\theta \psi(x)\,dx = \sum_{k=2}^{m+1} \int_C \alpha_1(x)a_k(x) \operatorname{grad} \psi(x) \cdot \frac{x-z_k}{|x-z_k|^m}\,dx.$$

Fix $k \in \langle 2, m+1 \rangle$ and define $F(x)=\alpha_1(x)a_k(x)$. Then $F \in D(z_k)$ and denoting $K=\max |F|$ we obtain for any $\psi \in D$ with $|\psi| \leqq 1$

$$\int_C F(x) \operatorname{grad} \psi(x) \cdot \frac{x-z_k}{|x-z_k|^m}\,dx = I_1 + I_2,$$

where

$$I_1 = \int_C \text{grad}\,(F(x)\psi(x))\cdot\frac{x-z_k}{|x-z_k|^m}\,dx \leqq Kv_\infty^C(z_k),$$

$$I_2 = -\int_C \psi(x)\,\text{grad}\,F(x)\cdot\frac{x-z_k}{|x-z_k|^m}\,dx$$

$$\leqq \int_C |\text{grad}\,F(x)|\cdot|x-z_k|^{1-m}\,dx < \infty.\text{(}^2\text{)}$$

2.11. Remark. It follows from 2.2, 2.9, and 2.10 that (2.12) is a necessary and sufficient condition to secure continuous dependence (with respect to uniform convergence) of $W\psi(z)$ on ψ for every $z \notin B$. For this reason we agree to impose (2.15) on C throughout the rest of the present paragraph.

Let us recall that $\theta \in \Gamma$ is called the exterior normal of C at y in the sense of Federer provided the symmetric difference of C and the half-space

$$R^m \cap \{x : (x-y)\cdot\theta < 0\}$$

has m-dimensional density 0 at y (cf. [F1]).

In what follows the term exterior normal is always to be interpreted in this sense. We put $n^C(y)=n(y)=\theta$ if θ is the exterior normal of C at y; otherwise $n(y)$ denotes the zero vector. The set of all y with $n(y)\neq 0$ is called the reduced boundary of C and will be denoted by $\hat{B}$. It is known from [DG2] and [F3] that

$$H_{m-1}(\hat{B}) < \infty$$

and

$$\int_C \text{div}\,w(x)\,dx = \int_{\hat{B}} w(y)\cdot n(y)\,dH_{m-1}(y)$$

for every vector-valued function $w=[w_1,\dots,w_m]$ with components $w_j \in D$.

2.12. Lemma. *For every $z \in R^m$*

$$v_\infty^C(z) = \int_B \frac{|n(y)\cdot(y-z)|}{|y-z|^m}\,dH_{m-1}(y). \tag{2.16}$$

If $v_\infty^C(z)<\infty$ and $M\subset B$ is a Borel set, then

$$\nu_z(M) = \int_M \frac{n(y)\cdot(y-z)}{|y-z|^m}\,dH_{m-1}(y).$$

Proof. Fix $z \in R^m$. Let $\psi \in D(z)$ and put $w(z)=O$ ($\in R^m$),

$$w(x) = \psi(x)\frac{x-z}{|x-z|^m}, \qquad x \neq z.$$

(2) The author is indebted to Herbert Federer for simplification of this proof.

Then

$$W_\psi(z) = \int_C \operatorname{div} w(x)\, dx = \int_B \psi(y) \frac{n(y)\cdot(y-z)}{|y-z|^m}\, dH_{m-1}(y)$$

and (2.16) follows from (2.6). Let now $v_\infty^C(z) < \infty$. As we have just seen,

$$\int_B f\, d\nu_z = \int_B f(y) \frac{n(y)\cdot(y-z)}{|y-z|^m}\, dH_{m-1}(y)$$

provided $f \in D(z)$; it is easily seen that this formula extends to any bounded Baire function f.

The following result will be useful below:

2.13. THEOREM. *Let*

$$V^C = \sup \{v_\infty^C(y) : y \in B\}.$$

Then $v_\infty^C(z) \leqq A + V^C$ *for every* $z \in R^m$.

Proof. We may assume $V^C < \infty$. Fix $z \in R^m - B$ and let d be an arbitrary number less than $v_\infty^C(z)$. Then there exist mutually disjoint closed parallelepipeds $K_1, \ldots, K_q$ such that

$$\sum_{j=1}^{q} |\nu_z(B \cap K_j)| > d.$$

Put $\sigma_j = \operatorname{sign} \nu_z(B \cap K_j)$ and consider the function

$$h(x) = \sum_{j=1}^{q} \sigma_j \nu_x(B \cap K_j),$$

which is harmonic on

$$R^m - \bigcup_{j=1}^{q} B \cap K_j \supset R^m - B.$$

Fix an arbitrary $y \in B$. If $y \notin \bigcup_{j=1}^q K_j$, then

$$\lim_{x \to y} h(x) = h(y) \leqq \|\nu_y\| \leqq V^C.$$

In the opposite case we may assume that $y \in K_1$, so that

$$\lim_{x\to y} \sum_{j=2}^{q} \sigma_j \nu_x(B \cap K_j) = \sum_{j=2}^{q} \sigma_j \nu_y(B \cap K_j) \leqq \|\nu_y\| \leqq V^C$$

and, by Proposition 2.6,

$$\sup_x |\nu_x(B \cap K_1)| \leqq A.$$

We see that

$$\limsup_{x \to y;\, x \notin B} h(x) \leqq A + V^C.$$

Noting that $h(x) \to 0$ as $|x| \to \infty$ we conclude that $h \leqq A + V^C$ on $R^m - B$. In particular, $d < h(z) \leqq A + V^C$.

2.14. COROLLARY. *If $r > 0$ and $z \in R^m$, then*

$$H_{m-1}(\Omega_r(z) \cap \hat{B}) \leqq m(m+1)^m(A+V^C)r^{m-1}. \tag{2.17}$$

Proof. To prove (2.17) we may clearly assume that $z=0$. Noting that V^C is invariant with respect to dilations of C we observe that it is sufficient to establish (2.17) for $r=1$ only. Let e^i denote the point in R^m all of whose coordinates vanish with the exception of the ith one which is equal to $m+1$. We have then for $\theta \in \Gamma$ and $y \in \Omega = \Omega_1(0)$

$$\sum_{i=1}^{m} |\theta \cdot (y - e^i)| \geqq 1,$$

so that

$$\begin{aligned} H_{m-1}(\hat{B} \cap \Omega) &\leqq \sum_{i=1}^{m} \int_B |n(y)\cdot(y-e^i)| \, dH_{m-1}(y) \\ &\leqq (m+1)^m \sum_{i=1}^{m} \int_B \frac{|n(y)\cdot(y-e^i)|}{|y-e^i|^m} \, dH_{m-1}(y) \\ &= (m+1)^m \sum_{i=1}^{m} v_\infty^C(e^i) \leqq m(m+1)^m(A+V^C). \end{aligned}$$

2.15. THEOREM. *Let $C(B)$ denote the Banach space of all continuous functions f on B with the norm $\|f\| = \sup |f|$. If Wf is bounded on $R^m - B$ for every $f \in C(B)$ then*

$$V^C < \infty. \tag{2.18}$$

If

$$C_i = R^m \cap \{z : d_C(z) = i\} \qquad (i = 0, 1)$$

and (2.18) *holds, then Wf is bounded and uniformly continuous on each of the sets C_0, C_1 and*

$$\lim_{z \to y;\, z \in C_1} Wf(z) = Wf(y) + A(1 - d_C(y))f(y) \qquad \text{for } y \in B \cap \text{cl } C_1, \tag{2.19}$$

$$\lim_{z \to y;\, z \in C_0} Wf(z) = Wf(y) - A d_C(y) f(y) \qquad \text{for } y \in B \cap \text{cl } C_0 \tag{2.20}$$

whenever $f \in C(B)$.

Proof. If $Wf(z) = \langle f, \nu_z \rangle$ is a bounded function of z on $Q \subset R^m$ for every $f \in C(B)$ then, by the uniform boundedness principle, $\|\nu_z\| = v_\infty^C(z)$ is bounded on Q. In view of 2.9, $v_\infty^C(z)$ must be bounded on cl Q as well. For $Q = R^m - B$ we get the first part of our theorem. Assume (2.18) and fix $y \in B$. If $f \equiv 1$ on B then (2.19), (2.20) follow from 2.6, 2.7. It is therefore sufficient to prove (2.19), (2.20) assuming $f \in C(B)$, $f(y)=0$. For every k we have the decomposition $f=f_k+g_k$, where $f_k \in C(B)$ vanishes in some neighborhood of y in B and $\|g_k\| \leqq 1/k$. Then Wf_k is continuous at y and $|Wg_k| \leqq (A+V^C)/k$. We see that $Wf=\lim_{k\to\infty} Wf_k$ is continuous at y. The rest is obvious.

3. The Fredholm radius of an operator.

3.1. *Notation.* As in the introduction, G will stand for a fixed open set with a compact boundary B in R^m. We put $C=R^m-G$ and write $v_r(y)=v_r^G(y)\ (=v_r^C(y))$, $V=V^C$ (cf. 1.6, 1.7, (1.12), 2.13). We always assume

$$V < \infty. \tag{3.1}$$

In view of 1.13,

$$NU : \mu \to NU\mu \tag{3.2}$$

is a bounded linear operator on $C^*(B)$. By 2.7, G has an m-dimensional density $d_G(y)$ at any $y \in R^m$.

3.2. LEMMA. *If f is a bounded Baire function on B then*

$$\langle f, NU\delta_y\rangle = Ad_G(y)f(y)+W^Cf(y), \qquad y \in B. \tag{3.3}$$

Proof. It is sufficient to prove (3.3) for $f \in D$ only. Employing 1.3, 2.6, and 2.7 we obtain

$$\begin{aligned}\langle f, NU\delta_y\rangle &= \int_G \operatorname{grad} f(x)\cdot\frac{y-x}{|y-x|^m}\,dx\\ &= Af(y)+\int_C \operatorname{grad} f(x)\cdot\frac{x-y}{|x-y|^m}\,dx\\ &= Ad_G(y)f(y)+W^Cf(y).\end{aligned}$$

3.3. DEFINITION. If $f \in C(B)$ we define

$$\overline{W}f(y) = \langle f, NU\delta_y\rangle-\tfrac{1}{2}Af(y), \qquad y \in B. \tag{3.4}$$

3.4. LEMMA. *$\overline{W}f \in C(B)$ whenever $f \in C(B)$. The operator*

$$\overline{W} : f \to \overline{W}f \tag{3.5}$$

is bounded on $C(B)$ and the operator (3.2) *is adjoint to $\frac{1}{2}AI+\overline{W}$, where I is the identity operator on $C(B)$. If $f \in C(B)$ and C_1 has the meaning described in* 2.15, *then*

$$\overline{W}f(y) = \lim_{z\to y;\ z\in C_1} W^Cf(z)-\tfrac{1}{2}Af(y), \qquad y \in B\cap \operatorname{cl} C_1, \tag{3.6}$$

$$\begin{aligned}\overline{W}f(y) &= W^Cf(y)+A(d_G(y)-\tfrac{1}{2})f(y)\\ &= \lim_{z\to y;\ z\in G} W^Cf(z)+\tfrac{1}{2}Af(y), \qquad y \in B.\end{aligned} \tag{3.7}$$

Proof. (3.7), (3.6) follow from (3.4), (3.3), and (2.19), (2.20). By (3.7), $\overline{W}f \in C(B)$ for $f \in C(B)$. If ν_y has the meaning described in 2.2 and

$$\bar{\nu}_y = A(d_G(y)-\tfrac{1}{2})\delta_y+\nu_y, \tag{3.8}$$

then

$$\overline{W}f(y) = \langle f, \bar{\nu}_y\rangle, \qquad f \in C(B), \qquad y \in B, \tag{3.9}$$

whence

$$\|\overline{W}\| = \sup_{y\in B} \|\bar{\nu}_y\| = \sup_{y\in B} (A|d_G(y)-\tfrac{1}{2}|+v_\infty(y)). \tag{3.10}$$

By 1.13, the formula (1.2) holds for any $\psi \in C(B)$. This together with (3.4) implies

$$NU = (\tfrac{1}{2}AI+\overline{W})^*, \tag{3.11}$$

where $(\cdots)^*$ denotes the operator adjoint to $(\cdots)$.

3.5. REMARK. In §4 we shall be engaged with the Neumann problem in the following formulation: Given $\nu \in C^*(B)$ find a $\mu \in C^*(B)$ with $NU\mu=\nu$. By (3.11), this problem reduces to solving the equation

$$(\tfrac{1}{2}AI+\overline{W})^*\mu = \nu.$$

In connection with this equation it is useful to know the Fredholm radius of $\overline{W}$, i.e., the reciprocal of

$$\omega\overline{W} = \inf_T \|\overline{W}-T\|,$$

where T ranges over all compact operators on $C(B)$ (cf. [RS]). Our main objective in §3 is to express $\omega\overline{W}$ in terms of geometric quantities connected with G and investigate relations between $\omega\overline{W}$ and regularity of B.

3.6. THEOREM. *Let I_B denote the set of all isolated points of B and put $E=B-I_B$ if I_B is finite, $E=B$ in the opposite case. Let $V_r=0$ or*

$$V_r = \sup_{y\in E} [A|\tfrac{1}{2}-d_G(y)|+v_r(y)]$$

according as $E=\varnothing$ or not and define

$$V_0 = \lim_{r\to 0+} V_r.$$

Then $\omega\overline{W}=V_0$.

Proof will be divided into two steps.

Step 1. We first prove that

$$\omega\overline{W} \leqq V_r \tag{3.12}$$

for every $r>0$ satisfying

$$H_{m-1}(\hat{B} \cap \{z : |z-y| = r\}) = 0 \qquad \text{for all } y, \tag{3.13}$$

where $\hat{B}$ is the reduced boundary defined in 2.11. If R is the set of all $r>0$ enjoying (3.13) then $(0, \infty)-R$ is at most countable, because spherical shells with different radii meet each other in a set of H_{m-1}-measure zero and $H_{m-1}(\hat{B})<\infty$. Hence $V_0=\inf\{V_r : r \in R\}$ and

$$\omega \overline{W} \leqq V_0 \tag{3.14}$$

will follow from (3.12). So let us fix $r \in R$. If I_B is finite we assume, as we may, $r<\operatorname{dist}(I_B, E)=\inf\{\operatorname{dist}(z, E) : z \in I_B\}$. Let c_y denote the characteristic function of $B-(\Omega_r(y) \cap E)$ and put

$$W_r f(y) = \int_B c_y f \, d\bar{\nu}_y, \qquad f \in C(B),$$

where $\bar{\nu}_y$ is defined by (3.8). Absolute values of all the functions in

$$\{W_r f : f \in C(B), \|f\| \leqq 1\} \tag{3.15}$$

are bounded by $\sup_{y\in B} \|\bar{\nu}_y\| \leqq \frac{1}{2}A+V$. If $f \in C(B)$ and x, y are arbitrary points in E with $|x-y|=d \leqq \frac{1}{2}r$, then we obtain from 2.12

$$W_r f(x) - W_r f(y) = J_1(f)+J_2(f),$$

where

$$J_1(f) = \int_B f(z)[c_x(z)-c_y(z)] \frac{n(z)\cdot(z-x)}{|z-x|^m} \, dH_{m-1}(z),$$

$$J_2(f) = \int_B f(z)c_y(z)\left[\frac{z-x}{|z-x|^m}-\frac{z-y}{|z-y|^m}\right]\cdot n(z) \, dH_{m-1}(z).$$

Denoting

$$\alpha(d) = \sup_z H_{m-1}[(\operatorname{cl} \Omega_{r+d}(z)-\Omega_{r-d}(z)) \cap \hat{B}]$$

we get for $\|f\| \leqq 1$

$$|J_1(f)| \leqq (\tfrac{1}{2}r)^{1-m}\alpha(d),$$

$$|J_2(f)| \leqq (m+1)d(\tfrac{1}{2}r)^{-m}.$$

Since $r \in R$, an easy compactness argument yields

$$\lim_{d\to 0+} \alpha(d) = 0.$$

We see that all the functions in (3.15) are equicontinuous on E; noting that $B-E$ is finite we conclude that they are equicontinuous on B as well and the operator

$$W_r : f \to W_r f$$

is compact. Hence

$$\omega \overline{W} \leqq \|\overline{W}-W_r\|.$$

If $f \in C(B)$ then

$$W_r f(y) = \overline{W} f(y) \qquad \text{for } y \in B - E$$

while (3.9) shows that, for $y \in E$,

$$(\overline{W} - W_r) f(y) = \int f \, d\bar{\nu}_y$$

with the integral extended over $B \cap \Omega_r(y)$. Consequently,

$$\|\overline{W} - W_r\| = \sup_{y \in E} |\bar{\nu}_y|(\Omega_r(y) \cap B) = V_r$$

and (3.12) is established.

Step 2. Now we are going to prove the inequality

$$\omega \overline{W} \geqq V_0 \tag{3.16}$$

which is trivial if $E = \varnothing$. Therefore we assume $E \neq \varnothing$, so that E is infinite. A point $y \in B$ will be termed a discontinuity for a $\mu \in C^*(B)$ if $\mu(\{y\}) \neq 0$. By the Radon theorem, every compact operator on $C(B)$ can be arbitrarily closely approximated by operators of finite rank. If Q is such an operator, sending $f \in C(B)$ into

$$Qf = \sum_{k=1}^{q} g_k \langle f, m_k \rangle, \tag{3.17}$$

where $g_k \in C(B)$ and $m_k \in C^*(B)$, then every m_k can be arbitrarily closely (in the norm of $C^*(B)$) approximated by $\bar{m}_k \in C^*(B)$ having only a finite number of discontinuities. Defining

$$\overline{Q} \cdots = \sum_{k=1}^{q} g_k \langle \cdots, \bar{m}_k \rangle$$

we see that the deviation $\|Q - \overline{Q}\|$ can be made as small as we want. It follows from these observations that, in order to prove (3.16), it is sufficient to show that

$$\|\overline{W} - Q\| \geqq V_0 \tag{3.18}$$

for every Q of the type (3.17), where $m_k \in C^*(B)$ have only a finite number of discontinuities each. Let us fix such a Q and denote by K the (finite) set of all $y \in B$ which represent a discontinuity for some of the measures m_k $(k = 1, \ldots, q)$. Every m_k splits into m_k^1 having no discontinuities and a finite combination of Dirac measures, to be denoted by m_k^2. Since y is the only possible discontinuity for $\bar{\nu}_y$, we have for $y \in B - K$

$$\left\|\bar{\nu}_y - \sum_k g_k(y) m_k\right\| = \left\|\bar{\nu}_y - \sum_k g_k(y) m_k^1\right\| + \left\|\sum_k g_k(y) m_k^2\right\|,$$

whence

$$\|\overline{W} - Q\| \geqq \sup\left\{\left\|\bar{\nu}_y - \sum_k g_k(y) m_k^1\right\| : y \in E - K\right\}.$$

Since the operator

$$f \to \left\langle f, \bar{\nu}_y - \sum_k g_k(y) m_k^1 \right\rangle$$

sends each $f \in C(B)$ into a continuous function of y we conclude that

$$a_r(y) = \left| \bar{\nu}_y - \sum_k g_k(y) m_k^1 \right| (\Omega_r(y) \cap B)$$

is a lower semicontinuous function of y for every $r > 0$. Consequently,

$$\begin{aligned} \|\overline{W} - Q\| &\geqq \sup \{a_r(y) : y \in E - K\} \\ &= \sup \{a_r(z) : z \in E - (I_B \cap K)\}. \end{aligned} \tag{3.19}$$

Consider now an arbitrary $y \in E \cap I_B \cap K$ and note that $E \cap I_B \neq \varnothing$ implies

$$\varnothing \neq E \cap (I_B - K) \subset E - (I_B \cap K).$$

If

$$r < \operatorname{dist}(I_B \cap K, E - (I_B \cap K)),$$

then

$$\begin{aligned} \left| \sum_k g_k(y) m_k^1 \right| (\Omega_r(y) \cap B) &= \left| \sum_k g_k(y) m_k^1 \right| (\Omega_r(y) \cap I_B) = 0, \\ a_r(y) &= |\bar{\nu}_y| (\Omega_r(y) \cap I_B) = \tfrac{1}{2} A. \end{aligned} \tag{3.20}$$

On the other hand, we have for any $z \in I_B$

$$\tfrac{1}{2} A \leqq |A(\tfrac{1}{2} - d_G(z)\delta_z)| (\Omega_r(z) \cap B) + \left| \nu_z - \sum_k g_k(z) m_k^1 \right| (\Omega_r(z) \cap B) = a_r(z),$$

because $\nu_z - \sum_k g_k(z) m_k^1$ has no discontinuities. Combining this with (3.20) we get

$$a_r(y) \leqq \sup \{a_r(z) : z \in E \cap (I_B - K)\} \leqq \sup \{a_r(z) : z \in E - (I_B \cap K)\}.$$

We have thus for small $r > 0$

$$\sup \{a_r(z) : z \in E - (I_B \cap K)\} = \sup \{a_r(y) : y \in E\}. \tag{3.21}$$

Note that

$$V_r = \sup_{y \in E} |\bar{\nu}_y| (\Omega_r(y) \cap B).$$

If $M = \max \{|g_k(x)| : x \in B,\ 1 \leqq k \leqq q\}$, then

$$\sup \{a_r(y) : y \in E\} \geqq V_r - M \sum_k \sup_{y \in E} |m_k^1| (\Omega_r(y) \cap B). \tag{3.22}$$

Since m_k^1 $(k=1, \dots, q)$ have no discontinuities,

$$\lim_{r\to 0+} \sup_{y\in B} |m_k^1|(\Omega_r(y) \cap B) = 0.$$

Making $r \to 0+$ in (3.22) and using (3.21), (3.19) we arrive at (3.18).

REMARK. The basic idea of the above proof goes back to J. Radon (cf. [RS]).

3.7. LEMMA. *Let us define $\hat{B}$ as in* 2.11 *and put*

$$B^* = B \cap \{y : |d_G(y)-\tfrac{1}{2}| < \tfrac{1}{2}\}.$$

Then $\hat{B}$ is dense in B^* (moreover, every ball of center in B^* meets $\hat{B}$ in a set of positive H_{m-1}-measure) and

$$H_{m-1}(B^*-\hat{B}) = 0.$$

Proof. If $y \in B^*$ then there is an $\varepsilon > 0$ such that

$$H_m(\Omega_r(y) \cap G) > \varepsilon H_m(\Omega_r(y)),$$

$$H_m(\Omega_r(y) \cap C) > \varepsilon H_m(\Omega_r(y))$$

for $0 < r < \varepsilon$. By the relative isoperimetric inequality for sets with finite perimeter (cf. Theorem (4.3) in [MI]; general isoperimetric inequalities for currents may be found in [FF, §6]) we conclude that

$$H_{m-1}(\Omega_r(y) \cap \hat{B}) \geqq \alpha r^{m-1}, \qquad 0 < r < \varepsilon,$$

where $\alpha > 0$ does not depend on r. Hence it follows by [F2, §3] that

$$H_{m-1}(B^*-\hat{B}) = 0.$$

3.8. *Notation.* For $z \in R^m$, $r > 0$ and $\theta \in \Gamma$ we put

$$\Omega_r(z, \theta) = \Omega_r(z) \cap \{x : (x-z)\cdot\theta > 0\}.$$

We denote by $a(\theta, \eta) = \arccos(\theta\cdot\eta)$ the nonoriented angle enclosed by θ, $\eta \in \Gamma$. It is easily seen that

$$\frac{a(\theta, \eta)}{2\pi} = \frac{H_m(\Omega_r(z, \theta) \cap \Omega_r(z, -\eta))}{H_m(\Omega_r(z))}. \tag{3.23}$$

The symbol n will always have the meaning described in 2.11. The symmetric difference of P, $Q \subset R^m$ will be denoted by $P \dot{-} Q$.

3.9. LEMMA. *Let $z \in \hat{B}$, $\theta = n(z)$. Then*

$$H_m(\Omega_r(z, \theta) \cap \operatorname{int} C) + H_m(\Omega_r(z, -\theta) \cap G) \leqq H_m(\Omega_r(z)) \frac{v_r(z)}{A}.$$

Proof. Let

$$\gamma_1 < \frac{H_m(\Omega_r(z, \theta) \cap \operatorname{int} C)}{H_m(\Omega_r(z))}, \qquad \gamma_2 < \frac{H_m(\Omega_r(z, -\theta) \cap G)}{H_m(\Omega_r(z))}.$$

Put $\Gamma_+ = \Gamma \cap \{\eta : \eta \cdot \theta > 0\}$, $\Gamma_- = \Gamma \cap \{\eta : \eta \cdot \theta < 0\}$, $S(\rho) = \{x : |x-z| = \rho\}$ and define K_z, L_z as in 2.6. There are $\rho_1, \rho_2 \in (0, r)$ such that

$$H_{m-1}(S(\rho_1) \cap \Omega_r(z, \theta) \cap \operatorname{int} C) > \gamma_1 H_{m-1}(S(\rho_1)), \tag{3.24}$$

$$H_{m-1}(S(\rho_2) \cap \Omega_r(z, -\theta) \cap G) > \gamma_2 H_{m-1}(S(\rho_2)). \tag{3.25}$$

By virtue of 2.7

$$H_{m-1}(L_z \cap \Gamma_+) = \tfrac{1}{2}A \lim_{\rho \to 0+} \frac{H_m(\Omega_\rho(z, \theta) \cap C)}{H_m(\Omega_\rho(z, \theta))} = 0,$$

$$H_{m-1}(L_z \cap \Gamma_-) = \tfrac{1}{2}A \lim_{\rho \to 0+} \frac{H_m(\Omega_\rho(z, -\theta) \cap C)}{H_m(\Omega_\rho(z, -\theta))} = \tfrac{1}{2}A.$$

We see that L_z is equivalent (H_{m-1}) with Γ_- and K_z is equivalent (H_{m-1}) with Γ_+. If $\eta \in L_z$ and $\{z + \rho\eta : 0 < \rho < r\} \cap G \neq \varnothing$ then, with the notation from 1.6, $n_r(\eta, z) \geqq 1$. Employing (3.25) we obtain

$$\int_{L_z} n_r(\eta, z)\, dH_{m-1}(\eta) > \gamma_2 A.$$

Similarly, (3.24) implies

$$\int_{K_z} n_r(\eta, z)\, dH_{m-1}(\eta) > \gamma_1 A,$$

so that

$$v_r(z) = \int_\Gamma n_r(\eta, z)\, dH_{m-1}(\eta) > (\gamma_1 + \gamma_2)A.$$

3.10. Lemma. *Let $N \in \Gamma$, $y \in R^m$, $r > 0$ and suppose that*

$$\sup_{z \in B} v_r(z) \leqq u_0 A, \tag{3.26}$$

$$H_m(\Omega_r(y, N) \cap C) \leqq u_1 H_m(\Omega_r(y)), \tag{3.27}$$

$$H_m(\Omega_r(y, -N) \cap \operatorname{cl} G) \leqq u_2 H_m(\Omega_r(y)). \tag{3.28}$$

If $s = u_0 + u_1 + u_2 < \frac{1}{2}$, then for every $\gamma > s$ there is a $\delta > 0$ (depending on $(\gamma - s)r$ only) such that

$$a(n(z), N) \leqq \pi\gamma \qquad \text{for } z \in \hat{B} \cap \Omega_\delta(y).$$

Proof. Let $\gamma = s + 2\varepsilon$, $\varepsilon > 0$, and consider a $\theta \in \Gamma$ with $a(N, \theta) > \gamma\pi$. We have by (3.23)

$$H_m(\Omega_r(y, -N) \cap \Omega_r(y, \theta)) > \tfrac{1}{2}\gamma H_m(\Omega_r(y)),$$

$$H_m(\Omega_r(y, N) \cap \Omega_r(y, -\theta)) > \tfrac{1}{2}\gamma H_m(\Omega_r(y)).$$

Let us fix $\delta>0$ small enough to secure

$$H_m(\Omega_r(z,\eta)\dot{-}\Omega_r(y,\eta)) < \varepsilon H_m(\Omega_r(y))$$

for $|z-y|<\delta$ and any $\eta\in\Gamma$. We have then for $z\in\Omega_\delta(y)$

$$H_m(\Omega_r(y,-N)\cap\Omega_r(z,\theta)) > \tfrac{1}{2}sH_m(\Omega_r(z)),$$

$$H_m(\Omega_r(y,N)\cap\Omega_r(z,-\theta)) > \tfrac{1}{2}sH_m(\Omega_r(z)),$$

whence we obtain on account of (3.27), (3.28)

$$\begin{aligned}H_m(\Omega_r(z,\theta)\cap \operatorname{int} C) &\geqq H_m(\Omega_r(z,\theta)\cap\Omega_r(y,-N))-H_m(\Omega_r(y,-N)\cap \operatorname{cl} G)\\ &> (\tfrac{1}{2}s-u_2)H_m(\Omega_r(z)),\end{aligned} \tag{3.29}$$

$$\begin{aligned}H_m(\Omega_r(z,-\theta)\cap G) &\geqq H_m(\Omega_r(z,-\theta)\cap\Omega_r(y,N))\\ &-H_m(\Omega_r(y,N)\cap C) > (\tfrac{1}{2}s-u_1)H_m(\Omega_r(z)).\end{aligned} \tag{3.30}$$

Suppose now that $z\in\hat{B}$ and $\theta=n(z)$. Employing (3.29), (3.30) and Lemma 3.9 we arrive at $v_r(z)>u_0A$, which violates (3.26).

3.11. *Notation.* Let P_N stand for the orthogonal projection of R^m onto $R^m\cap\{x : x\cdot N=0\}$. With every $\alpha\in(0,\tfrac{1}{2})$ we associate $B(\alpha)\subset B$ as follows. We let $y\in B(\alpha)$ if for every $\gamma\in(\alpha,\tfrac{1}{2})$ there is a neighborhood Q of y in B and an $N\in\Gamma$ such that $|P_N(x)-P_N(z)|\geqq|x-z|\cos\pi\gamma$ whenever $x, z\in Q$. By Theorem 5.1 in [MI] we get the following corollary of Lemma 3.10:

3.12. Corollary. *If* (3.26), (3.27), (3.28) *hold and* $s=u_0+u_1+u_2<\tfrac{1}{2}$, *then* $y\in B(s)$; *moreover, for every* $\gamma\in(s,\tfrac{1}{2})$ *there is a* $\delta>0$ (*depending on* $r(\gamma-s)$ *only*) *such that* $B\cap\Omega_\delta(y)\subset B(\gamma)$.

3.13. Theorem. *If* $V_0<\tfrac{1}{2}A$ *then* I_B *is finite and*

$$H_{m-1}(B-B(V_0/A)) = 0. \tag{3.31}$$

If $V_0<\tfrac{1}{4}A$ *then* $B=B(2V_0/A)$.

Proof. Let $V_0<\tfrac{1}{2}A$. Then I_B must be finite, $B-I_B=E\subset B^*$ and, by 3.7, $H_{m-1}(B-\hat{B})=0$. To prove (3.31) it is therefore sufficient to show that

$$\hat{B}\subset B\left(3\varepsilon+\frac{V_0}{A}\right)$$

for every small $\varepsilon>0$. Fix $y\in\hat{B}$, $N=n(y)$ and $\varepsilon>0$, $3\varepsilon+V_0/A<\tfrac{1}{2}$. We have then for sufficiently small $r>0$

$$H_m(\Omega_r(y,N)\cap C) \leqq \varepsilon H_m(\Omega_r(y)),$$

$$\begin{aligned}H_m(\Omega_r(y,-N)\cap \operatorname{cl} G) &= H_m(\Omega_r(y,-N)\cap G)\\ &\leqq \varepsilon H_m(\Omega_r(y)),\end{aligned}$$

$$V_r < V_0+\varepsilon A. \tag{3.32}$$

Employing 3.12 we get $y \in B(3\varepsilon + V_0/A)$. Suppose now that $\alpha = V_0/A < \frac{1}{4}$ and fix an $r > 0$ with (3.32). By 3.9 we have for all $y \in \hat{B}$

$$H_m(\Omega_r(y, n(y)) \cap C) + H_m(\Omega_r(y, -n(y)) \cap \text{cl } G) < (\alpha+\varepsilon) H_m(\Omega_r(y)).$$

By 3.12 there is a $\delta > 0$ independent of y such that $\Omega_\delta(y) \cap B \subset B(3\varepsilon + 2\alpha)$ for every $y \in \hat{B}$. It remains to note that $\hat{B}$ is dense in E by 3.7.

3.14. Corollary. *If $V_0 < \frac{1}{4}A$ then*

$$\lim_{r \to 0+} \sup \{\rho^{1-m} H_{m-1}(\Omega_\rho(y) \cap B) : y \in B, 0 < \rho < r\} \leqq b_{m-1} \sec (2V_0\pi/A),$$

where b_{m-1} denotes the volume of the unit ball in R^{m-1}.

3.15. Theorem. *Let $V_0 = 0$ (which means that $\overline{W}$ is compact). Then I_B is finite and $E = B - I_B$ is a surface of class C^1.*

Proof. For every $\varepsilon > 0$, $\varepsilon < \frac{1}{4}$ there is an $r > 0$ such that $V_r < \varepsilon A$. By 3.9 (note also that $H_m(B) = 0$)

$$H_m(\Omega_r(y, n(y)) \cap C) + H_m(\Omega_r(y, -n(y)) \cap \text{cl } G) < \varepsilon H_m(\Omega_r(y))$$

for all $y \in \hat{B}$. Employing 3.10 we get a $\delta > 0$ depending on $r\varepsilon$ only such that, for every couple of points $y, z \in \hat{B}$, $a(n(z), n(y)) \leqq 3\varepsilon\pi$ whenever $|y - z| < \delta$. We see that n is uniformly continuous on $\hat{B}$. By 3.7, n extends to a continuous function N on $E = \text{cl } \hat{B}$ and, for every $y \in E$,

$$N(y) = \lim_{r \to 0+} \frac{\int N(z)\, dH_{m-1}(z)}{H_{m-1}(\Omega_r(y) \cap E)}$$

with the integral extended over $\Omega_r(y) \cap E$. Hence it follows by [DG3, Theorem III] (see also definition of the reduced boundary presented in [DG3, p. 10]) that E is a surface of class C^1.

Remark. The main results of this paper (such as Theorem 1.13 or Theorem 3.6) are expressed in terms of the quantity $v_r(y)$. In the definition of $v_r(y)$ one considers all half-lines issuing at y, i.e., orthogonal trajectories of the level surfaces of the Green function with a fixed pole at y. This suggests the possibility of generalizing these results to the case of a Green space in the sense of [BC].

4. Boundary value problems.

Notation. We shall keep the notation and assumptions introduced in §3. Besides that we always assume here that $m > 2$ (see Remark 4.10 below dealing with $m = 2$). We shall start with investigation of solutions of the equations

$$(4.1) \qquad (\tfrac{1}{2}AI + \overline{W})f = 0 \qquad \text{over } C(B),$$

$$(4.2) \qquad (\tfrac{1}{2}AI + \overline{W})^*\mu = 0 \qquad \text{over } C^*(B).$$

$C_0(B)$ will denote the class of all $f \in C(B)$ satisfying (4.1) and $C^*(B)$ will stand for the set of all $\mu \in C^*(B)$ satisfying (4.2). We agree to use M as a generic notation for a Borel set. If μ is a signed Borel measure in R^m and $R \subset R^m$ is a fixed Borel set, we define $\mu \cap R$ by

$$\mu \cap R(M) = \mu(M \cap R), \qquad M \subset R^m.$$

Recalling the definition of $\bar{\nu}_y$ presented in (3.8) we obtain from (3.9) that, for every $\mu \in C^*(B)$,

$$\overline{W}^*\mu(M) = \int_B \bar{\nu}_y(M)\, d\mu(y), \qquad M \subset B. \tag{4.3}$$

It follows from (3.10) that

$$\|W^*\| \leqq \tfrac{1}{2}A + V, \tag{4.4}$$

where $V = V^C$ has been defined in 2.13.

4.1. LEMMA. *If $\mu \in C_0^*(B)$ then $|\mu|(I_B) = 0$ (see 3.6 for notation).*

Proof. Let $\mu \in C_0^*(B)$, $z \in I_B$ and denote by f the characteristic function of $\{z\}$. We have by 3.4, 1.13, and (3.3)

$$0 = NU\mu(\{z\}) = \int_B (Ad_G(y)f(y) + W^Cf(y))\, d\mu(y).$$

It follows from (2.12) that $W^Cf = 0$, so that $Ad_G(y)f(y) + W^Cf(y) = Af(y)$ for all $y \in B$. Hence $\mu(\{z\}) = 0$.

REMARK. A refinement of the preceding argument may be used to show that, for every $\mu \in C_0^*(B)$, $\mu \cap M$ is absolutely continuous with respect to $H_{m-1} \cap \hat{B}$ provided $d_G(y) > 0$ for all $y \in M$.

As it follows from 4.1, $C_0^*(B)$ contains only trivial measure in case $B = I_B$. In what follows *we always exclude the trivial case of a finite B.*

4.2. LEMMA. *Fix $z \in B$, $\mu \in C^*(B)$ and put for $t > 0$*

$$R_t = B \cap \Omega_t(z), \tag{4.5}$$

$$\alpha(t) = H_{m-1}(R_t \cap \hat{B}), \tag{4.6}$$

$$\beta(t) = |\mu|(R_t). \tag{4.7}$$

Let $0 < \rho < \delta < \Delta$ and suppose that

$$\mu \cap (R_\Delta - R_\delta) = \mu.$$

Then

$$|\overline{W}^*\mu|(R_\rho) \leqq \frac{\alpha(\rho)\beta(\Delta)}{(\Delta-\rho)^{m-1}} + (m-1)\alpha(\rho)\int_\delta^\Delta \frac{\beta(t)\, dt}{(t-\rho)^m}. \tag{4.8}$$

Proof. Let g denote the characteristic function of $\hat{B} \cap R_\rho$. By 2.12 we obtain for $y \in R_\Delta - R_\delta$ and $M \subset R_\rho$

$$|\bar{\nu}_y(M)| = |\nu_y(M)| \leqq \int_M g(x)|y-x|^{1-m}\, dH_{m-1}(x),$$

whence it follows easily by (4.3)

$$(4.9) \qquad |\overline{W}^*\mu|(R_\rho) \leqq \iint\limits_{B\times B} g(x)|y-x|^{1-m}\, dH_{m-1}(x)\, d|\mu|(y).$$

Since

$$\int_B |y-x|^{1-m}\, d|\mu|(y) \leqq \int_\delta^\Delta (t-|x|)^{1-m}\, d\beta(t)$$

$$\leqq \frac{\beta(\Delta)}{(\Delta-|x|)^{m-1}} + (m-1)\int_\delta^\Delta \frac{\beta(t)\, dt}{(t-|x|)^m},$$

(4.9) implies (4.8).

4.3. Lemma. *Fix $z \in B$, $r>0$ and put, with the notation from* 4.2,

$$R = R_r(=\Omega_r(z) \cap B),$$

$$V(R) = \sup\{|\bar{\nu}_y|(R) : y \in R\},$$

$$Q(R) = \sup\{\rho^{1-m}\alpha(\rho) : 0 < \rho < r\},$$

$$K(R) = \inf\left\{V(R)k^{m-2} + Q(R)\left[\left(\frac{k}{k-1}\right)^{m-1} - 1\right] : k > 1\right\}.$$

Define

$$\overline{W}_R^*\mu = (\overline{W}^*\mu) \cap R, \qquad \mu \in C^*(B).$$

Let C_R^* denote the set of all $\mu \in C^*(B)$ enjoying

$$J(\mu) = \int_0^r \rho^{1-m}|\mu|(R_\rho)\, d\rho < \infty$$

and

$$(4.10) \qquad |\mu|(B-R) = 0$$

and put

$$\|\mu\|_R = \frac{1}{m-2} r^{2-m}\|\mu\| + J(\mu), \qquad \mu \in C_R^*.$$

Then $\mu \in C_R^*$ implies $\overline{W}_R^*\mu \in C_R^*$ and

$$\|\overline{W}_R^*\mu\|_R \leqq K(R)\|\mu\|_R.$$

Proof. Fix $\mu \in C_R^*$ and $k>1$. We have with the notation from (4.7)

$$(4.11) \qquad J(\mu) = \int_0^r \rho^{1-m}\beta(\rho)\, d\rho.$$

Let now $0<\rho<r/k$ and define

$$\mu_\rho = \mu \cap R_{k\rho}, \qquad \mu^\rho = \mu - \mu_\rho.$$

In view of (4.3)

$$\|\overline{W}_R^*\mu_\rho\| \leqq V(R)\beta(k\rho). \tag{4.12}$$

Employing 4.2 we obtain

$$|\overline{W}_R^*\mu^\rho|(R_\rho) \leqq \frac{\alpha(\rho)\beta(r)}{(r-\rho)^{m-1}}+(m-1)\alpha(\rho)\int_{k\rho}^{r}\frac{\beta(t)\,dt}{(t-\rho)^m}. \tag{4.13}$$

On account of (4.12), (4.13) we get for $0<\rho<r/k$

$$\rho^{1-m}|\overline{W}_R^*\mu|(R_\rho) \leqq V(R)\rho^{1-m}\beta(k\rho)+Q(R)\frac{\beta(r)}{(r-\rho)^{m-1}}+(m-1)Q(R)\int_{k\rho}^{r}\frac{\beta(t)\,dt}{(t-\rho)^m},$$

while, by (4.3),

$$\rho^{1-m}|\overline{W}_R^*\mu|(R_\rho) \leqq V(R)\rho^{1-m}\beta(r) \qquad \text{for } r/k \leqq \rho < r.$$

Using (4.11) we obtain after simple calculation

$$\begin{aligned} J(\overline{W}_R^*\mu) &= \int_0^{r/k}\rho^{1-m}|\overline{W}_R^*\mu|(R_\rho)\,d\rho+\int_{r/k}^{r}\rho^{1-m}|\overline{W}_R^*\mu|(R_\rho)\,d\rho \\ &\leqq \frac{\beta(r)}{m-2}r^{2-m}\left(V(R)(k^{m-2}-1)+Q(R)\left[\left(\frac{k}{k-1}\right)^{m-2}-1\right]\right) \\ &\quad +J(\mu)\left(V(R)k^{m-2}+Q(R)\left[\left(\frac{k}{k-1}\right)^{m-1}-1\right]\right). \end{aligned}$$

Since, by virtue of (4.3),

$$\|\overline{W}_R^*\mu\| \leqq V(R)\beta(r),$$

we get finally

$$\begin{aligned} \|\overline{W}_R^*\mu\|_R &= \frac{1}{m-2}r^{2-m}\|\overline{W}_R^*\mu\|+J(\overline{W}_R^*\mu) \\ &\leqq \|\mu\|_R\left(V(R)k^{m-2}+Q(R)\left[\left(\frac{k}{k-1}\right)^{m-1}-1\right]\right). \end{aligned}$$

4.4. *Notation.* Let

$$Q_r = \sup\{\rho^{1-m}H_{m-1}(\Omega_\rho(z)\cap \hat{B}) : z\in B, 0<\rho<r\}, \qquad r>0,$$

$$Q_0 = \lim_{r\to 0+} Q_r.$$

Further define V_0 as in 3.6 and put

$$K_0 = \inf\left\{V_0k^{m-2}+Q_0\left[\left(\frac{k}{k-1}\right)^{m-1}-1\right] : k>1\right\}. \tag{4.14}$$

In what follows we shall always assume that

$$K_0 < \tfrac{1}{2}A. \tag{4.15}$$

4.5. REMARK. The inequality (4.15) implies

$$V_0 < \tfrac{1}{4}A. \tag{4.16}$$

Indeed, since B is infinite and $V_0 < \frac{1}{2}A$, (3.7) secures $H_{m-1}(\hat{B}) > 0$. It is known from [DG2], [F3] that for (H_{m-1}) almost all $y \in \hat{B}$

$$\lim_{\rho \to 0+} \rho^{1-m} H_{m-1}(\Omega_\rho(y) \cap \hat{B}) = b_{m-1},$$

where b_{m-1} denotes the volume of the unit ball in R^{m-1}. Hence $Q_0 \geqq b_{m-1}$ and k minimizing

$$V_0 k^{m-2} + Q_0 \left[\left(\frac{k}{k-1} \right)^{m-1} - 1 \right]$$

must satisfy

$$\tfrac{1}{2}A > Q_0 \left[\left(\frac{k}{k-1} \right)^{m-1} - 1 \right] \geqq b_{m-1} \frac{m-1}{k-1},$$

which guarantees $k^{m-2} > 2$.

On the other hand, if (4.16) holds, then 3.14 provides an estimate for Q_0 in terms of V_0. Clearly, (4.15) is fulfilled whenever V_0 is sufficiently small.

In view of (4.16) and 3.6, the Fredholm theory applies to the pair of adjoint equations

$$(\tfrac{1}{2}AI + \overline{W})f = g,$$

$$(\tfrac{1}{2}AI + \overline{W})^* \mu = \nu.$$

4.6. LEMMA. *If $\mu \in C_0^*(B)$ then $U|\mu|$ (see the introduction for notation) is bounded on B.*

Proof. Define V_r and E as in 3.6 and fix $r > 0$ and $k > 1$ such that

$$K = V_{2r} k^{m-2} + Q_r \left[\left(\frac{k}{k-1} \right)^{m-1} - 1 \right] < \tfrac{1}{2}A,$$

$$r < \operatorname{dist}(E, B - E).$$

Fix an arbitrary $z \in E$ and define $R = \Omega_r(z) \cap B$. We have then with the notation from 4.3

$$V(R) \leqq V_{2r}, \qquad Q(R) \leqq Q_r,$$

$$K(R) \leqq K < \tfrac{1}{2}A. \tag{4.17}$$

Let $\mu \in C_0^*(B)$, $\|\mu\| \leqq 1$ and put

$$\mu_0 = \mu \cap R, \qquad \mu^0 = \mu - \mu_0.$$

In view of (4.2)

$$\tfrac{1}{2}A\mu_0+\overline{W}^*\mu_0 = -\tfrac{1}{2}A\mu^0-\overline{W}^*\mu^0. \tag{4.18}$$

Restricting all measures occurring in (4.18) to R we obtain

$$(I+2A^{-1}\overline{W}_R^*)\mu_0 = -2A^{-1}\overline{W}_R^*\mu^0 \tag{4.19}$$

where, of course, I is the identity operator. Employing 4.2 with $\delta=r$ and $\Delta=r+\operatorname{diam} B$ we obtain easily for $0<\rho\leqq r/2$

$$\rho^{1-m}|\overline{W}^*\mu^0|(R_\rho) \leqq Q_r 2^m r^{1-m}\|\mu^0\| \leqq Q_r 2^m r^{1-m}.$$

On the other hand, we have for $\rho>r/2$

$$\rho^{1-m}|\overline{W}^*\mu^0|(R_\rho) \leqq 2^{m-1}r^{1-m}\|\overline{W}^*\|\cdot\|\mu^0\| \leqq 2^{m-1}r^{1-m}(\tfrac{1}{2}A+V),$$

so that

$$J(\overline{W}_R^*\mu^0) \leqq 2^{m-2}r^{2-m}(2Q_r+\tfrac{1}{2}A+V).$$

Since

$$\|\overline{W}_R^*\mu^0\| \leqq \|\overline{W}^*\mu^0\| \leqq \tfrac{1}{2}A+V$$

we arrive at

$$\|\overline{W}_R^*\mu^0\|_R \leqq \gamma_r,$$

where

$$\gamma_r = \frac{1}{m-2}r^{2-m}(\tfrac{1}{2}A+V)+2^{m-2}r^{2-m}(2Q_r+\tfrac{1}{2}A+V).$$

We see that $\overline{W}_R^*\mu^0 \in C_R^*$. It is easily seen that C_R^*, equipped with the norm $\|\cdots\|_R$, is a Banach space. In view of (4.3) and (4.17)

$$\|\overline{W}_R^*\|_R \leqq K < \tfrac{1}{2}A.$$

Hence we conclude by virtue of (4.19) that $\mu_0 \in C_R^*$ and

$$\|\mu_0\|_R \leqq \left(1-\frac{2K}{A}\right)^{-1}2A^{-1}\gamma_r = a_r.$$

Since a_r is independent of $z\in E$, we have, in particular,

$$\sup_{z\in E}\int_0^r \rho^{1-m}|\mu|(\Omega_\rho(z)\cap B)\,d\rho < \infty,$$

whence it follows easily

$$\sup_{z\in E}\int_0^\infty \rho^{1-m}|\mu|(\Omega_\rho(z)\cap B)\,d\rho < \infty.$$

Noting that

$$U|\mu|(z) = \frac{1}{m-2}\int_B |x-z|^{2-m}\, d|\mu|(x)$$

$$= \frac{1}{m-2}\int_0^\infty |\mu|(B\cap\{x : |x-z|^{2-m} > t\})\, dt$$

$$= \int_0^\infty \rho^{1-m}|\mu|(\Omega_\rho(z)\cap B)\, d\rho$$

we see that $U|\mu|$ is bounded on E. Since, by 4.1, spt $\mu\subset E$ and $B-E$ has a positive distance from E, $U|\mu|$ is bounded on B as well.

4.7. *Notation.* It follows easily from (4.16) and 3.13 that G has only a finite number of components; their closures are mutually disjoint. We shall denote by $q(0\leqq q<\infty)$ the number of bounded components of G. G_0 will stand for the unbounded component of G (if any); the bounded components of G will be denoted by $G_1,\ldots, G_q$.

Employing 4.6 we obtain by standard reasoning the following.

4.8. LEMMA. *The dimension of $C_0^*(B)$ does not exceed q.*

Proof. Let $\mu\in C_0^*(B)$. By 4.6, $U|\mu|$ is bounded on B. Hence it follows that μ has finite energy [B, p. 122] and

$$\int_{R^m} |\text{grad } U\mu(x)|^2\, dx = A\int_B U\mu(y)\, d\mu(y) < \infty$$

(see [B, pp. 131, 132]). In particular, there are $\phi_k\in D$ such that

$$\int_{R^m} |\text{grad } \phi_k(x) - \text{grad } U\mu(x)|^2\, dx \to 0 \qquad \text{as } k\to\infty.$$

(4.2) means that $NU\mu=0$ (see 3.4), so that

$$\int_G \text{grad } \phi_k(x)\cdot\text{grad } U\mu(x)\, dx = 0$$

for each k; making $k\to\infty$ we obtain

$$\int_G |\text{grad } U\mu(x)|^2\, dx = 0.$$

We see that $U\mu$ is constant on each G_j and vanishes on G_0. Next we prove the following assertion:

(a) *If $U\mu=0$ on G then $\mu=0$.*

Indeed, let $\mu=\mu_1-\mu_2$ be the Jordan decomposition of μ and assume that $U\mu_1$ and $U\mu_2$ coincide on G. Since G has a positive m-dimensional density at any $z\in B$, every fine neighborhood of z (in the Cartan topology) meets G (compare

[B, p. 78, paragraph 2, §3 and p. 84, paragraph 6]) and we conclude from the Cartan Theorem [B, p. 86; see also p. 84] that $U\mu_1(z)=U\mu_2(z)$. Since $U\mu_1$ and $U\mu_2$ coincide on B, they must coincide on R^m, by the domination principle [B, p. 123]. We have thus $U\mu=0$ on R^m, whence $\mu=0$ [B, p. 122].

If $q=0$ then (a) completes the proof of 4.8. Assume now $q>0$. With every $\mu \in C_0^*(B)$ we may associate the q-tuple $c(\mu)=[c_1(\mu), \ldots, c_q(\mu)]$, where $c_j(\mu)$ is the value taken on by $U\mu$ in G_j. The map

$$c: \mu \to c(\mu)$$

is an injection of $C_0^*(B)$ into R^q. Indeed, $c(\mu)=0$ means that $U\mu=0$ on G and (a) implies $\mu=0$.

4.9. PROPOSITION. *Let f_j denote the characteristic function of* fr G_j $(1 \leqq j \leqq q)$. *Then $\{f_1, \ldots, f_q\}$ is a basis in $C_0(B)$.*

Proof. Let us fix $j \in \langle 1, q\rangle$ and put $H=R^m-G_j$. Employing 2.12 and 2.6 we obtain for any $z \in R^m - \mathrm{cl}\, G \subset \mathrm{int}\, H$

$$W^C f_j(z) = \nu_z^H(\mathrm{fr}\, G_j) = 0,$$

whence it follows by (3.6)

$$(\tfrac{1}{2}AI+\overline{W})f_j = 0,$$

so that $f_j \in C_0(B)$. Since the dimension of $C_0(B)$ coincides with the dimension of $C_0^*(B)$ which is known to be $\leqq q$ and $f_1, \ldots, f_q$ are linearly independent, the proof is complete.

4.10. REMARK. Combining the above proposition and Fredholm's theorems one obtains Theorems 4.11–4.13 below.

If $m=2$ then 4.9 holds under more general assumptions on B. It is sufficient to require that E (see 3.6) consists of mutually disjoint simple closed curves and $V_0<\frac{1}{2}A$ (compare [K3], where further references may be found).

4.11. THEOREM. *Let $\nu \in C^*(B)$. Then $\nu=NU\mu$ for some $\mu \in C^*(B)$ if and only if*

$$\nu(\mathrm{fr}\, G_j) = 0, \qquad j = 1, \ldots, q.$$

Proof. This follows at once from 4.9 and the Fredholm Theorem.

4.12. THEOREM. *Let $\{f_1, \ldots, f_q\}$ be a basis in $C_0(B)$. Given $g \in C(B)$ there are $f \in C(B)$ and constants α_j $(j=1, \ldots, q)$ such that, for every $y \in B$, $Wf(x)$ tends to*

$$g(y) - \sum_{j=1}^{q} \alpha_j f_j(y)$$

as $x \to y$, $x \in \mathrm{int}\, C$. The constants α_j are uniquely determined and f is determined modulo $C_0(B)$.

Proof. Let $\{\mu_1, \ldots, \mu_q\}$ and $\{f_1, \ldots, f_q\}$ be dual bases in $C_0^*(B)$ and $C_0(B)$, respectively. Given $g \in C(B)$ we can find α_k so that

$$\left\langle g - \sum_{k=1}^{q} \alpha_k f_k, \mu_j \right\rangle = 0$$

for all j; clearly, $\alpha_k = \langle g, \mu_k \rangle$. Then

$$\left\langle g - \sum_{k=1}^{q} \alpha_k f_k, C_0^*(B) \right\rangle = 0$$

and the Fredholm Theorem yields an $f \in C(B)$ such that

$$(\tfrac{1}{2}AI + \overline{W})f = g - \sum_{k=1}^{q} \alpha_k f_k.$$

The rest follows from (3.6).

Standard reasoning yields also the following.

4.13. THEOREM. *Fix $x_j \in G_j$ ($j = 1, \ldots, q$). Given $g \in C(B)$ there are $f \in C(B)$ (determined modulo $C_0(B)$) and uniquely determined constants a_j such that, for every $y \in B$,*

$$Wf(x) + \sum_{j=1}^{q} a_j |x - x_j|^{2-m}$$

tends to $g(y)$ as $x \to y$, $x \in \operatorname{int} C$.

Proof. Define g_k by

$$g_k(x) = \frac{1}{m-2} |x - x_k|^{2-m}.$$

Then $\langle g_k, \mu \rangle = U\mu(x_k)$ for every $\mu \in C^*(B)$. It follows from (3.6) that

$$Wf + \sum_{j=1}^{q} \alpha_j g_j \, (f \in C(B), \alpha_j \in R^1)$$

represents a solution of the Dirichlet problem for C and the boundary condition g if and only if

$$(\tfrac{1}{2}AI + \overline{W})f = g - \sum_{j=1}^{q} \alpha_j g_j \tag{4.20}$$

on B. For the existence of an $f \in C(B)$ satisfying (4.20) it is necessary and sufficient that

$$\left\langle g - \sum_{j=1}^{q} \alpha_j g_j, C_0^*(B) \right\rangle = 0,$$

i.e.,

$$\sum_{j=1}^{q} \alpha_j U\mu(x_j) = \langle g, \mu \rangle, \qquad \mu \in C_0^*(B). \tag{4.21}$$

We know from the proof of 4.8 (note also that $C_0^*(B)$ has dimension q) that

$$\mu \rightarrow [U\mu(x_1), \ldots, U\mu(x_q)]$$

is an isomorphism of $C_0^*(B)$ onto R^q. Consequently, (4.21) determines α_j uniquely. The rest is obvious.

REMARK. Results related to some of those proved in the present paper were announced without proofs in [K1] (for the plane), [BMS] and [MS] (for a domain bounded by a simple closed surface in 3-space), [K2] (for a domain bounded by a hyper-surface in m-space) and in Abstract 630-197, (Theorem 1.13), Notices Amer. Math. Soc. **13** (1966).

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STATEMENT OF OWNERSHIP, MANAGEMENT AND CIRCULATION

(Act of October 23, 1962; Section 4369, Title 39, United States Code)

1. *Date of filing*: September 23, 1966
2. *Title of Publication*: Transactions of the American Mathematical Society
3. *Frequency of issue*: Monthly
4. *Location of known office of publication*: P. O. Box 6248, Providence, R. I. 02904
5. *Location of the headquarters or general business offices of the publishers*: P. O. Box 6248, Providence, R. I. 02904
6. *Names and addresses of publisher, editor and managing editor. Publisher*: American Mathematical Society, P. O. Box 6248, Providence, R. I. 02904. *Editor*: Louis Nirenberg, Chairman of the Editorial Committee, P. O. Box 6248, Providence, R. I. 02904. *Managing editor*: None
7. *Owner*: None
8. *Known bondholders, mortgagees, and other security holders owning or holding 1 percent or more of total amount of bonds, mortgages or other securities*: None
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